

Domination when the Stars Are Out

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Abstract

We algorithmize the recent structural characterization for claw-free graphs by Chudnovsky and Seymour. Building on this result, we show that Dominating Set on claw-free graphs is (i) fixed-parameter tractable and (ii) even possesses a polynomial kernel. To complement these results, we establish that Dominating Set is not fixed-parameter tractable on the slightly larger class of graphs that exclude $K_{1,4}$ as an induced subgraph. Our results provide a dichotomy for Dominating Set in $K_{1,\ell}$ -free graphs and show that the problem is fixed-parameter tractable if and only if $\ell \leq 3$. Finally, we show that our algorithmization can also be used to show that the related Connected Dominating Set problem is fixed-parameter tractable on claw-free graphs.

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1 Introduction

The dominating set problem is the problem of determining whether a given graph G has a dominating set of a specified size k . (A subset $D \subseteq V(G)$ is dominating if every vertex in G is either contained in D or adjacent to some vertex in D .) Dominating sets play a prominent role both in algorithmics and in combinatorics: There are books [27, 28] dedicated entirely to dominating sets, while Hedetniemi and Laskar list over 300 papers related to domination in graphs [29]. In complexity theory, Dominating Set was one of the first problems recognized as NP-complete [32], and one of the first examples of an NP-hard optimization problem with an approximation algorithm [31] whose approximation factor guarantee is tight under reasonable assumptions [21]. Dominating Set also plays a central role in parameterized complexity. It motivated the definition of the W-hierarchy, being the first natural example of a problem complete for the class W[2], and has been used since then in many reductions for showing intractability of other problems [16]. In particular, unless $\text{FPT}=\text{W}[2]$, Dominating Set is not solvable by an algorithm with running time $f(k) \cdot n^{O(1)}$ for any computable function f [15].

Since the dominating set problem is hard in its decision, approximation, and parameterized versions, research has focused on finding special graph classes for which the problem becomes tractable. In this paper we consider the class of claw-free graphs. A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbors, i.e. if it does not contain $K_{1,3}$ as an induced subgraph. The class of claw-free graphs contains several well-studied graph classes as a special case, including line graphs, unit interval graphs, the complements of triangle-free graphs, and graphs of several polyhedra and polytopes. Throughout the years, this graph class attracted much interest, and is by now the subject of hundreds of mathematical research papers and surveys [19].

In the context of algorithms, initial study was directed towards extending algorithms developed for line graphs. The foremost examples are the two independent results by Shibi [41] and Minty [35] (the latter corrected by Nakamura and Tamura [37]), which extend Edmond's classical polynomial-time algorithm for Maximum Independent Set on line graphs [17], better known as the maximum matching, to the class of claw-free graphs. In contrast to the independent set problem, Dominating set on line graphs, also known as Edge Dominating Set, is NP-complete [43]. Nevertheless, Fernau recently showed that this problem has an $f(k) \cdot n^{O(1)}$ time algorithm, where k is the of the solution [23]. Whether this result extends to claw-free graphs has been left an open question.

There has also been considerable study devoted to graphs excluding $K_{1,\ell}$ for $\ell > 3$. These types of graphs appear frequently when considering geometric intersection graphs of various types. For instance, unit square graphs are $K_{1,5}$ -free, and unit disk graphs are $K_{1,6}$ -free. Marx showed that Dominating Set is W[1]-hard on unit squares graphs, implying that it is also hard on $K_{1,5}$ -free graphs [34]. Note that the problem becomes easy on $K_{1,2}$ -free graphs, since these graphs are just disjoint unions of cliques. Thus, the remaining cases left open are $K_{1,3}$ -free and $K_{1,4}$ -free graphs. It is important to note that we exclude these graphs as an *induced* subgraph and not as a subgraph. In the latter case, Dominating Set is actually known to be fixed-parameter tractable due to a result by Philip et al. [38].

1.1 Our results In this paper, we show how to extend the algorithm of Fernau [23] from line graphs to claw-free graphs. For this, we use a recent highly nontrivial structural characterization of claw-free graphs due to Chudnovsky and Seymour. This characterization shows that every claw-free graph can be built by applying certain gluing operations to certain atomic structures. The proof of this characterization is contained in a sequence of seven papers (called *Claw-free graphs I* through *VII*). The survey [8] gives a compact accessible summary of the main ideas in the proof.

The original proof of the Chudnovsky and Seymour characterization theorem for claw-free

graphs is essentially nonalgorithmic. Thus, the main challenge in our approach was to provide an algorithmic version of this theorem. We fully meet and settle this challenge in Section 2.

Several attempts to characterize claw-free graphs and its subclasses have been made before the full decomposition theorem of Chudnovsky and Seymour appeared. For instance, Fouquet [24] showed that certain claw-free graphs are either so-called quasi-line graphs or possesses a vertex with a C_5 as an induced subgraph in its neighborhood. Inspired by this approach and drawing on ideas from the Chudnovsky-Seymour characterization, Faenza et al. [20] recently developed a new polynomial-time algorithm for Maximum Independent Set on claw-free graphs. For Dominating Set, however, the core difficulty of the problem persists even in these quasi-line graphs. Hence such a much finer decomposition is needed, which we develop in this paper. Moreover, we believe that having such a fine decomposition will prove useful in attacking other optimization problems on claw-free graphs.

Using our algorithmic claw-free decomposition, we establish the following:

- Dominating Set on claw-free graphs is fixed-parameter tractable. To be precise, we show that we can decide the existence of a dominating set of size at most k in $O^*(9^k)$ time (Section 3).
- Connected Dominating Set on claw-free graphs is fixed-parameter tractable and solvable in $O^*(36^k)$ time (Section 3). This resolves an open question due to Misra et al. [36].
- Dominating Set on claw-free graphs has a polynomial kernel with $O(k^4)$ vertices (Section 5).

To complement our results, we show that Dominating Set is W[1]-hard on $K_{1,4}$ -free graphs (see Section 6 for the proof). Thus, we completely determine the parameterized complexity status of Dominating Set in $K_{1,\ell}$ -free graphs for all values of ℓ .

2 An Algorithmic View of the Structure of Claw-Free Graphs

We give algorithms to find the decomposition of claw-free graphs implied by the structural characterization of claw-free graphs by Chudnovsky and Seymour. Moreover, we show that the characterization can be significantly simplified if we assume that the size of the maximum independent set of the graph is greater than three. Due to space limitations, we only state this theorem here and defer its full proof to the appendix.

2.1 Basic Definitions To understand the structure theorem, we need to introduce a significant amount of notation. All notions and definitions are essentially the same as in [10].

We work with a more general type of graph, a so-called *trigraph*. A *trigraph* is a graph with a distinguished subset of edges, that are called *semi-edges* and form a matching. Two vertices are called *semiadjacent* if there is a semi-edge between them, *strongly adjacent* if there is an edge between them that is not a semi-edge, and *strongly antiadjacent* if there is no edge between them. A graph is then simply a trigraph that has no semi-edges. We now say that u, v are *adjacent* if u, v are either strongly adjacent or semiadjacent, and u, v are *antiadjacent* if u, v are either strongly antiadjacent or semiadjacent. In a similar manner, we can distinguish (strong) neighborhoods, completeness, cliques, simplicial vertices, etc., as well as (strong) anti-neighborhoods, anti-completeness, stable sets, etc. We use $\alpha(G)$ to denote the maximum size of a stable set of G .

A trigraph G is a *thickening* of a trigraph G' if for every $v \in V(G')$ there is a nonempty set $X_v \subseteq V(G)$, such that X_v is a strong clique in G for each $v \in V(G')$, $X_u \cap X_v = \emptyset$ for all distinct $u, v \in V(G')$, and $\bigcup_{v \in V(G')} X_v = V(G)$. Moreover, if u, v are strongly adjacent in G' , then X_u is strongly X_v -complete in G ; if u, v are strongly antiadjacent in G' , then X_u is strongly X_v -anticomplete in G ; and if u, v are semiadjacent in G' , then X_u is neither strongly X_v -complete nor strongly X_v -anticomplete in G .

Note that if G is a thickening of G' and G' is a thickening of G'' , then G is also a thickening of G'' . Also note that if a graph G is a thickening of trigraph G' and u, v are semiadjacent in G' , then $|X_u| + |X_v| \geq 3$.

A strong clique X of a trigraph G is *homogeneous* if every vertex in $G \setminus X$ is either strongly complete or strongly anticomplete to X . A trigraph G admits *twins* if G has a homogeneous strong clique of size 2. A pair of strong cliques (A, B) is *homogeneous* if every vertex $v \in V(G) \setminus (A \cup B)$ is either strongly complete or strongly anticomplete to A , and is either strongly complete or strongly anticomplete to B . A homogeneous pair of cliques (A, B) is a *W-join* if A is not strongly complete nor strongly anticomplete to B , and A or B has size at least 2. A W-join is *proper* if no member of A is strongly complete or strongly anticomplete to B and no member of B is strongly complete or strongly anticomplete to A .

Observe that by thickening a trigraph to a graph, one creates twins and W-joins. Hence given a graph G , we can find a trigraph G' such that G is a thickening of G' by contracting twins into a single vertex and replacing W-joins by semi-edges.

2.2 Strips and Stripes A *strip-graph* H consists of disjoint finite sets $V(H)$ and $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (i.e. a subset of $V(H) \times E(H)$). For any $F \in E(H)$, let \overline{F} denote the set of $h \in V(H)$ incident with F . Note that a strip graph is essentially a hypergraph, except that we allow multiple edges and empty edges.

A *strip-structure* (H, η) of a trigraph G is a strip-graph H with $E(H) \neq \emptyset$ and a function η such that for each $F \in E(H)$, $\eta(F) \in 2^{V(G)}$ and for each $h \in \overline{F}$, $\eta(F, h) \subseteq \eta(F)$, satisfying:

- The sets $\eta(F)$ ($F \in E(H)$) are nonempty, pairwise disjoint, and have union $V(G)$.
- For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a strong clique of G .
- For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G , then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.
- For each $F \in E(H)$, the strip corresponding to F is claw-free.

Here the strip corresponding to $F \in E(H)$, where $\overline{F} = \{h_1, \dots, h_k\}$, is defined as follows. Let z_1, \dots, z_k be new vertices and let J be the trigraph obtained from $G[\eta(F)]$ by adding z_1, \dots, z_k and for each i making z_i strongly complete to $\eta(F, h_i)$ and strongly anticomplete to $J \setminus \eta(F, h_i)$. Then $(J, \{z_1, \dots, z_k\})$ is the *strip* corresponding to F . Observe that there is a direct correspondence between h_i and z_i and between $\eta(F, h_i)$ and $N(z_i)$. Moreover, given just a strip-structure, it is easy to reconstruct the graph it is based on.

We define $\eta(h) = \bigcup_{F | h \in \overline{F}} \eta(F, h)$ for all $h \in V(H)$.

We discern two special types of strips. A strip (J, Z) is a *spot* if J has three vertices, say v, z_1, z_2 , and v is strongly adjacent to z_1, z_2 , and z_1 is strongly antiadjacent to z_2 , and $Z = \{z_1, z_2\}$. A strip (J, Z) is a *stripe* if J is a claw-free trigraph and $Z \subseteq V(J)$ is a set of strongly simplicial vertices, such that Z is strongly stable and no vertex of $V(J) \setminus Z$ is adjacent to more than one vertex of Z .

We say that a stripe (J, Z) is a *thickening* of a stripe (J', Z') if J is a thickening of J' with sets X_v ($v \in V(J')$), such that $|X_z| = 1$ for each $z \in Z'$ and $Z = \bigcup_{z \in Z'} X_z$.

2.3 Special Claw-Free Trigraphs As mentioned before, the structural characterization shows that claw-free graphs can be decomposed into several base classes. Below we define those classes used in our structure theorem.

An arc is a connected subset of the sphere \mathbb{S}_1 . A *circular-arc graph* is the intersection graph of a set \mathcal{I} of arcs. If the arcs do not cover \mathbb{S}_1 , this is an *interval graph*. A circular-arc graph is *proper* if no arc of \mathcal{I} contains another arc of \mathcal{I} .

A *line trigraph* G of some graph H is a trigraph where $V(G) = E(H)$ and $e, f \in E(H)$ are adjacent in G if and only if e and f share an endpoint in H . Moreover, e, f are strongly adjacent if e and f share an endpoint of degree at least three.

Let G be the trigraph with $V(G) = \{v_1, \dots, v_{13}\}$ such that $\{v_1, \dots, v_6\}$ is an induced cycle; v_7 is strongly adjacent to v_1 and v_2 ; v_8 is strongly adjacent to v_4, v_5 , and possibly adjacent to v_7 ; v_9 is strongly adjacent to v_1, v_2, v_3 , and v_6 ; v_{10} is strongly adjacent to v_3, v_4, v_5 , and v_6 , and adjacent to v_9 ; v_{11} is strongly adjacent to v_1, v_3, v_4, v_6, v_9 , and v_{10} ; v_{12} is strongly adjacent to v_2, v_3, v_5, v_6, v_9 , and v_{10} ; v_{13} is strongly adjacent to v_1, v_2, v_4, v_5, v_7 , and v_8 . Then $G \setminus X$ for any $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$ is an *XX-trigraph*.

The following trigraphs are called *near-antiprismatic* trigraphs. Let G be a trigraph that is the disjoint union of three n -vertex strong cliques A, B, C for $n \geq 2$ and two vertices a_0, b_0 . Let $X \subseteq A \cup B \cup C$ with $|C \setminus X| \geq 2$ and let the graph have the following adjacencies. For $1 \leq i, j \leq n$, a_i and b_j are adjacent if and only if $i = j$, and c_i is antiadjacent to a_j and b_j if and only if $i = j$. All (anti-)adjacencies are strong, except that possibly a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$; a_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $b_i \in X$; b_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $a_i \in X$. Moreover, a_0 is strongly complete to A , b_0 is strongly complete to B , and a_0 is antiadjacent to b_0 . Then $G \setminus X$ is near-antiprismatic.

We also define certain special stripes. Let J be near-antiprismatic, let a_0, b_0 be as in the above definition, with a_0, b_0 strongly antiadjacent, and let $Z = \{a_0, b_0\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_2 .

Let H be a graph and let h_1, \dots, h_5 be a path in H , such that h_1 and h_5 have degree one and every edge of H is incident with one of h_2, h_3, h_4 . Let J be obtained from a line trigraph of H by making the vertices corresponding to edges h_2h_3 and h_3h_4 either semiadjacent or strongly antiadjacent, and let $Z = \{h_1h_2, h_4h_5\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_3 .

Let J be the trigraph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ such that $\{a_0, a_1, a_2\}$, $\{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are strong cliques, b_2, c_1 are strongly adjacent, b_2, c_2 are semiadjacent, and b_3, c_1 are semiadjacent. Let $Z = \{a_0, b_0\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_4 .

Let J be an XX-trigraph, let v_1, \dots, v_{13}, X be as in the definition of XX-trigraphs, let v_7, v_8 be strongly antiadjacent in J , and let $Z = \{v_7, v_8\} \setminus X$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_5 .

2.4 Algorithmic Structure Theorem Our main result is the following algorithmic structure theorem of claw-free graphs.

Theorem 2.1 *Let G be a connected claw-free graph, such that G does not admit twins or proper W -joins and $\alpha(G) > 3$. Then either*

- G is a thickening of an XX-trigraph, or G is a proper circular-arc graph, or
- G admits a strip-structure such that each strip (J, Z) either is a spot or is a stripe with $1 \leq |Z| \leq 2$ for which either
 - J is a proper circular-arc graph and $|Z| = 1$,
 - J is a proper interval graph and $|Z| = 2$,
 - (J, Z) is a thickening of a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$, or
 - $|Z| = 1$, $\alpha(J) \leq 3$, and $J \setminus N[Z] \neq \emptyset$.

Moreover, we can distinguish the cases and find the strip-structure in polynomial time.

The proof is given in the appendix.

3 Application to Fixed-Parameter Algorithms: Dominating Set

Using Theorem 2.1, we show that Dominating Set and Connected Dominating Set on claw-free graphs are fixed-parameter tractable. Due to space limitations, we defer the proof for Connected Dominating Set to the appendix.

Let $\gamma(G)$ denote the minimum size of a dominating set of G . More generally, let $\gamma(G, A)$, where $A \subseteq V(G)$, denote the size of a smallest subset of $V(G)$ dominating all vertices in $V(G) \setminus A$. We often implicitly use the following result of Allan and Laskar [1]. Let $i(G)$ denote the minimum size of an independent dominating set of G , that is, of any subset of $V(G)$ that is both an independent set and a dominating set of G .

Theorem 3.1 ([1]) *If G is a claw-free graph, then $i(G) = \gamma(G)$.*

Allan and Laskar also give an algorithm to turn any dominating set into an independent dominating set of the same or smaller size. The consequence for this paper is that we can assume throughout w.l.o.g. that any (minimum) dominating set that we consider is also an independent set.

The idea of how to establish the fixed-parameter tractability of Dominating Set on claw-free graphs is as follows. We first show that we can remove twins and proper W -joins from G without changing the size of its minimum dominating set. If $\alpha(G) \leq 3$, then $\gamma(G) \leq 3$, and we can easily find a minimum dominating set by exhaustive enumeration. Otherwise, if $\alpha(G) > 3$, we can apply Theorem 2.1. Then G either belongs to some basic class, or it can be decomposed into strips. If G belongs to a basic class, we can again easily find a minimum dominating set. If G can be decomposed into strips, we solve Minimum Dominating Set separately on each strip. We then develop a fixed-parameter algorithm to stitch the solutions of the strips together.

3.1 Easy Cases and Further Structure We first show that we can remove twins and (proper) W -joins from a graph without changing the size of its minimum dominating set. Moreover, the reductions preserve claw-freeness.

Lemma 3.2 *Let a, b be twins of a graph G . Then $\gamma(G) = \gamma(G - a)$.*

Proof: Let D be any minimum dominating set of G . Since $N[a] = N[b]$, at most one of a, b is in D . If $a \in D$, replace a by b . The resulting set is a dominating set of $G - a$ of the same size.

Let D be any minimum dominating set of $G - a$. As $D \cap N[b] \neq \emptyset$ and $N[a] = N[b]$, D is a dominating set of G as well. \square

Lemma 3.3 *Let (A, B) be a W -join of a graph G and let $X \subseteq V(G)$ be such that $A \cap X \neq \emptyset$ implies $A \subseteq X$ and the same for B . Construct a graph G' as follows:*

1. *if $a_0 \in A$ is strongly complete to B and $b_0 \in B$ is strongly complete to A , remove $A \setminus \{a_0\}$ and $B \setminus \{b_0\}$;*
2. *if no $a_0 \in A$ is strongly complete to B and no $b_0 \in B$ is strongly complete to A , remove all edges between A and B and remove all but one vertex from both A and B ;*
3. *if $a_0 \in A$ is strongly complete to B and no $b_0 \in B$ is strongly complete to A , let $a_1 \in A$ be strongly antiadjacent to some $b_0 \in B$, and remove all vertices of $A \setminus \{a_0, a_1\}$ and all vertices of $B \setminus \{b_0\}$.*

Now let $X' = X \cap V(G')$. Then $\gamma(G, X) = \gamma(G', X')$.

Proof: There are three cases to consider.

1. Note that a_0 and b_0 are adjacent in G' .
 Let $D' \subseteq V(G')$ dominate $V(G') \setminus X'$. From the definition of a W-join, recall that every vertex of $V(G) \setminus (A \cup B)$ is either A -complete or A -anticomplete and either B -complete or B -anticomplete. Moreover, A and B are cliques. If $a_0 \in D'$ (resp. $b_0 \in D'$), then a_0 (resp. b_0) dominates both A and B in G . Clearly, D' is also a subset of $V(G)$ dominating $V(G) \setminus X$. Let $D \subseteq V(G)$ dominate $V(G) \setminus X$. If $D \cap A \neq \emptyset$, we can assume that $a_0 \in D$. Similarly, if $D \cap B \neq \emptyset$, we can assume that $b_0 \in D$. By the definition of a W-join, it follows that $D \cap A \subseteq \{a_0\}$, that $D \cap B \subseteq \{b_0\}$, and thus that D is also a subset of $V(G')$ dominating $V(G') \setminus X'$.
2. Let a_0 and b_0 be the vertices of A and B respectively that were not removed. Note that a_0 and b_0 are not adjacent in G' .
 Let $D' \subseteq V(G')$ dominate $V(G') \setminus X'$. Clearly, D' is also a subset of $V(G)$ dominating $V(G) \setminus X$ and $|D| = |D'|$.
 Let $D \subseteq V(G)$ dominate $V(G) \setminus X$. Note that no vertex of A (resp. B) dominates all of B (resp. A). Now let D' be the set containing all vertices of D that are also in G' . If $A \cap D \neq \emptyset$ (resp. $B \cap D \neq \emptyset$), add a_0 (resp. b_0) to D' . By the definition of a W-join, it follows that D' is a subset of $V(G')$ dominating $V(G') \setminus X'$ and $|D'| \leq |D|$.
3. Note that a_0 is adjacent to both a_1 and b_0 in G' and that a_1 and b_0 are not adjacent in G or G' .
 Let $D' \subseteq V(G')$ dominate $V(G') \setminus X'$. If $a_0 \in D'$, then D' dominates $N[A]$ and B in G . If $a_1 \in D'$, then D' dominates $N[A]$ in G , but not all of B . Hence there must be a vertex to ensure that B gets dominated. However, as a_1 does not dominate b_0 in G , D' contains such a vertex. A similar argument holds for the case when $b_0 \in D'$. Then, from the definition of a W-join, D' is also a subset of $V(G)$ dominating $V(G) \setminus X$.
 Let $D \subseteq V(G)$ dominate $V(G) \setminus X$. Now let D' be the set containing all vertices of D that are also in G' . If $A \cap D \neq \emptyset$ (resp. $B \cap D \neq \emptyset$, replace $A \cap D$ by a_0 (resp. $B \cap D$ by b_0) and call the resulting set D' . Note that the set of vertices dominated in G by a_0 is a superset of the set of vertices dominated in G by any other vertex of A . Moreover, no vertex of B is complete to A and any vertex in $N[A] \setminus B$ is complete to A , and thus any vertex of $B \cap D$ is replaceable by any other vertex of B . Clearly, D' is a subset of $V(G')$ dominating $V(G') \setminus X'$.

The lemma follows. \square

The second case of the lemma implies that we can remove proper W-joins. These reductions then allow us to use the structure theorem.

We now show that if a graph G is in a ‘basic class’, a minimum dominating set can be computed in polynomial time.

Theorem 3.4 ([30]) *Let G be a circular-arc graph. Then $\gamma(G)$ can be computed in linear time.*

Lemma 3.5 *Let G be a graph that is a thickening of an XX -trigraph. Then $\gamma(G)$ can be computed in polynomial time.*

Proof: Consider a graph $G' \in \mathcal{S}_2$ such that G is a thickening of G' . In G' , make v_7, v_8 and v_9, v_{10} strongly antiadjacent and call the resulting graph G'' . If D is a dominating set in G'' , then D corresponds to a dominating set in G of equal size. Observe now that v_2, v_4, v_6 dominate all vertices of G'' . Hence $\gamma(G)$ is constant and thus can be computed in polynomial time. \square

Given a strip-structure of a graph, we want to compute a minimum dominating set efficiently for all its strips and be able to stitch these solutions together in an optimal fashion. To this end, we need to parameterize the minimum dominating set of a strip (J, Z) by what a minimum dominating set D of G would look like relative to this strip. Let F be the edge of the strip structure corresponding to (J, Z) . Then for each $h \in \overline{F}$ corresponding to some $z \in Z$, there are three possible cases, depending on whether or not $\eta(h)$ or $\eta(F, h)$ contains a vertex of D or not. We model this by considering two disjoint sets $X, Y \subseteq Z$. We put $z \in Z$ into X to model the situation where $\eta(h) \cap D = \emptyset$ and we put z into Y to model the situation where $(\eta(h) \setminus \eta(F, h)) \cap D \neq \emptyset$. Then, for any class of strips we have, we have to show how to compute $\gamma(J \setminus (X \cup Y), N[Y])$ efficiently for any disjoint $X, Y \subseteq Z$. Observe that the case when $z \in Z$ is neither in X nor in Y correctly models the case when $\eta(F, h) \cap D \neq \emptyset$, because, since z is simplicial, we can always assume that any dominating set of J contains a neighbor of z instead of z .

Lemma 3.6 *Let (J, Z) be a stripe such that (J, Z) is a connected circular-arc graph. For any disjoint $X, Y \subseteq Z$, $\gamma(J \setminus (X \cup Y), N[Y])$ can be computed in linear time.*

Proof: Find a set of arcs I_1, \dots, I_n of the sphere \mathbb{S}_1 such that this is a representation of J as a circular arc graph. This can be done in linear time [14]. Since each $z \in Z$ is strongly simplicial, there is a point $p_z \in \mathbb{S}_1$ such that the arcs containing p_z are precisely those corresponding to $N[z]$. We can assume that p_z is contained in the interior of each of these intervals.

Consider some disjoint $X, Y \subseteq Z$. For each $z \in Y$, remove the interval $[p_z - \epsilon, p_z + \epsilon]$ from each arc for some infinitesimally small $\epsilon > 0$. Let $I'_1, \dots, I'_{n'}$ be the resulting set of arcs and J' the intersecting graphs of these arcs. Note that $n' = n + |N[Y]|$ and that J' is a circular arc graph.

Consider some $z \in Y$. Observe that both copies of z in J' correspond to either a leftmost or a rightmost interval of the representation. For each copy, add a vertex to J' adjacent to only to z . Let J'' be the graph that is obtained after doing this for all $z \in Y$. Moreover, we remove all $z \in X$. By the preceding observation, J'' is still a circular arc graph.

The vertices added to J' ensure that both copies of z for each $z \in Y$ will be in some minimum dominating set of J'' . Hence $\gamma(J'') - 2|Y| = \gamma(J \setminus (X \cup Y), N[Y])$. As J'' is a circular arc graph, $\gamma(J'')$ can be computed in linear time following Theorem 3.4. \square

Lemma 3.7 *Let (J, Z) be a thickening of a stripe in \mathcal{Z}_5 such that J is a graph. For any disjoint $X, Y \subseteq Z$, $\gamma(J \setminus (X \cup Y), N[Y])$ can be computed in polynomial time.*

Proof: Consider some disjoint $X, Y \subseteq Z$. For any $z \in Y$, add a new vertex adjacent to z only. Remove all $z \in X$. The resulting graph J' has $\gamma(J') - |Y| = \gamma(J \setminus (X \cup Y), N[Y])$. Observe now that one of $\{v_4, v_6, v_7\}$, $\{v_2, v_6, v_8\}$, $\{v_3, v_6, v_7, v_8\}$, $\{v_2, v_4, v_6\}$ is to a dominating set of J' using similar arguing as in Lemma 3.5. Hence $\gamma(J') \leq 4$ and thus $\gamma(J')$ can be computed in polynomial time. \square

Lemma 3.8 *Let (J, Z) be a stripe with $1 \leq |Z| \leq 2$, $\alpha(J) \leq 3$, $J \setminus N[Z] \neq \emptyset$, and J is a graph. For any disjoint $X, Y \subseteq Z$, $\gamma(J \setminus (X \cup Y), N[Y])$ can be computed in polynomial time.*

Proof: Consider some disjoint $X, Y \subseteq Z$. For any $z \in Y$, add a new vertex adjacent to z only. Remove all $z \in X$. The resulting graph J' has $\gamma(J') - |Y| = \gamma(J \setminus (X \cup Y), N[Y])$. Since $\alpha(J') \leq 5$, $\gamma(J')$ can be computed in polynomial time. \square

We observe that if a stripe (J, Z) is a thickening of a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$, then $\alpha(J) \leq 3$. Hence the above lemma also applies to such stripes.

Lemma 3.9 *Let (J, Z) be a spot. For any disjoint $X, Y \subseteq Z$, $\gamma(J \setminus (X \cup Y), N[Y])$ can be computed in constant time.*

This is trivial.

We now investigate the strip-structure given by Theorem 2.1 in relation to the minimum dominating set problem. It follows from the strip-structure that we can see H as a multigraph with loops. The loops are precisely those $F \in E(G)$ for which $|\overline{F}| = 1$. We bicolor the edges of H as follows. Color an edge $F \in E(H)$ black if the strip (J, Z) corresponding to F satisfies that $V(J) \setminus N[Z] \neq \emptyset$, or that $V(J)$ is a union of two strong cliques, $|V(J)| \geq 4$, and $|Z| = 2$. All other edges are colored white. We can observe from Theorem 2.1 that strips corresponding to white edges of H are either spots or have exactly one edge and two vertices.

Lemma 3.10 *Strips corresponding to white edges of H are either spots or have exactly one edge and two vertices.*

Proof: Let F be a white edge and let (J, Z) be the corresponding strip. Suppose that $|Z| = 1$. Then $V(J) = N[Z]$. But since any two vertices of $N(Z)$ are twins, it follows that $V(J) = 2$. So suppose that $|Z| = 2$. Since $V(J) = N[Z]$, $V(J)$ is a union of two strong cliques. But then $|V(J)| \leq 3$. But then (J, Z) cannot be a stripe and thus it must be a spot by Theorem C.20. \square

Lemma 3.11 *If $\gamma(G) \leq k$, then the subgraph H_B of H induced by the black edges has at most $2k$ vertices.*

Proof: Let D be a minimum dominating set of G . We say that an edge $F \in E(H_B)$ is *marked* if $D \cap \eta(F) \neq \emptyset$. Suppose that $|V(H_B)| > 2k$. Then there is a vertex $h \in V(H_B)$ that is not incident to a marked edge. Let F be any edge of H_B incident to h and let (J, Z) be the strip corresponding to F . If $V(J) \setminus N[Z] \neq \emptyset$, then D must contain a vertex of $\eta(F)$, a contradiction to the fact that F is unmarked. Hence $V(J)$ is a union of two strong cliques. But then for D to be a dominating set of G , $\eta(F) \cap D \neq \emptyset$, or $\eta(h) \cap D \neq \emptyset$ and $\eta(h') \cap D \neq \emptyset$, where $\overline{F} = \{h, h'\}$. Since F is unmarked and h is not incident to a marked edge, neither is the case. But then D is not a dominating set of G , a contradiction. Hence $|V(H_B)| \leq 2k$. \square

Lemma 3.12 *Let D be a dominating set of G with $|D| \leq k$. Let H' denote the graph obtained from H by removing all black edges and removing all $h \in V(H)$ incident to a black edge $F \in E(H)$ for which $\eta(F, h) \cap D \neq \emptyset$. Suppose that there are k' vertices h for which $\eta(F, h) \cap D \neq \emptyset$ for some black edge F . Then D induces a vertex cover of H' of size at most $2(k - k')$.*

Proof: We say that an edge $F \in E(H')$ is *marked* if $D \cap \eta(F) \neq \emptyset$. Now construct a vertex cover C as follows. For any marked edge, add both endpoints to C . It is clear that $|C| \leq 2(k - k')$. Moreover, it is clear that any edge of H' that is not incident to a vertex of C corresponds to a vertex not dominated by D . \square

3.2 FPT algorithm We now show how to stitch the results of the stripes together. Our approach extends ideas of Fernau [23] for parameterized Edge Dominating Set.

Theorem 3.13 *Let G be a claw-free graph and $k \geq 0$ an integer. Then we can decide in $O^*(9^k)$ time whether $\gamma(G) \leq k$.*

Proof: Following Lemma 3.2 and 3.3, we can assume that G does not admit twins or proper W-joins. Note that all twins in a graph can be found in polynomial time (see appendix), while proper W-joins can be found in $O(n^2m)$ time [33]. If $\alpha(G) \leq 3$, then $\gamma(G) \leq 3$, which we can determine in polynomial time by exhaustive enumeration. Therefore we can apply Theorem 2.1. Consider the various cases. If G is a proper circular-arc graph or if G is a thickening of an XX-trigraph, then $\gamma(G)$ can be computed in polynomial time by Theorem 3.4 and Lemma 3.5.

Otherwise, consider the strip-structure (H, η) found. Let D be a minimum dominating set of G . Then for any $F \in E(H)$, the way $\eta(F)$ is dominated is essentially determined by $D \cap (\bigcup_{h \in \overline{F}} \eta(h))$. If we could guess this information, we can use the preceding lemmas to complete the theorem.

Lemmas 3.11 and 3.12 suggest the following approach to guessing this information. Let S_1 be any subset of the vertices of H that are incident to a black edge. Remove S_1 and all black edges from H , and call the remaining graph H' . Let S_2 be any minimal vertex cover of H' of size at most $2k - |S_1|$.

Let $S = S_1 \cup S_2$. For each such set S , we will determine a dominating set D such that $D \cap \eta(h) \neq \emptyset$ for each $h \in S$. To this end, we construct an auxiliary multigraph G' with vertices v_h for each $h \in S$, a weight function w on the edges of G' , and an integer k' . The idea is that k' is the number of vertices that any dominating set D of G must have if $D \cap \eta(h) \neq \emptyset$ for each $h \in S$. Then we use the multigraph and its associated edge weight function to decide which strips should be made responsible for ensuring that $D \cap \eta(h) \neq \emptyset$ for each $h \in S$, while minimizing $|D|$. Initially, G' consists of just the vertices v_h and no edges, and $k' = 0$.

Consider some edge $F \in E(H)$ and let (J, Z) be the strip corresponding to F . Suppose that $\overline{F} = \{h\}$ for some $h \in V(H)$, i.e. that $|Z| = 1$. Note that (J, Z) must be a stripe, as spots have $|Z| = 2$. If $h \notin S$, then the strip is itself responsible for dominating all vertices in $\eta(F)$. So add $\gamma(J \setminus Z)$ to k' . Otherwise, i.e. if $h \in S$, then some vertex of $\eta(h)$ will be in the dominating set and it could potentially also be in $\eta(F)$. So add a vertex v_F to G' , add an edge e_F between v_F and v_h to G' , with weight $\gamma(J) - \gamma(J \setminus Z, N[Z])$. This models the additional cost of having a vertex of $\eta(F, h)$ in the dominating set. Observe that since $N[Z]$ is a strong clique, any dominating set for J can be assumed to have a vertex in $N(Z)$, i.e. in $\eta(F, h)$. Finally, add $\gamma(J \setminus Z, N[Z])$ to k' .

Suppose that $\overline{F} = \{h, h'\}$ for distinct $h, h' \in V(H)$, i.e. that $|Z| = 2$. If (J, Z) is a spot, then $\overline{F} \cap S \neq \emptyset$, or the vertex in $\eta(F)$ will never be dominated. Add an edge e_F to G' . If $h, h' \in S$, then e_F runs between v_h and $v_{h'}$. If only one of h and h' belongs to S , say $h \in S$, add a new vertex v_F to G' and let e_F run between v_h and v_F . The weight of e_F is 1.

If (J, Z) is a stripe, let $Z = \{z, z'\}$, where z corresponds to h and z' to h' . If $\overline{F} \cap S = \emptyset$, add $\gamma(J \setminus Z)$ to k' . If $\overline{F} \cap S = \{h\}$, then add a vertex v_F to G' and an edge e_F with weight $\gamma(J \setminus \{z'\}) - \gamma(J \setminus Z, N[z])$ between v_F and v_h . Moreover, add $\gamma(J \setminus Z, N[z])$ to k' . The case when $\overline{F} \cap S = \{h'\}$ is similar.

So assume that $\overline{F} \cap S = \{h, h'\}$. Add two vertices $v_F^h, v_F^{h'}$ and three edges $e_F^h, e_F^{h'}, e_F^{h, h'}$ to G' . Let $e_F^{h, h'}$ run between v_h and $v_{h'}$, and set its weight to $\gamma(J) - \gamma(J \setminus Z, N[Z])$. Let e_F^h run between v_F^h and v_h , and set its weight to $\gamma(J \setminus \{z'\}, N[z']) - \gamma(J \setminus Z, N[Z])$. Let $e_F^{h'}$ run between $v_F^{h'}$ and $v_{h'}$, and set its weight to $\gamma(J \setminus \{z\}, N[z]) - \gamma(J \setminus Z, N[Z])$. Finally, add $\gamma(J \setminus Z, N[Z])$ to k' .

Observe that each edge e added to G' for some $F \in E(H)$ in the above construction corresponds to a particular way to ensure that a dominating set of the strip F has a vertex in $\eta(F, h)$, where e is incident to v_h in G' . The weight on the edge corresponds to the number of extra vertices it would cost to ensure this, compared to the cost of having no vertex in $\eta(F, h)$. Therefore we want to find a subset of the edges of minimum total weight that covers all vertices v_h of G' . This clearly corresponds to a smallest dominating set D of G for which $D \cap \eta(h) \neq \emptyset$ for each $h \in S$.

Hence it remains to find a subset of $E(G')$ of minimum total weight that covers all vertices v_h

of G' . We first consider any parallel edges and remove them all except one of minimum weight. Hence we only need to solve this problem on a graph. This problem is precisely an instance of the minimum generalized weighted edge cover problem, where given a graph, a subset R of its vertices, and an edge weight function, one is asked to find a minimum-weight subset of edges covering R . This problem can be solved in cubic time [39, 23].

The above algorithm yields a smallest dominating set D with $D \cap \eta(h) \neq \emptyset$ for each $h \in S$. We repeat this procedure for all $S = S_1 \cup S_2$. The number of possible choices for S_1 is at most $\sum_{i=0}^{2k} \binom{2k}{i}$. For each set S_1 , we spend $O^*(2^{2k-|S_1|})$ time to enumerate all minimal vertex covers [13]. Since $\sum_{i=0}^{2k} \binom{2k}{i} 2^{2k-i} = 9^k$ by the binomial theorem, the above algorithm decides whether or not $\gamma(G) \leq k$ in $O^*(9^k)$ time. \square

4 Connected Dominating Set

Let $\gamma_c(G)$ denote the size of a minimum connected dominating set of G . Let $\gamma_c(G, A)$, where $A \subseteq V(G)$, denote the size of the smallest subset D of $V(G)$ dominating $V(G) \setminus N[A]$, such that each connected component of $G[D]$ contains at least one vertex of A .

It is trivial to show the following relation between $\gamma(G)$ and $\gamma_c(G)$.

Lemma 4.1 *For any graph G , $\gamma_c(G) \leq 3 \cdot \gamma(G)$.*

The approach we follow for Connected Dominating Set is similar to the one we followed previously for Dominating Set.

4.1 Removing Twins and Proper W-joins We first show that we can remove twins and proper W-joins.

Lemma 4.2 *Let a, b be twins of a graph G . Then $\gamma_c(G) = \gamma_c(G - a)$.*

Proof: Clearly, if D is a minimum connected dominating set of $G - a$, then, as $N[a] = N[b]$, D is a connected dominating set of G as well.

Suppose that D is a minimum connected dominating set of G . If $a \in D$, then we can replace a by b and D would still be a connected dominating set of G . But then D is also a connected dominating set of $G - a$. \square

This is quite similar as for Minimum Dominating Set. For proper W-joins, the situation is somewhat differently, as removing all edges crossing the W-join might disconnect the graph.

Lemma 4.3 *Let (A, B) be a proper W-join of a graph G . Let G' be the graph obtained from G by removing A, B and adding vertices a, a', b, b' . Connect a, a' to $N(A) \setminus B$, b, b' to $N(B) \setminus A$, a to a' , b to b' , and a to b . Then $\gamma_c(G) = \gamma_c(G')$. Moreover, G' is claw-free if G is claw-free.*

Proof: Suppose that D is a connected dominating set of G . We first show that we can assume that $|D \cap A|, |D \cap B| \leq 1$. For suppose that $|D \cap A| \geq 2$. Note that $N[a] \setminus B$ is the same for each $a \in A$. Hence we can remove all but one vertex of $D \cap A$, except if the vertices of $D \cap A$ are the only vertices dominating B . But then we can replace a vertex of $D \cap A$ by a vertex of B adjacent to a second vertex of $D \cap A$. It follows that we can assume that $|D \cap A|, |D \cap B| \leq 1$. Now since (A, B) is a proper W-join, no vertex of A (resp. B) dominates all of B (resp. A). Hence $D \cap (N[A] \setminus B) \neq \emptyset$ and $D \cap (N[B] \setminus A) \neq \emptyset$. So let D' be the set containing all vertices of D that are also in G' . If $D \cap A \neq \emptyset$, add a to D' . If $D \cap B \neq \emptyset$, add b to D' . Clearly, D' is a connected dominating set of G' and $|D'| \leq |D|$.

Suppose that D' is a connected dominating set of G' . From the definition of a W-join, recall that every vertex of $V(G) \setminus (A \cup B)$ is either A -complete or A -anticomplete and is either B -complete or B -anticomplete. Moreover, A and B are cliques. Let D be the set containing all vertices of D' that are also in G . If one of $a, a' \in D'$ and one of $b, b' \in D'$, add any two adjacent vertices, one of A and one of B , to D . Otherwise, if one of $a, a' \in D'$, add any vertex of A to D , and if one of $b, b' \in D'$, add any vertex of B to D . Then D is a dominating set of G and $|D| \leq |D'|$.

Suppose that G is claw-free and that G' has a claw (w, x, y, z) with center w . If at most one vertex of w, x, y, z is equal to one of a, a', b, b' , then (w, x, y, z) directly corresponds to a claw in G . Because a, a' and b, b' form a clique, $N[a] \setminus \{b\} = N[a']$, and $N[b] \setminus \{a\} = N[b']$, at most two of w, x, y, z can be equal to one of a, a', b, b' . If $w \in V(G) \cap V(G')$, then w.l.o.g. $x = a'$ and y is one of b, b' . But then we can replace x and y by any antiadjacent pair of vertices of A and B to get a claw in G . If $w \in \{a, a', b, b'\}$, then w.l.o.g. $w = a$ and $x = b$. But then we can replace w and x by any adjacent pair of vertices of A and B to get a claw in G . Hence G' is claw-free. \square

4.2 Easy Cases

Theorem 4.4 ([5]) *Let G be a circular arc graph. Then $\gamma_c(G)$ can be computed in linear time.*

Lemma 4.5 *Let G be a graph that is a thickening of a member of \mathcal{S}_2 . Then $\gamma_c(G)$ can be computed in polynomial time.*

Proof: Consider a graph $G' \in \mathcal{S}_2$ such that G is a thickening of G' . In G' , make v_7, v_8 and v_9, v_{10} strongly antiadjacent and call the resulting graph G'' . If D is a connected dominating set in G'' , then D corresponds to a connected dominating set in G of equal size. Observe now that v_1, v_2, v_3, v_4 form a connected dominating set of G'' . Hence $\gamma_c(G)$ is constant and thus can be computed in polynomial time. \square

Lemma 4.6 *Let (J, Z) be a stripe such that $(J, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$ and J is a connected graph. Then we can compute $\gamma_c(J)$, $\gamma_c(J, Z)$, and $\gamma_c(J \setminus \{z\})$ for any $z \in Z$ in polynomial time.*

Proof: By Theorem 4.4, it suffices to show how to compute $\gamma_c(J, Z)$. If $|Z| = 1$, then this is just $\gamma_c(J)$ if $V(J) \neq N[Z]$ and 0 otherwise. So assume that $|Z| = 2$. Find a set of intervals I_1, \dots, I_n of the line such that this is a representation of J as an interval graph. This can be done in linear time [4]. We can assume that $v_1 \in Z$ and $v_n \in Z$. Consider any $i < j$ such that no arc lies between I_i and I_j and no interval contains either I_i or I_j . Construct J' from J by removing all intervals ending to the right of I_i , and by adding an interval overlapping the right endpoint of I_i and no other intervals of J' . Construct J'' from J by removing all intervals starting to the left of I_j , and by adding an interval overlapping the left endpoint of I_j and no other intervals of J' . Now compute $\gamma_c(J', v_1) + \gamma_c(J'', v_n)$ in polynomial time. The minimum over all i, j is equal to $\gamma_c(J, Z)$. \square

Lemma 4.7 *Let (J, Z) be a thickening of a stripe in \mathcal{Z}_5 and J is a connected graph. Then we can compute $\gamma_c(J)$, $\gamma_c(J, Z)$, and $\gamma_c(J \setminus \{z\})$ for any $z \in Z$ in polynomial time.*

Proof: We can argue in a manner similar to Lemma 4.5 that $v_1, v_2, v_3, v_4, v_7, v_8$ yields a connected dominating set of J . Then the lemma follows by exhaustively enumerating all possible solutions. \square

Lemma 4.8 *Let (J, Z) be a stripe with $1 \leq |Z| \leq 2$, $\alpha(J) \leq 3$, $J \setminus N[Z] \neq \emptyset$, and J is a graph. Then we can compute $\gamma_c(J)$, $\gamma_c(J, Z)$, and $\gamma_c(J \setminus \{z\})$ for any $z \in Z$ in polynomial time.*

Proof: It is easy to see that $\gamma_c(J) \leq 3 \cdot \gamma(J) \leq 3 \cdot \alpha(J) \leq 9$. Then the lemma follows by exhaustively enumerating all possible solutions. \square

4.3 Complexity Results

Theorem 4.9 *Let G be a claw-free graph and $k \geq 0$ an integer. Then we can decide in $O^*(36^k)$ time whether $\gamma_c(G) \leq k$.*

Proof: According to Lemma 4.2 and 4.3, we can assume that G does not admit twins or proper W-joins. Suppose that $\alpha(G) \leq 3$. Then $\gamma(G) \leq 3$. Observe that $\gamma_c(G) \leq 3 \cdot \gamma(G) \leq 9$. But then we can apply exhaustive enumeration to find a minimum connected dominating set of G in polynomial time.

Now apply Theorem C.20. Consider the various cases. If $G \in \mathcal{S}_3$ or if G is a thickening of a member of \mathcal{S}_2 , then $\gamma_c(G)$ can be computed in polynomial time by Theorem 4.4 and Lemma 4.5.

Otherwise, let (H, η) be the strip-structure for G that was found. Observe that if D is a connected dominating set of G of size at most k , then a subset of D is a minimal dominating set of G of size at most k . In the proof of Theorem 3.13, we essentially showed how to enumerate all possible 'templates' of minimal dominating sets of G of size at most k . To decide whether $\gamma_c(G) \leq k$, we enumerate all such templates again, but now we have to ensure connect any dominating set that follows the template. Where we used a minimum-weight edge cover in Theorem 3.13, we use a minimum edge-weighted Steiner tree here.

So let S_1 be any subset of the vertices of H that are incident to a black edge. Remove S_1 and all black edges from H let S_2 be any minimal vertex cover of the remainder of H of size at most $2k - |S_1|$. Let $S = S_1 \cup S_2$ and initialize k' to 0. Note that we can assume that each black edge is incident to a vertex of S .

Consider any $F \in E(H)$ for which $|\overline{F}| = 1$. Suppose that $\overline{F} = \{h\}$ for some $h \in V(H)$ and let (J, Z) be the strip corresponding to F . Observe that $h \in S$. Remove F and, if $V(J) \setminus N[Z] \neq \emptyset$, add $\gamma_c(J)$ to k' . Note that remaining graph is still connected, that it has no loops, and that all remaining white edges are spots.

Consider a black edge F and let $\overline{F} = \{h, h'\}$. If say $h' \notin S$, we are not allowed to use vertices of $\eta(h')$ in the connected dominating set. Hence we add $\gamma_c(J \setminus \{z'\})$ to k' , where z' is the vertex of Z corresponding to h and remove F from H . Since we assumed that $S \cap \overline{F} \neq \emptyset$ in the choice of S , it suffices to consider the case that $h, h' \in S$. Then we set the weight of F to $\gamma_c(J) - \gamma_c(J, Z)$ and add $\gamma_c(J, Z)$ to k . Finally, we set the weight of all remaining white edges to 1.

Let H' be the remaining multigraph and w the weight function we defined on it. Note that the weight function can be computed in polynomial time using Lemma 4.6, 4.7, and 4.8. If H' is not connected, we continue to the next S . Otherwise, observe that a Steiner tree of H' with terminal set S corresponds to a connected dominating set of G . In particular, if k' plus the weight of this Steiner tree is at most k , then G has a connected dominating set of size at most k . Therefore we want to find a Steiner tree of which the edges have minimum total weight. This can be done in $O^*(2^{|S|})$ time [3]. Using the analysis of Theorem 3.13 and since $|S| \leq 2k$, it follows that we can decide whether $\gamma_c(G) \leq k$ in $O^*(36^k)$ time. \square

5 Polynomial Kernel for Dominating Set

We show that Dominating Set has a polynomial kernel on claw-free graphs. A *kernelization algorithm* for a parameterized problem Π computes in polynomial time, given an instance (x, k) of Π , a new instance (x', k') of Π , such that $(x', k') \in \Pi$ if and only if $(x, k) \in \Pi$ and $|x'| \leq f(k)$ for some computable function f . The instance (x', k') is called a *kernel* of Π , and it is called a *polynomial kernel* if f is a polynomial. Not every fixed-parameter tractable problem admits a polynomial kernel, unless the polynomial hierarchy collapses to the third level [2].

The basic idea of our kernel is to replace each stripe of the strip-structure given in Theorem 2.1 by a stripe of size at most linear in k . If we then reduce the strip-structure itself to have a polynomial number of vertices and edges, we obtain a kernel.

5.1 Reducing Stripes We first consider stripes (J, Z) with $|Z| = 2$. As a first step, we need to show that if a stripe (J, Z) is a thickening of a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$, we can distinguish of a member of which class (J, Z) is thickening of. We provide recognition algorithms for each class in the appendix. Now that we know the ‘type’ of (J, Z) , we can reduce it.

5.2 When $|Z| = 2$

5.2.1 \mathcal{Z}_1

Lemma 5.1 *Let (J, Z) be a proper interval graph for which $\gamma(J) \leq k$. Then there is a proper interval graph (J', Z) , such that $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J' \setminus (X \cup Y), N[Y])$ for any disjoint $X, Y \subseteq Z$ and $|V(J')| \leq 18k + 2$.*

Proof: We make two crucial observations. First, we note that $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J \setminus (X \cup N[Y]))$ for any disjoint $X, Y \subseteq Z$. For consider any $v_i \in N[Y]$ that is in a minimum dominating set D of $J \setminus (X \cup Y)$. We may assume that v_i is adjacent to v_0 . Then v_i must have a neighbor v_j that is not in $N[Y]$, nor in the dominating set. Let $v_k \neq v_j$ be any neighbor of v_i that is not in $N[Y]$, nor in the dominating set. Since v_j, v_k are not adjacent to v_0 , $j, k > i$. But then $i < j < k$ or $i < k < j$, and thus v_j and v_k are adjacent. Hence $(D \setminus \{v_i\}) \cup \{v_j\}$ is also a dominating set of $J \setminus (X \cup Y)$. By replacing in this way, we can show that $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J \setminus (X \cup N[Y]))$ for any disjoint $X, Y \subseteq Z$.

Second, we note that for any linear interval graph there is always a minimum dominating set with a special structure. A simple greedy strategy for finding a minimum dominating set in a linear interval graph is to recursively consider the undominated vertex v_i of lowest index and add a neighbor of v_i of maximum index to the dominating set. It is an easy exercise to prove that this always yields a minimum dominating set of J .

We now show the following. Consider arbitrary disjoint $X, Y \subseteq Z$. Let D denote the dominating set for $\gamma(J \setminus (X \cup N[Y]))$ as found by the above strategy and let D^u denote the undominated vertices of lowest index as considered by the strategy. We claim that for any $A \subseteq V(J)$, $\gamma((J[A \cup D \cup D^u]) \setminus (X \cup N[Y])) = \gamma(J \setminus (X \cup N[Y]))$. This follows immediately from the optimality of the greedy strategy.

We now construct J' as follows. For each disjoint $X, Y \subseteq Z$, add the vertices of D and D^u as defined above to J' . Also ensure that the vertices of Z are added. Since $\gamma(J) \leq k$ and there are nine choices for X, Y , it follows that $|V(J')| \leq 18k + 2$. \square

5.2.2 \mathcal{Z}_2

Lemma 5.2 *Let (J, Z) be a stripe such that J is connected and J does not admit twins. Then we can decide in polynomial time whether or not (J, Z) is a thickening of some $(J', Z') \in \mathcal{Z}_2$. Moreover, we can find such a stripe (J', Z') , if it exists.*

The proof of this lemma is deferred to the appendix.

Lemma 5.3 *Let (J, Z) be a stripe such that (J, Z) is a thickening of a member of \mathcal{Z}_2 and J is a graph. Then we can find a claw-free stripe (J', Z) with $|V(J')| \leq 26$, such that $\gamma(J \setminus (X \cup Y), N(Y)) = \gamma(J' \setminus (X \cup Y), N(Y))$ for any disjoint $X, Y \subseteq Z$.*

Proof: Using Lemma 5.2, we can find $(J'', Z'') \in \mathcal{Z}_2$ such that (J, Z) is a thickening of (J'', Z'') . In particular, we can find A, B, C and a labeling of the vertices as in the definition of \mathcal{Z}_2 . Using Lemma 3.3, we know how to reduce the at most three W -joins in J /semiadjoint pairs of vertices in J'' . It remains to reduce the number of other vertices of J . Consider the graph G' as constructed in the above lemma and look at the different types of connected components it admits.

Look at the set S_{AC} of all connected components that consist of a single edges ac with $a \in A, c \in C$. Pick some edge $a^*c^* \in S_{AC}$. Note that for any minimum set D dominating $J \setminus (N[Y] \cup X)$ for some disjoint $X, Y \subseteq Z$, we can always replace a vertex in $D \cap S_{AC} \cap A$ by a^* and a vertex in $D \cap S_{AC} \cap C$ by c^* . Moreover, if a^* is dominated by some vertex v outside of S_{AC} , then all vertices in $S_{AC} \cap A$ are dominated by v . Similarly, if c^* is dominated by some vertex v not in S_{AC} , then v dominates all vertices in $S_{AC} \cap C$. It follows that we can remove the vertices of all components of S_{AC} from J , except a^*c^* .

In a similar manner, we can reduce the number of components with one edge between A and B and those with one edge between B and C . The number of single vertex components is at most three, as J does not admit twins.

It remains to reduce the number of triangles. Let T denote the set of all triangles in G' . Suppose that there are at least three triangles and consider two distinct triangles in T : $a_i b_i c_i$ and $a_j b_j c_j$. Note that $\gamma(J \setminus (X \cup Y), N(Y)) \leq 2$ for any choice of disjoint $X, Y \subseteq Z$. So consider any minimum set D dominating $J \setminus (X \cup N[Y])$ for some choice of disjoint $X, Y \subseteq Z$. If $|D \cap T| = 2$, we can always use a_i and b_j instead to dominate J . If $|D \cap T| = 1$, we can always replace any vertex in $D \cap T \cap A$ by a_i , any vertex in $D \cap T \cap B$ by b_j , and any vertex in $D \cap T \cap C$ by c_i . If D has $D \cap T \subseteq \{a_i, b_i, c_i, a_j, b_j, c_j\}$ and dominates both a_i and a_j , then it dominates all vertices in $T \cap A$. This is trivial if $D \cap A \neq \emptyset$. Otherwise, there is some $c \in D \cap C$, which must dominate $A \setminus \{a_i, a_j\}$, and thus D dominates $T \cap A$. Similar arguments show that if D dominates both b_i and b_j (respectively c_i and c_j), it dominates all vertices in $T \cap B$ (respectively $T \cap C$).

This reduction gives J' . Note that $|V(J')| \leq 26$. \square

5.2.3 \mathcal{Z}_3

Lemma 5.4 *Given a stripe (J, Z) for which J does not admit twins and $V(J)$ is not a union of two strong cliques, we can check in polynomial time whether (J, Z) is a thickening of a member of \mathcal{Z}_3 . Moreover, we can find such a member if it exists.*

The proof of this lemma is deferred to the appendix.

Lemma 5.5 *Let (J, Z) be a thickening of a member of \mathcal{Z}_3 , such that J is a graph, J does not admit twins, and $V(J)$ is not a union of two strong cliques. Then there is a (J', Z) , that is a thickening of a member of \mathcal{Z}_3 , such that J' is a graph, $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J' \setminus (X \cup Y), N[Y])$ for any disjoint $X, Y \subseteq Z$, and $|V(J')| \leq 33$.*

Proof: Apply the recognition algorithm as described above and let H be the graph underlying J . Let $P = \{h_1, \dots, h_5\}$. Note that any vertex of $X_{h_2 h_3}$ together with any vertex of $X_{h_3 h_4}$ dominates $J \setminus (X \cup Y)$ for any choice of disjoint $X, Y \subseteq Z$. Hence $\gamma(J) \leq 2$.

Consider any two edges e, f incident with pendant vertices of H that are not in P and that are also incident with the same vertex of P . Then $N_J[e] = N_J[f]$. But then e and f are twins in J , a contradiction. Hence the number of pendant vertices of H in $V(H) \setminus P$ is at most three.

Let T_{23} denote the set of vertices in $V(H) \setminus P$ of degree two that are adjacent to h_2 and h_3 . Suppose that $|T_{23}| > 2$ and let v_1, v_2 be two distinguished vertices of T_{23} . Consider any minimum set $D \subseteq V(J) \setminus (X \cup Y)$ dominating $V(J) \setminus (X \cup N[Y])$ for some disjoint $X, Y \subseteq Z$. Note that

$1 \leq |D| \leq 2$. If $|D| = 2$, then we can assume that $D = \{h_2h_3, h_3h_4\}$. If $|D| = 1$, then D does not contain a vertex v corresponding to an edge incident to a vertex of T_{23} , as there would be a vertex u corresponding to a suitably chosen incident to another vertex of T_{23} that is not dominated by v , unless $u \in N[Y]$. But then we can replace v by h_3h_4 . Moreover, if D dominates all vertices corresponding to edges incident with any two vertices of T_{23} , then it must dominate all vertices corresponding to edges incident with a vertex of T_{23} . Hence we can reduce T_{23} to at most two. We can use similar arguments to reduce the size of T_{24} and T_{34} to at most two. By a similar argument, we can reduce the number of vertices in $V(H) \setminus P$ of degree three to at most two.

Using Lemma 3.3 to reduce the size of any W-joins left in the graph, we can construct a claw-free stripe (J', Z') such that J' is a graph, $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J' \setminus (X \cup Y), N[Y])$ for any disjoint $X, Y \subseteq Z$, and $|V(J')| \leq 33$. \square

5.2.4 \mathcal{Z}_4

Lemma 5.6 *Given a stripe (J, Z) for which J does not admit twins, we can check in polynomial time whether (J, Z) is a thickening of a member of \mathcal{Z}_4 . Moreover, we can find such a member if it exists.*

The proof of this lemma is deferred to the appendix.

Lemma 5.7 *Let (J, Z) be a thickening of a member of \mathcal{Z}_4 , such that J is a graph. Then there is a claw-free stripe (J', Z) , such that J' is a graph, $\gamma(J \setminus (X \cup Y), N[Y]) = \gamma(J' \setminus (X \cup Y), N[Y])$ for any disjoint $X, Y \subseteq Z$, and $|V(J')| \leq 11$.*

Proof: Apply the recognition algorithm as described above. Now apply Lemma 3.3 to the W-joins (X_{c_1}, X_{b_3}) and (X_{c_2}, X_{b_2}) . \square

5.2.5 \mathcal{Z}_5

Lemma 5.8 *Let (J, Z) be a stripe such that J does not admit twins. Then we can decide in polynomial time whether (J, Z) is a thickening of a member of \mathcal{Z}_5 . Moreover, we can find such a (J', Z') if it exists.*

The proof of this lemma is deferred to the appendix.

Lemma 5.9 *Let (J, Z) be a stripe such that J is a graph, J does not admit twins, and (J, Z) is a thickening of a member of \mathcal{Z}_5 . Then we can find a claw-free stripe (J', Z) such that $|V(J')| \leq 13$ and $\gamma(J \setminus (X \cup Y), N(Y)) = \gamma(J' \setminus (X \cup Y), N(Y))$ for any disjoint $X, Y \subseteq Z$.*

Proof: Use Lemma 5.8 to find $(J'', Z'') \in \mathcal{Z}_5$ such that (J, Z) is a thickening of (J'', Z'') . If J'' has a semiadjacent pair of vertices, then J has a W-join. We reduce this W-join following Lemma 3.3. Let J' be the resulting graph. It is easily seen that $|V(J')| \leq 13$. \square

5.3 When $|Z| = 1$ For stripes (J, Z) with $|Z| = 1$, we can follow a simpler approach.

Lemma 5.10 *Let (J, Z) be a thickening of an almost-unbreakable stripe such that $|Z| = 1$ and J is a graph. Then there is a claw-free stripe (J', Z') and an integer $k' \geq 0$, such that $\gamma(J \setminus Z, N[Z]) = \gamma(J' \setminus Z', N[Z']) + k'$, $\gamma(J \setminus Z) = \gamma(J' \setminus Z') + k'$, $\gamma(J) = \gamma(J') + k'$, $|Z'| = 1$, and $|V(J')| \leq 4$.*

Proof: Let $Z = \{z\}$. First, we observe that $\gamma(J \setminus Z, N[Z]) \leq \gamma(J \setminus Z) \leq \gamma(J)$. The first inequality is immediate from the definition of $\gamma(\cdot, \cdot)$. To see the second inequality, recall that $N[Z]$ is a strong clique. Consider any dominating set D of J . If $z \in D$, replace it by any vertex in $N(z)$. The resulting set is still a dominating set of J , and by construction, also of $J \setminus Z$.

Secondly, observe that $0 \leq |\gamma(J)| - |\gamma(J \setminus Z, N[Z])| \leq 1$, because any dominating set of $J \setminus Z$ where $N[Z]$ does not need to be dominated can be made into a dominating set of J by simply adding z to it.

Hence there are three cases to consider. First, assume that $\gamma(J \setminus Z, N[Z]) + 1 = \gamma(J \setminus Z) = \gamma(J)$. Let J' be a two-vertex path and let $Z' = \{z'\}$, where z' is one of the vertices of J' . Let $k' = \gamma(J \setminus Z, N[Z])$. Note that $\gamma(J') = \gamma(J' \setminus Z') = 1$ and that $\gamma(J' \setminus Z', N[Z']) = 0$. The lemma follows.

Suppose instead that $\gamma(J \setminus Z, N[Z]) = \gamma(J \setminus Z) = \gamma(J) - 1$. Let J' be a four-vertex path and let $Z' = \{z'\}$, where z' is one of the end vertices of the path. Let $k' = \gamma(J \setminus Z, N[Z]) - 1$. Note that $\gamma(J') = 2$ and that $\gamma(J' \setminus Z', N[Z']) = \gamma(J' \setminus Z') = 1$. Moreover, $k' \geq 0$, because $\gamma(J \setminus Z) < \gamma(J)$, implying that any minimum dominating set D for $J \setminus Z$ satisfies $D \cap N(Z) = \emptyset$, even though the vertices in $N(Z)$ have to be dominated. The lemma follows.

Finally, suppose that $\gamma(J \setminus Z, N[Z]) = \gamma(J \setminus Z) = \gamma(J)$. Let J' be a three-vertex path and let $Z' = \{z'\}$, where z' is one of the end vertices of the path. Let $k' = \gamma(J) - 1$. Note that $\gamma(J' \setminus Z', N[Z']) = \gamma(J' \setminus Z') = \gamma(J') = 1$. Moreover, clearly $\gamma(J) \geq 1$ and thus $k' \geq 0$. The lemma follows. \square

It follows that if for some stripe (J, Z) with $|Z| = 1$, we know $\gamma(J \setminus (X \cup Y), N(Y))$ for any disjoint $X, Y \subseteq Z$, then we can replace it with a claw-free stripe (J', Z') of constant size and $|Z'| = 1$. Moreover, given the sizes of the minimum dominating sets for the different X, Y , we can find (J', Z') in constant time.

5.4 Reducing the Number of Strips We again consider the strip-structure as a multigraph with loops and color its edges black or white as before. Consider the subgraph H_W of H induced by the white edges and flatten it by removing all parallel edges and replacing loops by pendant vertices. Then H_W has a vertex cover of size at most $2k$ by Lemma 3.12 if $\gamma(G) \leq k$. Now kernelize H_W using the Buss rule. Repeatedly do the following until no longer possible:

- Remove a vertex of degree 0.
- Remove a vertex v of degree greater than $2k$ in H_W and add v to a set M .

It is clear that the remaining graph H'_W must have at most $(2k)^2 + 2k$ vertices, or we can answer NO. Moreover, $|M| \leq 2k$, or we can answer NO.

Let H_B denote the subgraph of H induced by black edges. From Lemma 3.11, $|V(H_B)| \leq 2k$, or we can answer NO. Now consider the subgraph of H induced by $V(H'_W)$, M , and H_B . To each vertex $h \in M$, we add a loop F with $\bar{F} = \{h\}$, and set $\eta(F) = \eta(F, h) = v_h^*$ for some new vertex v_h^* . Call the resulting graph H_k . It is clear that $|V(H_k)| \leq 8k + 4k^2$.

Lemma 5.11 *Let G_k be the subgraph of G induced by H_k . Then $\gamma(G) \leq k$ if and only if $\gamma(G_k) \leq k$.*

Proof: Let D be a dominating set of G_k . Consider any vertex h in $V(H) - V(H_k)$ and any $F \in E(H)$ incident to h . Then $\bar{F} = \{h, h'\}$ for some $h' \in V(H)$ and $\eta(F)$ is a spot. In particular, since H is connected and by the construction of H_k , $h' \in M$. Because of v_h^* , $D \cap \eta(h') \neq \emptyset$. But then $\eta(F)$ is also dominated by D . It follows that D is also a dominating set of G .

Let D be a dominating set of G of size at most k . Then $D \cap \eta(h) \neq \emptyset$ for any $h \in M$. Moreover, for any $h' \in V(H) - V(H_k)$, any vertex of $\eta(h')$ must be in $\eta(h)$ for some $h \in M$. Hence we can replace any vertex v of $D \cap \eta(h')$ by v_h^* for the $h \in M$ for which $v \in \eta(h) \cap \eta(h')$. The resulting set D' is a dominating set of G and of G_k and has $|D'| \leq |D|$. \square

5.5 The Kernel

We now show the following.

Theorem 5.12 *Minimum Dominating Set on claw-free graphs has an $O(k^4)$ kernel.*

Proof: Consider an instance G, k of parameterized Minimum Dominating Set, where G is a claw-free graph. Following Lemma 3.2 and Lemma 3.3, we can assume that G does not admit twins or proper W-joins. Moreover, if $\alpha(G) \leq 3$, then $\gamma(G) \leq 3$ and we can determine $\gamma(G)$ in polynomial time. We can then determine whether or not $\gamma(G) \leq k$ and return a trivial YES- or NO-instance accordingly.

We now apply Theorem 2.1 and consider the various cases. If G is a proper circular-arc graph or if G is a thickening of an XX-trigraph, then $\gamma(G)$ can be computed in polynomial time by Theorem 3.4 and Lemma 3.5. We then determine whether or not $\gamma(G) \leq k$ and return a trivial YES- or NO-instance accordingly.

Otherwise, consider the strip-structure (H, η) found. It follows from Lemma 5.11 that either we can return a trivial NO-instance, or we can reduce (H, η) to a strip-structure (H_k, η_k) with $|V(H_k)| = O(k^2)$. Let G_k be the graph induced by (H_k, η_k) . Then moreover $\gamma(G) \leq k$ if and only if $\gamma(G_k) \leq k$ by Lemma 5.11. By abuse of notation, we can assume w.l.o.g. that $H = H_k$ and $\eta = \eta_k$.

We now consider different types of strips (J, Z) and show how to bound their number and size.

Suppose that $|Z| = 1$. Consider all strips for which additionally $|V(J)| = 2$. Note that each $h \in V(H)$ is incident to at most one $F \in E(H)$ that corresponds to such a strip, for any two would imply the existence of twins in G . Hence we can limit the number of such strips by k . If $|V(J)| > 2$, then as G does not admit twins, $J \setminus N[Z] \neq \emptyset$. Hence any dominating set of G must have at least one vertex in $J \setminus Z$. Following Theorem 2.1, either J is a proper circular-arc graph, or $\alpha(J) \leq 3$, or (J, Z) is a thickening of a member of \mathcal{Z}_5 . In all three cases, we can compute $\gamma(J \setminus (X \cup Y), N(Y))$ for any disjoint $X, Y \subseteq Z$ in polynomial time, following Theorem 3.4 and Lemmas 3.7 and 3.8. We then apply Lemma 5.10 to replace (J, Z) by a stripe (J', Z') of constant size. Since each dominating set of G must have at least one vertex in $J \setminus Z$, the number of these stripes is at most k .

So assume that $|Z| = 2$. Consider all stripes between h and h' for certain $h, h' \in V(H)$ that are a union of two strong cliques. Suppose that there are j such stripes and denote the two cliques of the i -th stripe by A_i and B_i . We can assume that $A_i \subseteq \eta(h)$ and $B_i \subseteq \eta(h')$. Hence $A = \bigcup_i A_i$ is a strong clique and so is $B = \bigcup_i B_i$. But then we can actually view all these stripes as a single stripe that is the union of the two strong cliques A, B and assume that $j \leq 1$. (Alternatively, one can reduce (A, B) to form a proper W-join in G if $j \geq 2$.) If $j = 1$, we can show that this stripe is a thickening of a stripe with four vertices and one semiadjacent pair of vertices (see Lemma C.4 in the appendix). Then we use Lemma 3.3 to reduce this stripe to a claw-free stripe (J', Z') with $|V(J')| \leq 5$ and $|Z'| = 2$. Since these stripes each correspond to a black edge of H , the total number of such stripes in H is at most $\binom{2k}{2}$ by Lemma 3.11.

Observe that any two spots incident to the same $h, h' \in V(H)$ form twins. But then H contains at most $\binom{8k+4k^2}{2}$ spots. Moreover, they already have constant size.

We conclude from Theorem 2.1 that the strips not considered thus far satisfy $J \setminus N[Z] \neq \emptyset$ and that $J \setminus Z$ thus must contain a vertex of any dominating set of G . Moreover, J is a proper interval graph or (J, Z) is a thickening of a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. Then it follows that we can ‘kernelize’ each such strip to have size $O(k)$, following Lemma 5.1, 5.3, 5.5, 5.7, and 5.9. Since $J \setminus Z$ must contain a vertex of any dominating set of G , there are at most k such strips.

Consider the strip structure (H', η') and strips as constructed above. Let G' be the graph induced by these strips. Then G' is claw-free, has $O(k^4)$ vertices by the above construction, and has a dominating set of size k if and only if G has one. \square

6 Hardness on $K_{1,4}$ -free Graphs

We show that k -DOMINATING SET is W[1]-hard when restricted to $K_{1,4}$ -free graphs. Our hardness result is obtained by the so-called “ k -MULTICOLORED CLIQUE *reduction technique*”, introduced and explained in [22]. The problem which we reduce from in this technique is defined as follows:

k -MULTICOLORED CLIQUE

Instance: A graph G and a vertex-coloring $c : V(G) \rightarrow \{1, 2, \dots, k\}$ for G .

Question: Does G have a clique including vertices of all k colors? That is, is there a subset of k vertices $K \subseteq V(G)$ such that for all distinct $u, v \in K$: $\{u, v\} \in E(G)$ and $c(u) \neq c(v)$?

Parameter: k .

Theorem 6.1 ([22]) k -MULTICOLORED CLIQUE is W[1]-complete.

The general idea in the k -MULTICOLORED CLIQUE reduction technique is to organize gadgets into three categories: Vertex-selection, edge-selection, and validation. The role of the first two is to encode the selection of k vertices and $\binom{k}{2}$ edges that together form the k -multicolored clique in the instance of the k -MULTICOLORED CLIQUE problem. The task of the validation gadget is, as its name suggests, to validate the selection of vertices and edges. That is, to make sure that the edges selected are in fact incident to the selected vertices. Below we give more detail on how this is done for k -DOMINATING SET in $K_{1,4}$ -free graphs.

Let us denote by (H, c, k) a given instance of k -MULTICOLORED CLIQUE, where H is a graph and $c : V(H) \rightarrow \{1, \dots, k\}$ is the vertex-coloring of H . We construct an instance (G, k') of k -DOMINATING SET as follows: The graph G will consist of three big cliques, each one corresponding to a different gadget (*i.e.* vertex-selection, edge-selection, and validation), and some additional dummy vertices. The vertex-selection clique is formed by the vertices $A := \{a_v : v \in V(H)\}$, the edge-selection clique is formed by $B := \{b_{\{u,v\}} : \{u, v\} \in E(H)\}$, and the validation clique by $C := \{c_{(u,v)}, c_{(v,u)} : \{u, v\} \in E(H)\}$. Note that there are two “directed” validation vertices, $c_{(u,v)}$ and $c_{(v,u)}$, for each edge $\{u, v\} \in E(H)$. The dummy vertices will be denoted by $X := \{x_i : 1 \leq i \leq k\}$ and $Y := \{y_{\{i,j\}} : 1 \leq i < j \leq k\}$. These are all the vertices of G , *i.e.* $V(G) := A \cup B \cup C \cup X \cup Y$.

We next describe the edges connecting the vertices of G . The first three sets of edges connect all vertices in the same clique:

- $E_1 := \{\{a_u, a_v\} : a_u, a_v \in A \text{ and } u \neq v\}$.
- $E_2 := \{\{b_{\{u,v\}}, b_{\{u',v'\}}\} : b_{\{u,v\}}, b_{\{u',v'\}} \in B \text{ and } \{u, v\} \neq \{u', v'\}\}$.
- $E_3 := \{\{c_{(u,v)}, c_{(u',v')}\} : c_{(u,v)}, c_{(u',v')} \in C \text{ and } (u, v) \neq (u', v')\}$.

The next two sets of edges connect the dummy vertices to vertices in the selection gadgets. These edges will ensure that exactly k vertices will be chosen from the vertex-selection gadget, one for each color, and exactly $\binom{k}{2}$ vertices will be chosen from the edge-selection gadget, one for each pair of colors.

- $E_4 := \{\{x_i, a_v\} : x_i \in X, a_v \in A, \text{ and } c(v) = i\}$.
- $E_5 := \{\{y_{\{i,j\}}, b_{\{u,v\}}\} : y_{\{i,j\}} \in Y, b_{\{u,v\}} \in B, \text{ and } \{c(u), c(v)\} = \{i, j\}\}$.

Finally, we add the two sets of edges which connect vertices in the selection gadgets to vertices in the validation gadget:

- $E_6 := \{\{a_v, c_{(v,u)}\} : a_v \in A \text{ and } c_{(v,u)} \in C\}$.

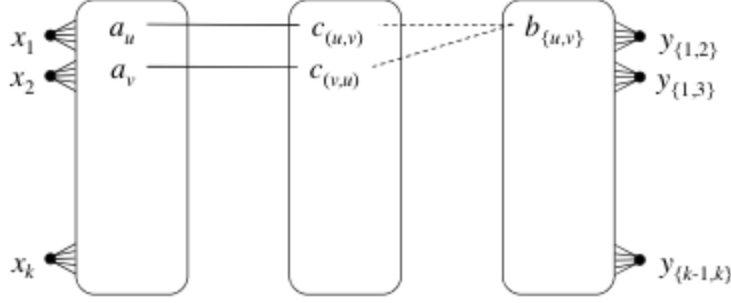


Figure 1: A graphical example of the construction used in the reduction. Here vertices u and v are adjacent vertices in H from different color classes, *i.e.* $c(u) \neq c(v)$. Solid lines in the figure represent edges, while dashed lines represent non-edges.

- $E_7 := \{\{b_{\{u,v\}}, c_{(u',v')}\} : b_{\{u,v\}} \in B, c_{(u',v')} \in C, \{c(u), c(v)\} = \{c(u'), c(v')\}, \text{ and } \{u, v\} \neq \{u', v'\}\}$.

Setting $E(G) := \bigcup_{1 \leq i \leq k} E_i$, and $k' := k + \binom{k}{2}$, completes the description of our construction. See Figure 1 below for a depictive example. It is clear that this reduction is a parameterized reduction. Furthermore, observe that G is indeed $K_{1,4}$ -free since the neighborhood of each vertex can be partitioned into at most three cliques. Thus, to complete our argument, we show that (H, c) has a multicolored clique of size k iff and G has dominating of size k' .

Lemma 6.2 *If (H, c) has a multicolored clique of size k then G has dominating of size k' .*

Proof: Let K be a multicolored clique of size k in H . Then for any $i \in \{1, \dots, k\}$, there is some $v \in K$ with $c(v) = i$, and for any distinct $u, v \in K$, we have $\{u, v\} \in E(H)$. We argue that $D := \{a_v : v \in K\} \cup \{b_{\{u,v\}} : u, v \in K, u \neq v\}$ is a dominating set in G . Indeed, all vertices of A and B are dominated by some $a_v \in D$ and some $b_{\{u,v\}} \in D$. For any $i \in \{1, \dots, k\}$, the dummy vertex x_i is dominated by $a_v \in D$, where $v \in K$ and $c(v) = i$, and for any distinct $i, j \in \{1, \dots, k\}$, the vertex $y_{\{i,j\}}$ is dominated by $a_{\{u,v\}} \in D$, where $u, v \in K$ and $\{c(u), c(v)\} = \{i, j\}$. Finally, take any vertex $c_{(u,v)} \in C$. If $u \in K$, then $a_u \in D$ dominates $c_{(u,v)}$, and otherwise, the vertex $b_{\{u',v'\}} \in D$ with $\{c(u'), c(v')\} = \{c(u), c(v)\}$ dominates $c_{(u,v)}$. \square

Lemma 6.3 *If G has dominating of size k' then (H, c) has a multicolored clique of size k .*

Proof: Let D be a dominating set of size k' in G . Then it is not difficult to see that due to the dummy vertices, for each $i \in \{1, \dots, k\}$, there is exactly one vertex $a_v \in D$ with $c(v) = i$, and for each distinct $i, j \in \{1, \dots, k\}$, exactly one vertex $b_{\{u,v\}} \in D$ with $\{c(u), c(v)\} = \{i, j\}$. Furthermore, there are no other vertices in D . Let $K := \{u : a_u \in D\}$. We argue that K is a multicolored clique of size k in H .

First observe that by the above arguments, K has exactly one vertex for each color $i \in \{1, \dots, k\}$, and so it suffices to argue that K forms a clique in H . For this, it is enough to show that for any pair of distinct vertices $u, v \in K$, we must have $b_{\{u,v\}} \in D$. Suppose that this is not the case, and let u and v denote two vertices in K with $b_{\{u,v\}} \notin D$. Let $b_{\{u',v'\}}$ be the vertex in D such that $\{c(u'), c(v')\} = \{c(u), c(v)\}$. Then $\{u', v'\} \neq \{u, v\}$, so w.l.o.g. $u' \neq u$ and $u' \neq v$. But then, by construction, the validation vertex $c_{(u',v')}$ is not dominated by D . Thus, we get that $b_{\{u,v\}} \in D$ for any pair of distinct vertices $u, v \in K$, which implies that $\{u, v\} \in E(H)$ for any pair of distinct vertices $u, v \in K$. Thus K forms a clique in H , and the lemma is proven. \square

7 Conclusion

We gave an algorithmic structure theorem for claw-free graphs from the structural characterization of these graphs by Chudnovsky and Seymour. This enabled us to determine the parameterized complexity of Dominating Set on claw-free graphs by showing it is fixed-parameter tractable and has a polynomial kernel. Combined with our hardness result for $K_{1,4}$ -free graphs, this settles the parameterized complexity of the dominating set problem on $K_{1,\ell}$ -free graphs for all values of ℓ .

References

- [1] Allan, R.B., Laskar, R., “On Domination and Independent Domination Numbers of a Graph”, *Discrete Mathematics* 23 (1978), pp. 73–76.
- [2] Bodlaender, H.L., Downey, R.G., Fellows, M.R., Hermelin, D., “On problems without polynomial kernels”, *Journal of Computer and System Sciences* 75:8 (2009), pp. 423–434.
- [3] Björklund, A., Husfeldt, T., Kaski, P., Koivisto, M., “Fourier meets Möbius: Fast Subset Convolution”, *STOC 2007*, ACM, 2007, pp. 67–74.
- [4] Booth, K.S., Lueker, G.S., “Testing for the consecutive ones property, interval graphs and graph planarity using PQ-tree algorithms”, *Journal of Computer and Systems Sciences* 13 (1976), pp. 335–379.
- [5] Chang, M.-W., “Efficient Algorithms for the Domination Problems on Interval and Circular-Arc Graphs”, *SIAM Journal on Computing* 27:6 (1998), pp. 1671–1694.
- [6] Charbit, P., de Montgolfier, F., Raffinot, M., “Linear Time Split Decomposition Revisited”, preprint, <http://arxiv.org/abs/0902.1700>.
- [7] Charbit, P., Habib, M., Trotignon, N., Vušković, K., “Detecting 2-joins faster”, preprint.
- [8] Chudnovsky, M., Seymour, P.D., “The Structure of Claw-Free Graphs” in *London Mathematical Society Lecture Note Series: Surveys in Combinatorics*, Volume 327, 2005, pp. 153–171.
- [9] Chudnovsky, M., Seymour, P.D., “Claw-free graphs. IV. Decomposition theorem”, *Journal of Combinatorial Theory, Series B* 98:5 (September 2008), pp. 839–938.
- [10] Chudnovsky, M., Seymour, P.D., “Claw-free graphs. V. Global structure”, *Journal of Combinatorial Theory, Series B* 98:6 (November 2008), pp. 1373–1410.
- [11] Cornuéjols, G., Cunningham, W.H., “Composition for perfect graphs”, *Discrete Mathematics* 55:3 (1985), pp. 245–254.
- [12] Dahlhaus, E., “Parallel Algorithms for Hierarchical Clustering and Applications to Split Decomposition and Parity Graph Recognition”, *Journal of Algorithms* 36:2 (2000), pp. 205–240.
- [13] Damaschke, P., “Parameterized Enumeration, Transversals, and Imperfect Phylogeny Reconstruction” in Downey, R., Fellows, M., Dehne, P. (eds.) *Proc. IWPEC 2004*, LNCS 3162, Springer-Verlag, Berlin, 2004, pp. 1–12.
- [14] Deng, X., Hell, P., Huang, J., “Linear-Time Representation Algorithms for Proper Circular-Arc Graphs and Proper Interval Graphs”, *SIAM Journal on Computing* 25:2 (1996), pp. 390–403.
- [15] Downey, R.G., Fellows, M.R., “Fixed-parameter tractability and completeness”, *Congressus Numerantium* 87, 1992, pp. 161–178.
- [16] Downey, R.G., Fellows, M.R., *Parameterized Complexity*, Springer, New York, 1999.
- [17] Edmonds, J., “Paths, trees, and flowers”, *Canadian Journal of Mathematics* 17, pp. 449–467.
- [18] Eisenbrand, F., Oriolo, G., Stauffer, G., Ventura, P., “The Stable Set Polytope of Quasi-Line Graphs”, *Combinatorica* 28:1 (2008), pp. 45–67.

- [19] Faudree, R., Flandrin, E., Ryjáček, Z., “Claw-free graphs – A survey”, *Discrete Mathematics* 164:1-3 (1997), pp. 87–147.
- [20] Faenza, Y., Oriolo, G., Stauffer, G., “An algorithmic decomposition of claw-free graphs leading to an $O(n^3)$ -algorithm for the weighted stable set problem”, accepted to SODA 2011.
- [21] Feige, U., “A Threshold of $\ln n$ for Approximating Set Cover”, *Journal of the ACM* 45:4 (1998), pp. 634–652.
- [22] Fellows, M.R., Hermelin, D., Rosamond, F.A., Vialette, S., “On the parameterized complexity of multiple-interval graph problems”, *Theoretical Computer Science* 410:1 (2009), pp. 53-61.
- [23] Fernau, H., “Edge Dominating Set: Efficient Enumeration-Based Exact Algorithms” in Bodlaender, H.L., Langston, M.A. (eds.) *Proc. IWPEC 2006*, LNCS 4169, Springer-Verlag, Berlin, 2006, pp. 142–153.
- [24] Fouquet, J.L., “A Strengthening of Ben Rebea’s Lemma”, *J. Combinat. Theory, Series B* 59:1 (1993), pp. 35–40.
- [25] Gabow, H.N., Tarjan, R.E., “Faster Scaling Algorithms for General Graph Matching Problems”, *Journal of the ACM* 38:4 (October 1991), pp. 815–853.
- [26] Harel, D., Tarjan, R.E., “Fast Algorithms for Finding Nearest Common Ancestors”, *SIAM Journal of Computing*, 13:2 (May 1984), pp. 338–355.
- [27] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., *Fundamentals of domination in graphs*, Marcel Dekker Inc., New York, 1998.
- [28] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., *Domination in graphs: Advanced Topics*, Marcel Dekker Inc., New York, 1998.
- [29] Hedetniemi, S.T., Laskar, R.C., “Bibliography on domination in graphs and some basic definitions of domination parameters”, *Discrete Mathematics* 86:1-3 (1990), pp. 257–277.
- [30] Hsu, W.-L., Tsai, K.-H., “Linear-time algorithms on circular arc graphs”, *Information Processing Letters* 40 (November 1991), pp. 123–129.
- [31] Johnson, D.S., “Approximation algorithms for combinatorial problems”, *Journal of Computer and System Sciences* 9 (1974), pp. 256–278.
- [32] Karp, R.M., “Reducibility Among Combinatorial Problems” in Miller, R.E., Thatcher, J.W. (eds.) *Complexity of Computer Computations*, Plenum, New York, pp. 85–103.
- [33] King, A.D., Reed, B.A., “Bounding χ in Terms of ω and Δ for Quasi-Line Graphs”, *Journal of Graph Theory* 59:3 (2008), pp. 215–228.
- [34] Marx, D., *Parameterized Complexity of Independence and Domination on Geometric Graphs* in *Proc. IWPEC 2006*, LNCS 4169, Springer-Verlag, Berlin, 2006, pp. 154–165.
- [35] Minty, G.J., “On maximal independent sets of vertices in claw-free graphs”, *Journal of Combinatorial Theory Series, B* 28:3 (1980), pp. 284–304.
- [36] Misra, N., Philip, G., Raman, V., Saurabh, S., “The effect of girth on the kernelization complexity of Connected Dominating Set” in *FSTTCS 2010*, to appear.
- [37] Nakamura, D., Tamura, A., “A revision of Minty’s algorithm for finding a maximum weighted stable set of a claw-free graph”, *Journal of the Operations Research Society of Japan* 44:2 (2001), pp. 194–204.
- [38] Philip, G., Raman, V., Sikdar, S., “Solving Dominating Set in Larger Classes of Graphs: FPT Algorithms and Polynomial Kernels” in Fiat, A., Sanders, P. (eds.) *Proc. ESA 2009*, LNCS 5757, Springer-Verlag, Berlin, 2009, pp. 694–705.
- [39] Plesník, J., “Constrained weighted matchings and edge coverings in graphs”, *Discrete Applied Mathematics* 92:2-3 (June 1999), pp. 229–241.

- [40] Roussopoulos, N.D., “A $\max\{m, n\}$ algorithm for determining the graph H from its line graph G ”, *Information Processing Letters* 2:4 (1973), pp. 108–112.
- [41] Sbihi, N., “Algorithme de recherche d’un stable de cardinalité maximum dans un graphe sans étoile”, *Discrete Mathematics* 29:1 (1980), pp. 53–76.
- [42] Tarjan, R.E., “Efficiency of a Good But Not Linear Set Union Algorithm”, *Journal of the ACM*, 22:2 (April 1975), pp. 215–225.
- [43] Yannakakis, M., Gavril, F., “Edge dominating sets in graphs”, *SIAM Journal on Applied Mathematics* 38:3 (1980), pp. 364–372.

A Definitions

We give formal definitions of all notions needed to understand the structural characterization of claw-free graphs by Chudnovsky and Seymour. The notation and definitions are the same as in [10].

A.1 Basic Definitions In this paper, we work with a more general type of graph, so-called trigraphs. All graphs are finite and simple. A *trigraph* G has a finite set of vertices $V(G)$ and an adjacency function $\phi_G : V(G) \times V(G) \rightarrow \{-1, 0, 1\}$, such that

- $\phi_G(v, v) = 0$ for all $v \in V(G)$,
- $\phi_G(u, v) = \phi_G(v, u)$ for all $u, v \in V(G)$,
- at most one of $\phi_G(u, v), \phi_G(u, w) = 0$ for all distinct $u, v, w \in V(G)$.

We say that distinct $u, v \in V(G)$ are *strongly adjacent* if $\phi_G(u, v) = 1$, *semiadjacent* if $\phi_G(u, v) = 0$, and *strongly antiadjacent* if $\phi_G(u, v) = -1$. We then say that u, v are *adjacent* if u, v are either strongly adjacent or semiadjacent and u, v are *antiadjacent* if u, v are either strongly antiadjacent or semiadjacent. Moreover, a vertex u is a *neighbor* of a vertex v if u, v are adjacent and u, v are *strong neighbors* if u and v are strongly adjacent. The notions of *antineighbor* and *strong antineighbor* are similarly defined. We denote by $N(v)$ the set of neighbors of a vertex v and define $N[v] = N(v) \cup \{v\}$. Similarly, we define $N(X) = \{v \mid v \in N(x) \setminus X, x \in X\}$ and $N[X] = N(X) \cup X$.

Note that, by definition, the set of semiadjacent pairs of vertices of a trigraph is a matching. Moreover, if G contains no semiadjacent pairs of vertices, then G is just a graph. Or, in the reverse, a graph G' regarded as a trigraph is the trigraph G where $V(G) = V(G')$ and $\phi_G(u, v) = 1$ if u, v are adjacent in G' and $\phi_G(u, v) = -1$ otherwise.

Given disjoint sets $A, B \subseteq V(G)$, we say that A is *complete* to B or *B-complete* if every vertex of A is adjacent to every vertex of B . We say that A is *strongly complete* to B or *strongly B-complete* if every vertex of A is strongly adjacent to every vertex of B . If we say that $a \in V(G)$ is (strongly) complete to B , we mean that $\{a\}$ is (strongly) complete to B . The notions of *anticomplete* and *strongly anticomplete* are defined similarly.

A set $C \subseteq V(G)$ is a *clique* if every pair of vertices of C is adjacent and a *strong clique* if every pair of vertices of C is strongly adjacent. A clique of three vertices is a *triangle*. A set $I \subseteq V(G)$ is *stable* if every pair of vertices of I is antiadjacent and *strongly stable* if every pair of vertices of I is strongly antiadjacent. Let $\alpha(G)$ denote the size of a largest subset of $V(G)$ that is strongly stable. A vertex v of a trigraph is *simplicial* if $N[v]$ is a clique and *strongly simplicial* if $N[v]$ is a strong clique.

A trigraph G is a *thickening* of a trigraph G' if for every $v \in V(G')$ there is a nonempty set $X_v \subseteq V(G)$, such that

- $X_u \cap X_v = \emptyset$ for all distinct $u, v \in V(G')$ and $\bigcup_{v \in V(G')} X_v = V(G)$,
- X_v is a strong clique in G for each $v \in V(G')$,
- if u, v are strongly adjacent in G' , then X_u is strongly X_v -complete in G ,
- if u, v are strongly antiadjacent in G' , then X_u is strongly X_v -anticomplete in G ,
- if u, v are semiadjacent in G' , then X_u is neither strongly X_v -complete nor strongly X_v -anticomplete in G .

Note that if G is a thickening of G' and G' is a thickening of G'' , then G is also a thickening of G'' . Also note that if a graph G is a thickening of trigraph G' and u, v are semiadjacent in G' , then $|X_u| + |X_v| \geq 3$.

For any $X \subseteq V(G)$, $G[X]$ is the trigraph *induced by* X , which is the trigraph with vertex set X and adjacency determined by the restriction of ϕ_G to $X \times X$. We say that H is an *induced*

subtrigraph of G if H is isomorphic to $G[X]$ for some $X \subseteq V(G)$. We define $G \setminus X = G[V(G) \setminus X]$. Isomorphism between trigraphs is defined as expected.

A *claw* is a trigraph with four vertices a, b, c, d , such that $\{b, c, d\}$ is stable and complete to a . Then a is the *center* of the claw. If no induced subtrigraph of a trigraph G is isomorphic to a claw, then G is *claw-free*.

A.2 Joins and Structures A strong clique X of a trigraph G is *homogeneous* if every vertex in $G \setminus X$ is either strongly complete or strongly anticomplete to X . A trigraph G admits *twins* if G has a homogeneous strong clique of size 2.

A pair of strong cliques (A, B) is *homogeneous* if every vertex $v \in V(G) \setminus (A \cup B)$ is either strongly complete or strongly anticomplete to A , and is either strongly complete or strongly anticomplete to B .

A homogeneous pair of cliques (A, B) is a *W-join* if A is not strongly complete nor strongly anticomplete to B , and A or B has size at least 2. A W-join is *proper* if no member of A is strongly complete or strongly anticomplete to B and no member of B is strongly complete or strongly anticomplete to A .

Observe that by thickening a trigraph, one creates twins and W-joins. Hence given a graph G , we can find a trigraph G' such that G is a thickening of G' by contracting twins into a single vertex and replacing W-joins by semiadjacent edges.

A partition (V_1, V_2) of $V(G)$ of a trigraph G is a *0-join* if V_1 is strongly anticomplete to V_2 and $V_1, V_2 \neq \emptyset$.

A trigraph that does not admit a 0-join is called *connected*.

A partition (V_1, V_2) of $V(G)$ of a trigraph G is a *1-join* if there are sets $A_1 \subseteq V_1, A_2 \subseteq V_2$ such that

- $A_1 \cup A_2$ is a strong clique,
- $V_1 \setminus A_1$ is strongly anticomplete to V_2 , and $V_2 \setminus A_2$ is strongly anticomplete to V_1 ,
- $A_i, V \setminus A_i \neq \emptyset$ for $i = 1, 2$.

A partition (V_1, V_2) of $V(G)$ of a trigraph G is a *pseudo-1-join* if there are sets $A_1 \subseteq V_1, A_2 \subseteq V_2$ such that

- $A_1 \cup A_2$ is a strong clique,
- $V_1 \setminus A_1$ is strongly anticomplete to V_2 , and $V_2 \setminus A_2$ is strongly anticomplete to V_1 ,
- both V_1 and V_2 are not strong stable sets.

A trigraph admitting a 1-join and no 0-join admits a pseudo-1-join, but the converse is false. We will denote (pseudo-) 1-joins either by the partition (V_1, V_2) of the vertices or the 'connecting subsets' (A_1, A_2) . Note that V_1, V_2 can be easily determined if we just know A_1, A_2 , and vice versa.

A partition (V_0, V_1, V_2) of $V(G)$ of a trigraph G forms a *generalized 2-join* if for $i = 1, 2$ there are disjoint sets $A_i, B_i \subseteq V_i$ such that

- V_0, V_1 and V_2 are pairwise strongly anticomplete, except that $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ form a strong clique,
- $A_i, B_i, V_i \setminus (A_i \cup B_i) \neq \emptyset$ for $i = 1, 2$.

If $V_0 = \emptyset$, we call it a *2-join*.

A partition (V_0, V_1, V_2) of $V(G)$ of a trigraph G forms a *pseudo-2-join* if for $i = 1, 2$ there are disjoint sets $A_i, B_i \subseteq V_i$ such that

- V_0, V_1 and V_2 are pairwise strongly anticomplete, except that $V_0 \cup A_1 \cup A_2$ and $V_0 \cup B_1 \cup B_2$ form a strong clique,
- both V_1 and V_2 are not strong stable sets.

A graph admitting a (generalized) 2-join and no 1-join admits a pseudo-2-join, but the converse is false. We will use both the notation (V_0, V_1, V_2) or (V_1, V_2) and the notation (A_1, A_2, B_1, B_2) for (generalized/pseudo-) 2-joins.

A partition (V_1, V_2, V_3, V_4) of $V(G)$ of a trigraph G forms a *biclique* if

- $V_1 \neq \emptyset$, and $V_1 \cup V_2$ and $V_1 \cup V_3$ form a strong clique,
- V_1 is strongly anticomplete to V_4 ,
- either $|V_1| \geq 2$ or $V_2 \cup V_3$ is not a strong clique,
- $V_2 \cup V_3 \cup V_4$ is not a strong stable set,
- if $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent, then they have the same neighbors in V_4 and v_2, v_3 are not semiadjacent to any vertex of V_4 .

A *three-cliqued trigraph* $(G; A, B, C)$ consists of a trigraph G and three pairwise disjoint strong cliques A, B, C in G such that $V(G) = A \cup B \cup C$.

A *hex-join* of two distinct three-cliqued trigraphs $(G_1; A_1, B_1, C_1)$ and $(G_2; A_2, B_2, C_2)$ is the three-cliqued trigraph $(G; A, B, C)$, where $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $C = C_1 \cup C_2$, and G is the trigraph with vertex set $V(G) = V(G_1) \cup V(G_2)$ and adjacency as follows:

- $G[V(G_1)] = G_1$ and $G[V(G_2)] = G_2$,
- A_1 is strongly complete to $V(G_2) \setminus B_2$, B_1 is strongly complete to $V(G_2) \setminus C_2$, and C_1 is strongly complete to $V(G_2) \setminus A_2$,
- the pairs (A_1, B_2) , (B_1, C_2) , (C_1, A_2) are strongly anticomplete.

A trigraph G admits a *hex-join* if G has three strong cliques A, B, C such that $(G; A, B, C)$ is a three-cliqued trigraph expressible as a hex-join.

A.3 Strips and Stripes A *strip-graph* H consists of disjoint finite sets $V(H)$ and $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (i.e. a subset of $V(H) \times E(H)$). For any $F \in E(H)$, let \overline{F} denote the set of $h \in V(H)$ incident with F . Note that the definition of strip graph is close to the definition of hypergraphs, except that we allow multiple edges and empty edges here.

Let G be a trigraph and let $Y \subseteq V(G)$. Then a family (X_1, \dots, X_k) of subsets of Y is a *circus* in Y if

- for $1 \leq i \leq k$ and $x \in X_i$, the set of neighbors of x in $Y \setminus X_i$ is a strong clique,
- for $1 \leq i < j \leq k$, $X_i \cap X_j$ is strongly anticomplete to $Y \setminus (X_i \cup X_j)$,
- for $1 \leq h < i < j \leq k$, $X_h \cap X_i \cap X_j = \emptyset$.

A *strip-structure* (H, η) of a trigraph G is a strip-graph H with $E(H) \neq \emptyset$ and a function η such that for each $F \in E(H)$, $\eta(F) \in 2^{V(G)}$ and for each $h \in \overline{F}$, $\eta(F, h) \subseteq \eta(F)$, satisfying:

- The sets $\eta(F)$ ($F \in E(H)$) are nonempty, pairwise disjoint, and have union $V(G)$.
- For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a strong clique of G .
- For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G , then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.
- For each $F \in E(H)$, the family $\eta(F, h)$ ($h \in \overline{F}$) is a circus in $\eta(F)$.

To simplify notation, we define $\eta(h) = \bigcup_{F|h \in \overline{F}} \eta(F, h)$ for all $h \in V(H)$.

Let (H, η) be a strip-structure of a trigraph G and let $F \in E(H)$, where $\overline{F} = \{h_1, \dots, h_k\}$. Let v_1, \dots, v_k be new vertices and let J be the trigraph obtained from $G[\eta(F)]$ by adding v_1, \dots, v_k and for each i making v_i strongly complete to $\eta(F, h_i)$ and strongly anticomplete to $J \setminus \eta(F, h_i)$. Then $(J, \{v_1, \dots, v_k\})$ is the *strip* corresponding to F .

Observe that if G is claw-free, stating that the family $\eta(F, h)$ ($h \in \overline{F}$) is a circus in $\eta(F)$ is equivalent to stating that the strip corresponding to F is claw-free.

A strip (J, Z) is a *spot* if J consists of three vertices v, z_1 , and z_2 such that v is strongly adjacent to z_1, z_2 , and z_1 is strongly antiadjacent to z_2 , and $Z = \{z_1, z_2\}$.

A strip (J, Z) is a *stripe* if J is a claw-free trigraph and $Z \subseteq V(J)$ is a set of strongly simplicial vertices, such that Z is strongly stable and no vertex of $V(J) \setminus Z$ is adjacent to more than one vertex of Z .

We say that a stripe (J, Z) is a *thickening* of a stripe (J', Z') if J is a thickening of J' with sets X_v ($v \in V(J')$) such that $|X_z| = 1$ for each $z \in Z'$ and $Z = \bigcup_{z \in Z'} X_z$.

The *nullity* of a strip-structure (H, η) is the number of pairs (F, h) with $F \in E(H)$, $h \in \overline{F}$, and $\eta(F, h) = \emptyset$.

Given a strip-structure (H, η) of a trigraph, we say that $F \in E(H)$ is *purified* if either the sets $\eta(F, h)$ ($h \in \overline{F}$) are pairwise disjoint, or $\overline{F} = \{h_1, h_2\}$, $|\eta(F)| = 1$, and $\eta(F, h_1) = \eta(F, h_2) = \eta(F)$. A strip-structure (H, η) is *purified* if each $F \in E(H)$ is purified.

Observe that saying that $F \in E(H)$ is purified is equivalent to saying that the strip corresponding to F is either a stripe or a spot.

A.4 Special Trigraphs In the definitions below, whenever adjacency between two vertices is not specified, they are strongly antiadjacent. Moreover, if two vertices are said to be adjacent, they can be either strongly adjacent or semiadjacent, unless otherwise specified. Similarly, if two vertices are said to be antiadjacent, they can be either strongly antiadjacent or semiadjacent, unless otherwise specified. Finally, if we say that $\{v_1, \dots, v_i\}$ is an induced cycle, we mean the cycle $v_1 \cdots v_i v_1$.

A *line trigraph* G of some graph H is a trigraph where $V(G) = E(H)$ and $e, f \in E(H)$ are adjacent in G if and only if e and f share an endpoint in H . Moreover, e, f are strongly adjacent if e and f share an endpoint of degree at least three. The class of all line trigraphs is denoted \mathcal{S}_0 .

We use $L(H)$ to denote the line graph of H .

The *icosahedron* is the (planar) graph G with $V(G) = \{v_1, \dots, v_{12}\}$ such that

- for $i = 1, \dots, 10$, v_i is adjacent to v_{i+1} and v_{i+2} (indices modulo 10),
- v_{11} is adjacent to v_1, v_3, v_5, v_7, v_9 ,
- v_{12} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$.

This graph regarded as a trigraph is denoted by G_0 . Let $G_1 = G_0 \setminus \{v_{12}\}$. Let G_2 be obtained from $G_1 \setminus \{v_{10}\}$ by possibly making v_1 semiadjacent to v_4 or making v_6 semiadjacent to v_9 . The class of trigraphs denoted by \mathcal{S}_1 consists precisely of G_0, G_1 , and the four possibilities for G_2 .

Let G be the trigraph with $V(G) = \{v_1, \dots, v_{13}\}$ such that

- $\{v_1, \dots, v_6\}$ is an induced cycle,
- v_7 is strongly adjacent to v_1 and v_2 ,
- v_8 is strongly adjacent to v_4, v_5 and possibly adjacent to v_7 ,
- v_9 is strongly adjacent to v_1, v_2, v_3 , and v_6 ,
- v_{10} is strongly adjacent to v_3, v_4, v_5 , and v_6 , and adjacent to v_9 ,
- v_{11} is strongly adjacent to v_1, v_3, v_4, v_6, v_9 , and v_{10} ,

- v_{12} is strongly adjacent to $v_2, v_3, v_5, v_6, v_9,$ and v_{10} ,
- v_{13} is strongly adjacent to $v_1, v_2, v_4, v_5, v_7,$ and v_8 .

Then $G \setminus X$ for any $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$ is an XX -trigraph. The class of all XX -trigraphs is denoted \mathcal{S}_2 .

Consider the sphere \mathbb{S}_1 and a set $\mathcal{I} = \{I_1, \dots, I_n\}$ of subsets of \mathbb{S}_1 , such that no distinct I_i, I_j share an endpoint and no three members of \mathcal{I} have union \mathbb{S}_1 . Let P be a finite subset of \mathbb{S}_1 and let G be the trigraph with $V(G) = P$ such that distinct $u, v \in P$ are adjacent in G if and only if $u, v \in I_i$ for some $i = 1, \dots, n$. Moreover, u, v are strongly adjacent if at least one of u, v is in the interior of I_i . Call such trigraphs *circular interval trigraphs* and denote the class of all circular interval trigraphs by \mathcal{S}_3 .

It is easy to prove that if a circular interval trigraph has no semiadjacent edges, then it is a proper circular arc graph.

Let H be a graph with $V(H) = \{v_1, \dots, v_7\}$ such that

- $\{h_1, \dots, h_5\}$ is an induced cycle,
- h_6 is adjacent to at least three of h_1, \dots, h_5 ,
- h_7 is adjacent to h_6 .

Let G be the graph obtained from the line graph of H by adding a new vertex adjacent to precisely those edges of $E(H)$ not incident with h_6 and regarding it as a trigraph. If h_4, h_5 both have degree two in H , possibly make the vertices of G corresponding to edges h_3h_4 and h_1h_5 semiadjacent. The class of trigraphs containing precisely these trigraphs is denoted \mathcal{S}_4 .

Let G be a trigraph that is the disjoint union of three n -vertex strong cliques A, B, C for $n \geq 2$ and five vertices $\{d_1, \dots, d_5\}$ such that for some $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$,

- for $1 \leq i, j \leq n$, a_i and b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j and b_j if and only if $i \neq j$,
- a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$,
- a_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $b_i \in X$,
- b_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $a_i \in X$,
- no two of $A \setminus X, B \setminus X, C \setminus X$ are strongly complete to each other,
- d_1 is strongly complete to $A \cup B \cup C$,
- d_2 is strongly complete to $A \cup B$ and adjacent to d_1 ,
- d_3 is strongly complete to $A \cup \{d_2\}$,
- d_4 is strongly complete to $B \cup \{d_2, d_3\}$,
- d_5 is strongly adjacent to d_3 and d_4 .

The class of all trigraphs $G \setminus X$ is denoted by \mathcal{S}_5 .

The following trigraphs are called *near-antiprismatic* or *antihat* trigraphs. Let G be a trigraph that is the disjoint union of three n -vertex strong cliques A, B, C for $n \geq 2$ and two vertices a_0, b_0 such that for some $X \subseteq A \cup B \cup C$ with $|C \setminus X| \geq 2$,

- for $1 \leq i, j \leq n$, a_i and b_j are adjacent if and only if $i = j$, and c_i is antiadjacent to a_j and b_j if and only if $i = j$. All such pairs are strongly adjacent or antiadjacent, except possibly
 - a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$,
 - a_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $b_i \in X$,
 - b_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $a_i \in X$.
- a_0 is strongly complete to A ,

- b_0 is strongly complete to B ,
- a_0 is antiadjacent to b_0 ,

The class of all trigraphs $G \setminus X$ is denoted by \mathcal{S}_6 .

A trigraph G is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and at least two pairs of vertices in X are strongly adjacent. The class of antiprismatic trigraphs is denoted by \mathcal{S}_7 .

A.5 Special Stripes Consider a trigraph J with vertex set $\{v_1, \dots, v_n\}$ with $n \geq 2$ such that for $1 \leq i < j < k \leq n$, if v_i and v_k are adjacent in J , then v_j is strongly adjacent to both v_i and v_k . This is a *linear interval trigraph*. Moreover, let v_1, v_n be strongly antiadjacent, let no vertex be adjacent to both v_1 and v_n , and let no vertex be semiadjacent to either v_1 or v_n . Let $Z = \{v_1, v_n\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_1 .

Let $J \in \mathcal{S}_6$, let a_0, b_0 be as in the definition of \mathcal{S}_6 , with a_0, b_0 strongly antiadjacent, and let $Z = \{a_0, b_0\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_2 .

Let H be a graph and let h_1, \dots, h_5 be a path in H , such that h_1 and h_5 have degree one and every edge of H is incident with one of h_2, h_3, h_4 . Let J be obtained from the line trigraph of H by making the vertices corresponding to edges h_2h_3 and h_3h_4 either semiadjacent or strongly antiadjacent, and let $Z = \{h_1h_2, h_4h_5\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_3 .

Let J be the trigraph with vertices $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ such that $\{a_0, a_1, a_2\}$, $\{b_0, b_1, b_2, b_3\}$, $\{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are strong cliques, b_2, c_1 are strongly adjacent, b_2, c_2 are semiadjacent, and b_3, c_1 are semiadjacent. Let $Z = \{a_0, b_0\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_4 .

Let $J \in \mathcal{S}_2$, let v_1, \dots, v_{13}, X be as in the definition of \mathcal{S}_2 , let v_7, v_8 be strongly antiadjacent in J , and let $Z = \{v_7, v_8\} \setminus X$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_5 .

Let $J \in \mathcal{S}_3$, let I_1, \dots, I_n be as in the definition of \mathcal{S}_3 , let $z \in V(G)$ belong to at most one of I_1, \dots, I_n and not be an endpoint of one of I_1, \dots, I_n . Then z is a strongly simplicial vertex of J . Let $Z = \{z\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_6 .

Let $J \in \mathcal{S}_4$, let H, h_1, \dots, h_7 be as in the definition of \mathcal{S}_4 , let e be the edge in H between h_6 and h_7 , and let $Z = \{e\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_7 .

Let $J \in \mathcal{S}_5$, let d_1, \dots, d_5, A, B, C be as in the definition of \mathcal{S}_5 , and let $Z = \{d_5\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_8 .

Let J be the trigraph where the vertex set is the union of five sets $\{z\}, A, B, C, D$ with $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ for some $n \geq 1$, such that

- $\{z\} \cup D$ is a strong clique,
- $A \cup C$ and $B \cup C$ are strong cliques,
- for $1 \leq i \leq n$, a_i and b_i are antiadjacent and every vertex of D is strongly adjacent to precisely one of a_i, b_i ,
- for $1 \leq i < j \leq n$, $\{a_i, b_i\}$ is strongly complete to $\{a_j, b_j\}$, and
- the adjacency between C and D is arbitrary.

Note that J is antiprismatic. Let $Z = \{z\}$. The class of all such stripes (J, Z) is denoted by \mathcal{Z}_9 .

A.6 Chudnovsky and Seymour Theorem A trigraph G is *indecomposable* if G neither admits twins, nor a W-join, nor a 0-join, nor a 1-join, nor a generalized 2-join, nor a hex-join.

Theorem A.1 (Chudnovsky and Seymour [9, 10]) *Every indecomposable trigraph belongs to $\mathcal{S}_0 \cup \dots \cup \mathcal{S}_7$.*

B Algorithms

We propose algorithms to find 0-joins, (pseudo-) 1-joins, and (pseudo-/generalized) 2-joins in polynomial time. We should note that in the literature, 1-joins and 2-joins are usually defined in a different manner than as they were defined here (see e.g. [11, 12, 6, 7]). On claw-free graphs however, they might be more closely related, which could lead to faster algorithms for finding these joins. For the current version of this paper, it is most important that we have polynomial-time algorithms for finding these joins, which we included below. We emphasize that on general graphs, the definitions of the different types of joins used in this paper and those found in the literature diverge, which lead us to studying them separately. The algorithms were obtained independently from the literature results. Moreover, our algorithms work on general graphs, and thus may be of independent interest.

Theorem B.1 *In $O(n + m)$ time, one can find a 0-join in a graph G , or report that G has no 0-join.*

Proof: Trivial, using depth/breadth-first search. \square

Theorem B.2 *In $O(m\Delta(G))$ time, one can find twins in a graph G or report that G does not admit twins.*

Proof: For each edge $(u, v) \in E(G)$, check if $N[u] = N[v]$. This takes $O(\Delta(G))$ time. \square

B.1 Finding 1-joins We now describe how to find 1-joins.

Lemma B.3 *Let (A_1, A_2) be a 1-join of a connected graph G . Then $G[V_1]$ and $G[V_2]$ are connected.*

Proof: Suppose that $G[V_1]$ has two connected components C and C' . Because $G[A_1]$ is a clique, at most one of C, C' contains vertices of A_1 . Suppose that C contains no vertices of A_1 . Because (A_1, A_2) is a 1-join, C is not connected to V_2 . This contradicts that G is connected. \square

A spanning tree T of a graph G is a *BFS-spanning tree* if it is obtained from a breadth-first search on G . The *root* of a BFS-spanning tree is the vertex that is the origin of the breadth-first search.

Lemma B.4 *Let T be a BFS-spanning tree of a connected graph G and let (A_1, A_2) be a 1-join. Suppose that the root r of T is in V_1 . Then there is a vertex $v \in A_1$ such that all vertices of A_2 are neighbors of v in T .*

Proof: This is immediate from the fact that T is a BFS-spanning tree. \square

For a tree T , call a pair of vertices (u, v) *diametral* if the length of the u - v -path in T is maximum among all pairs.

Lemma B.5 *Given a tree T on n vertices, a diametral pair of vertices can be found in $O(n)$ time.*

Proof: Let r be an arbitrary leaf of T and let u be a vertex furthest away in T from r . Let v be a vertex furthest away from u in T . Note that all these vertices can be found in $O(n)$ time. We claim that (u, v) is a diametral pair. Suppose that there is no diametral pair containing u and let (a, b) be one such pair. Then the u - r -path in T intersects the a - b -path, or (a, b) would not be diametral. But then $\text{dist}(a, u)$ or $\text{dist}(b, u)$ is at least as large as $\text{dist}(a, b)$, a contradiction. It follows that (u, v) is a diametral pair. \square

We require some auxiliary notions on trees. Given any rooted tree T , the *nearest common ancestor* of any two vertices $a, b \in V(T)$, denoted by $\text{nca}(a, b)$, is the vertex c in T that is an ancestor of both a and b and no child of c is an ancestor of a and b . If a is not an ancestor of b and vice versa, define the *a-nearest almost-common ancestor* of a and b , or $a\text{-nca}(a, b)$, as the child of $\text{nca}(a, b)$ that is an ancestor of a .

Lemma B.6 *Let T be a BFS-spanning tree of a connected graph G . Then for any diametral pair (u, v) , at most one of u, v is in $A_1 \cup A_2$ for any 1-join (A_1, A_2) .*

Proof: Let T be a BFS-spanning tree of a graph G , let (u, v) be any diametral pair, and let (A_1, A_2) be a 1-join. Suppose that the root r of T is in V_1 . We distinguish three cases.

(1) $u, v \in A_2$

Note that u, v must be leaves of T . From Lemma B.4, there is a vertex $w \in A_1$ adjacent in T to all vertices of A_2 . Since $G[V_2]$ is connected, there is a path in $T \cap (V_2 \cup \{w\})$ from w to a vertex $x \in V_2 \setminus A_2$ that is a leaf of T . But then $3 \geq \text{dist}_T(x, v) > \text{dist}_T(u, v) = 2$, contradicting that (u, v) is a diametral pair.

(2) $v \in A_1, u \in A_2$

Consider the vertex $w \in A_1$ adjacent to all vertices of A_2 . Since $G[V_2]$ is connected, there is a path from w to a vertex $x \in V_2 \setminus A_2$ that is a leaf of T . But then $\text{dist}_T(x, v) > \text{dist}_T(u, v)$, contradicting that (u, v) is a diametral pair.

(3) $u, v \in A_1$

Neither u nor v are neighbors in T of some vertex w in A_1 . For suppose that u has a neighbor $w \in A_1$ in T . Because T is a BFS-spanning tree, all vertices of A_2 are a neighbor of w and thus there is a vertex in $V_2 \setminus A_2$ that is a leaf of T and has larger distance to v .

Consider the vertex $w \in A_1$ neighboring all vertices in A_2 . Then $w \neq u, v$, as u and v are leaves. Moreover, w is not adjacent to u or v in T . Again there is a vertex $x \in V_2 \setminus A_2$ that is a leaf of T with $\text{dist}_T(w, x) \geq 2$ and thus $\text{dist}_T(r, x) \geq \text{dist}_T(r, w) + 2$. Let $a = \text{nca}(u, v)$, $b = \text{nca}(u, w)$ and $c = \text{nca}(v, w)$. Note that $\text{dist}_T(u, v) = \text{dist}_T(r, u) + \text{dist}_T(r, v) - 2 \cdot \text{dist}_T(r, a)$. If b and c are equal to a or an ancestor of a , then $\text{dist}_T(u, x) = \text{dist}_T(r, u) + \text{dist}_T(r, x) - 2 \cdot \text{dist}_T(r, b)$. Because $(v, w) \in E(G)$ and T is a BFS-spanning tree, $|\text{dist}_T(r, v) - \text{dist}_T(r, w)| \leq 1$. But then $\text{dist}_T(r, x) > \text{dist}_T(r, v)$. Since $\text{dist}_T(r, a) \geq \text{dist}_T(r, b)$, this implies that $\text{dist}_T(u, x) > \text{dist}_T(u, v)$. This contradicts that (u, v) form a diametral pair. Hence b or c is a descendant of a . Assume w.l.o.g. that c is a descendant of a . Then

$$\begin{aligned} \text{dist}_T(c, x) &= \text{dist}_T(r, x) - \text{dist}_T(r, c) \\ &\geq \text{dist}_T(r, w) + 2 - \text{dist}_T(r, c) \\ &> \text{dist}_T(r, v) - \text{dist}_T(r, c) \\ &= \text{dist}_T(c, v). \end{aligned}$$

This means that $\text{dist}_T(u, x) > \text{dist}_T(u, v)$, contradicting that (u, v) is diametral. \square

Theorem B.7 *In $O(n(n+m))$ time, one can find a 1-join in a connected graph G , or report that G does not have such a join.*

Proof: Consider any BFS-spanning tree of G and some diametral pair (r, r') of this tree. If G has a 1-join (A_1, A_2) , we know from the above lemma that at most one of r, r' is in $A_1 \cup A_2$. Assume w.l.o.g. that r is not in $A_1 \cup A_2$ (algorithmically we will actually try both r and r'). Construct a BFS-spanning tree T with r as its root.

Let $e = (u, v)$ be any edge of the spanning tree not incident with r . Assume that $u \in A_1$, $v \in A_2$, and v has a child w in T . By Lemma B.4 and B.3, we can assume such an edge to exist. Moreover, $w \in V_2 \setminus A_2$. Now find the set of all vertices in $N[u] \cap N[v]$. This set forms the candidate set $A := A_1 \cup A_2$. This can be done in $O(\Delta(G))$ time.

What remains is to verify that we have indeed found a 1-join. As a first step, we verify that A is a clique. This can be done in $O(|A|^2)$ or $O(m)$ time. Next, we find the partition V_1, V_2 . To this end, collect the set R of all vertices reachable from w in $G - A$, using say a breadth-first search. Let $A' = N(R)$. Note that $A' \subseteq A$. Then consider $N(A') \setminus A$ and continue the breadth-first search in $G - A$ from those vertices. Iteratively apply this procedure. If the search visits all vertices of $G - A$, then G has no 1-join with (u, v) as its basis. Otherwise, the set R of visited vertices forms $V_2 \setminus A_2$ and the vertices in $N(R) \cap A$ form A_2 . It is now easy to find A_1 and V_1 .

The running time for each edge of T is bounded by $O(n + m)$. Since we need to consider at most $n - 2$ edges of the spanning tree and at most two possible roots, the total running time of the algorithm is $O(n(n + m))$. \square

We will now speed-up the part of the algorithm responsible for finding A_1 and A_2 , which is the most expensive part of the above lemma. We assume a random access machine model with logarithmic costs.

A 1-join (A_1, A_2) is *minimal* if there is no $A \subset A_1$ such that $(A_1 \setminus A, A_2 \cup A)$ is 1-join as well.

Lemma B.8 *If (A_1, A_2) is a minimal 1-join, then $G[V_1] - E(G[A_1])$ is connected.*

Proof: For suppose not and let \mathcal{C} be the set of connected components of $G[V_1] - E(G[A_1])$. For any $C \in \mathcal{C}$, observe that $C \cap A_1 \neq \emptyset$, because $G[V_1]$ is connected. But since the components are pairwise disjoint, this implies that the components induce a partition of A_1 into nonempty subsets. As $|\mathcal{C}| \geq 2$, $(A_1 \setminus (A_1 \cap C), A_2 \cup (A_1 \cap C))$ is a 1-join for any $C \in \mathcal{C}$, contradicting the minimality of (A_1, A_2) . \square

Let $\alpha(i, j)$ denote the inverse Ackermann function.

Theorem B.9 *In $O(m(\Delta(G) + \alpha(m, \Delta(G))))$ time, one can find a 1-join in a connected graph G , or report that G does not have such a join.*

Proof: Consider again the rooted tree T and edge $e = (u, v)$ from the previous lemma. As before, we assume that $u \in A_1$, $v \in A_2$ for some 1-join (A_1, A_2) . Moreover, we may assume that u is closer to the root r of T than v . This in turn implies that all vertices of A_2 are further from r than u .

Given any rooted tree T and vertex $t \in V(T)$, define $T(t)$ as the subtree of T , rooted at t , containing t and all of its descendants. If T is a spanning tree of a graph G , define $h(t)$ as the nca(a, b) closest to r for any edge $(a, b) \in E(G)$ incident with a vertex in $T(t) \setminus \{t\}$. Clearly, $h(t)$ is either t or an ancestor of t .

Consider a graph G and a rooted spanning tree T of G . For any vertex p and its set of children C , we say that $c, c' \in C$ are *linked* if there is an edge $(a, b) \in E(G)$ for which $\text{dist}_T(a, b) \geq 3$, $a \in T(c)$, and $b \in T(c')$. We then say that $c, c' \in C$ are *sequentially linked* if there is a sequence $c = c_1, \dots, c_i = c'$ of children of C such that c_j is linked to c_{j+1} for any $j = 1, \dots, i - 1$. Observe

that if $c, c' \in A_1 \cup A_2$ for some 1-join (A_1, A_2) and c and c' are sequentially linked, then either both c and c' must be in A_1 or both must be in A_2 .

Now think back on the algorithm of Theorem B.7 and let A be the candidate set for the join. All vertices for which we have not yet decided whether they should be in A_1 or in A_2 must be children of u . We say a child c of u is of *type 1* if $h(c) \notin T(u)$ or it is incident with an edge $e \notin E(A)$ for which $\text{nca}(e) \notin T(u)$. Here $\text{nca}(e)$ is a shorthand for the nearest common ancestor of the two endpoints of e .

Now consider the following observation.

- (1) A child c is in A_1 for some minimal 1-join (A_1, A_2) if and only if it is of type 1 or sequentially linked to a child of type 1.

This follows immediately from Lemma B.8.

Using this observation, we can split A into sets A_1 and A_2 . Note that this choice is only difficult for children of u . Other vertices must belong to A_1 . For children of u , we use the above observation. If every child of u is, or is sequentially linked to, a vertex of type 1, then A cannot be split to form a 1-join.

To use these ideas algorithmically, we should be able to compute nca and h efficiently. By preprocessing T in linear time, we can compute nca and $a\text{-nca}$ in constant time for any pair of vertices using algorithms of Harel and Tarjan [26]. Define $x(t)$ for any $t \in V(T)$ as the $\text{nca}(t, t')$ closest to r for any $(t, t') \in E(G)$. By preprocessing in linear time, we know the height of each vertex in the tree. Then we can compute $x(t)$ in $O(m)$ time. Now set $h(t) = t$ for any leaf of T and set $h(t)$ equal to $\max\{x(t'), h(t')\}$ for any nonleaf t , where the maximum is over all children t' of t . This takes $O(n)$ time.

To determine whether two children c, c' of a vertex p are sequentially linked, we use preprocessing with a union-find data structure. For each vertex p , associate one such data structure, initially with each child c of p in a separate set. Now for any edge $(a, b) \in E(G)$, we determine $x = \text{nca}(a, b)$. If $x = a$ or $x = b$, then this edge is not relevant. Otherwise, determine $y = a\text{-nca}(a, b)$ and $z = b\text{-nca}(a, b)$. If $y = a$ and $z = b$, then again this edge is not relevant. Otherwise, we perform a union in the data structure associated with x on y and z . This takes $O(m \cdot \alpha(m, \Delta(G)))$ time in total [42]. Afterwards, we process all structures such that find-operations will take constant time. This takes $O(n)$ time. Using this data structure, we can answer in $O(1)$ time whether two children are sequentially linked.

This concludes the analysis of all preprocessing. Preprocessing takes $O(m \cdot \alpha(m, \Delta(G)))$ time in total.

We now analyze the algorithm itself. Finding the candidate set A can be done in $O(\Delta(G))$ time. Since we do this $n - 2$ times, this contributes $O(n \cdot \Delta(G))$ to the running time. To determine the contribution of the total time it takes to verify that candidate sets form a clique, we note that we check a nonedge at most once per candidate set, so this uses $O(n)$ time. An edge is checked only if both endpoints are in the closed neighborhood of the edge (u, v) that is the basis of the candidate set. Hence an edge is checked $O(\Delta(G))$ times during the course of the algorithm. This gives a contribution of $O(m \cdot \Delta(G))$ to the running time.

It remains to analyze the running time for splitting the candidate set A into A_1 and A_2 . For this, we only need to check whether a child c of u is of type 1 or sequentially linked to a child of type 1. Determining whether $h(c) \notin T(u)$ takes $O(1)$ time. Hence we spend no more than $O(n \cdot \Delta(G))$ time on this in the course of the algorithm. We determine $\text{nca}(e)$ for an edge at most $\Delta(G)$ times if one of its endpoints is a child of the current vertex u . Since this happens at most twice, we spend no more than $O(m \cdot \Delta(G))$ time on this. If we determine that a vertex is of type 1, we indicate in the set of the union-find data structure containing this vertex that it contains a vertex of type

1. Finally, the algorithm checks for each child whether it is or is sequentially linked to a vertex of type 1. This takes another $O(n \cdot \Delta(G))$ time over the course of the algorithm.

The conclusion of the analysis is a running time for the algorithm of $O(m \cdot \Delta(G) + m \cdot \alpha(m, \Delta(G)))$. \square

Corollary B.10 *In $O(m(\Delta(G) + \alpha(m, \Delta(G))))$ time, one can find a pseudo-1-join in a connected graph G , or report that G does not have such a join.*

Proof: Let (A_1, A_2) be a pseudo-1-join. Suppose that $V_1 \setminus A_1 = \emptyset$. Since $|A_1|$ must be at least two in this case, A_1 contains twins. Hence we should first check whether G has twins whose neighborhood is a clique. If no such twins exist, any pseudo-1-join of G also is a 1-join, which can be found using Theorem B.9. Using Theorem B.2, we can find all twins in $O(m \cdot \Delta(G))$ time. Using similar ideas as in the proof of Theorem B.9, we can test all neighborhoods of twins for being a clique in $O(m \cdot \Delta(G))$ time in total. \square

B.2 Finding 2-joins We now describe algorithms to find the various 2-joins.

Lemma B.11 *Let (A_1, A_2, B_1, B_2) be a (generalized) 2-join of a connected graph G that does not admit a 1-join. Then $G[V_1]$ and $G[V_2]$ are connected.*

Proof: Suppose that $G[V_1]$ has at least two connected components and let C be one of them. If C does not contain vertices of A_1 or B_1 , then C is a connected component of G , contradicting the connectedness of G . Assume w.l.o.g. that C contains vertices of A_1 . Then C contains no vertices of B_1 . It follows that $(A_1, A_2 \cup V_0)$ is a 1-join, a contradiction. \square

Theorem B.12 *In $O(nm(\Delta(G) + \alpha(m, \Delta(G))))$ time, one can find a 2-join in a connected graph G that does not admit a 1-join, or report that G does not have such a join.*

Proof: Observe that if (A_1, A_2, B_1, B_2) is a 2-join of G , then one of the connected components of $G - (A_1 \cup A_2)$ admits a 1-join. Furthermore, if T is a spanning tree of G , then there is an edge $e = (u, v)$ of T such that $u \in A_1, v \in A_2$ or $u \in B_1, v \in B_2$.

We now proceed as follows. Find a spanning tree of G . For each edge $(u, v) \in E(T)$, we remove all edges between vertices in $N[u] \cap N[v]$ from G and try to find a 1-join in one of the connected components of this graph that does not use any vertices of $N[u] \cap N[v]$. If no such join exists, then (u, v) cannot be a basis for a 2-join. Otherwise, let (B_1, B_2) be this 1-join. Now remove all edges between B_1 and B_2 from G and try to find a 1-join (A_1, A_2) in the remaining graph that does not use any vertices of $B_1 \cup B_2$ and for which neither $V_1 \setminus A_1$ nor $V_2 \setminus A_2$ equals B_1 or B_2 . If no such 1-join exists, then G does not have a 2-join with (u, v) as a basis. Otherwise, let (A_1, A_2) be the 1-join we just found. Then (A_1, A_2, B_1, B_2) is a 2-join.

Using the algorithm of Theorem B.9 as a subroutine, the described algorithm takes $O(nm(\Delta(G) + \alpha(m, \Delta(G))))$ time. \square

Theorem B.13 *In $O(nm(\Delta(G) + \alpha(m, \Delta(G))))$ time, one can find a generalized 2-join in a connected graph G that does not admit a 1-join, or report that G does not have such a join.*

Proof: We first try to find a 2-join (i.e. a generalized 2-join with $V_0 = \emptyset$), using Theorem B.12. If no 2-join exists, guess a vertex $v \in V_0$. Then V_0 is equal to the set of vertices $u \in N[v]$ for which $N[u] = N[v]$. We can now apply the strategy as in the proof of Theorem B.12 to $G \setminus V_0$ and ensure that both 1-joins found are in $N(V_0)$. \square

Theorem B.14 *In $O(nm(\Delta(G) + \alpha(m, \Delta(G))))$ time, one can find a pseudo-2-join in a connected graph G that does not admit a pseudo-1-join, or report that G does not have such a join.*

Proof: We first try to find a generalized 2-join using Theorem B.13. If one exists, G has a pseudo-2-join. Otherwise, suppose that G has a pseudo-2-join (V_0, V_1, V_2) for which $V_1 \setminus (A_1 \cup B_1) = \emptyset$. Suppose that there is no edge in G between A_1 and B_1 . Then w.l.o.g. $|A_1| \geq 2$. But then $(A_1, V(G) \setminus A_1)$ would form a pseudo-1-join of G , a contradiction. Hence we can just apply the same idea as in the proof of Theorem B.13, but when using Theorem B.12 inside of it, we do not insist that neither $V_1 \setminus A_1$ nor $V_2 \setminus A_2$ equals B_1 or B_2 . \square

C An Algorithmic Structure Theorem of Claw-Free Graphs

We obtain an algorithmic structure theorem of claw-free graphs following from the structural characterization of claw-free graphs by Chudnovsky and Seymour [10].

C.1 Strip-structures A stripe (J, Z) is *almost-unbreakable* if

- J neither admits a 0-join, nor a pseudo-1-join, nor a pseudo-2-join,
- there are no twins $u, v \in V(J) \setminus Z$,
- there is no W -join (A, B) in J such that $Z \cap A, Z \cap B = \emptyset$.

Theorem C.1 *Every claw-free trigraph G admits a purified strip-structure with nullity zero such that all its strips are either spots or thickenings of almost-unbreakable stripes. If G is a graph, such a strip-structure can be found in polynomial time.*

Proof: Let H be such that $V(H) = \emptyset$, $E(H) = \{F\}$, and the incidence relation is empty and let η be such that $\eta(F) = V(G)$. Then (H, η) is a purified strip-structure for G with nullity zero.

We now iteratively apply the following rules. We do not apply rule i until rule $i - 1$ cannot be applied to any strip. After applying a rule, we check for rule 1 again, etc., until no more rules can be applied to any strip. Consider any $F \in E(H)$ and its corresponding strip (J, Z) . Assume that (J, Z) is not a spot. Let $\overline{F} = \{h_1, \dots, h_k\}$ and let $Z = \{v_1, \dots, v_k\}$.

(1) F is not purified

If $k \leq 1$, then F is purified. So suppose that $k \geq 2$ and $\eta(F, h_1) \cap \eta(F, h_2) \neq \emptyset$. Let $W = \eta(F, h_1) \cap \eta(F, h_2)$. If $k \geq 3$, by (CS3), $W \cap \eta(F, h_i) = \emptyset$ for $3 \leq i \leq k$. Moreover, by (CS2), W is strongly anticomplete to $\eta(F) \setminus (\eta(F, h_1) \cup \eta(F, h_2))$. Finally, by (SD2), W is a strong clique of G .

Let $W = \{w_1, \dots, w_\ell\}$. Create a new edge F'_i for each w_i and construct a new strip-structure (H', η') for G as follows.

- $V(H') = V(H)$ and $E(H') = E(H) \cup \{F'_1, \dots, F'_\ell\}$,
- for each $F_0 \in E(H)$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are adjacent in H ,
- each F'_i is adjacent only with h_1, h_2 for all $1 \leq i \leq \ell$,
- for each $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$,
- $\eta'(F) = \eta(F) \setminus W$, $\eta'(F, h_1) = \eta(F, h_1) \setminus W$, $\eta'(F, h_2) = \eta(F, h_2) \setminus W$, and $\eta'(F, h_i) = \eta(F, h_i)$ for all $3 \leq i \leq k$,
- $\eta'(F'_i) = \{w_i\}$, $\eta'(F'_i, h_1) = \{w_i\}$, and $\eta'(F'_i, h_2) = \{w_i\}$ for all $1 \leq i \leq \ell$.

If $\eta(F) = W$, we remove F from H' . It can be quickly verified that (H', η') is a strip-structure for G . Furthermore, F'_1, \dots, F'_ℓ are all purified.

(2) $\eta(F, h_i) = \emptyset$ for some $h_i \in \overline{F}$

Construct a new strip-structure (H', η') for G as follows.

- $V(H') = V(H)$ and $E(H') = E(H)$,
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are adjacent in H ,
- for each $h \in V(H) \setminus \{h_i\}$, F is incident with h in H' if and only if they are adjacent in H ,
- for each $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$,

- $\eta'(F) = \eta(F)$ and $\eta'(F, h) = \eta(F, h)$ for all $h \in \overline{F} \setminus \{h_i\}$.

Clearly, (H', η') is a strip-structure for G .

After exhaustively applying (1) and (2), (H, η) is a purified strip-structure of nullity zero.

(3) J admits a 0-join (V_1, V_2)

For $j = 1, 2$, let $Q_j = Z \cap V_j$ and let $P_j = \{h_i \mid 1 \leq i \leq k \text{ and } v_i \in V_j\}$. Clearly, $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \overline{F}$. As (H, η) has nullity zero, each v_i has a neighbor in $J \setminus Z$. Hence $V_1 \setminus Q_1 \neq \emptyset$ and $V_2 \setminus Q_2 \neq \emptyset$.

Now consider two new edges F'_1 and F'_2 and let (H', η') be defined as follows.

- $V(H') = V(H)$ and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H
- for $j = 1, 2$ and $h \in V(H)$, F'_j is incident with h in H' if and only if $h \in P_j$
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$
- for $j = 1, 2$, $\eta'(F'_j) = V_j \setminus Q_j$ and $\eta'(F'_j, h) = \eta(F, h)$ for all $h \in P_j$

Observe that (H', η') is again a strip-structure for G .

(4) J admits a pseudo-1-join (V_1, V_2)

As J has no 0-join, $A_i \neq \emptyset$ for $i = 1, 2$. For $j = 1, 2$, since V_j is not strongly stable, $V_j \setminus Z \neq \emptyset$. For $j = 1, 2$, let $Q_j = V_j \cap Z$ and let $P_j = \{h_i \mid 1 \leq i \leq k \text{ and } v_i \in V_j\}$. Clearly, $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \overline{F}$. Moreover, $V_j \setminus Q_j \neq \emptyset$.

If $Z \cap (A_1 \cup A_2) \neq \emptyset$, suppose that $v_1 \in A_1$. Then since $A_1 \cup A_2$ is a strong clique and (H, η) is purified, $Z \cap (A_1 \cup A_2) = \{v_1\}$ and $Z \setminus \{v_1\}$ is strongly anticomplete to $A_1 \cup A_2$. Furthermore, v_1 is strongly anticomplete to $J \setminus (A_1 \cup A_2)$, since every vertex in this set has an antineighbor in $A_1 \cup A_2$ and v_1 is strongly simplicial.

Now consider two new edges F'_1 and F'_2 . If $Z \cap (A_1 \cup A_2) = \emptyset$, consider a new vertex h' as well. Otherwise, suppose that $v_1 \in A_1$ and let $h' = h_1$. Let (H', η') be defined as follows.

- $V(H') = V(H) \cup \{h'\}$, and $E(H') = (E(H) \setminus \{F\}) \cup \{F'_1, F'_2\}$
- for each $F_0 \in E(H) \setminus \{F\}$ and $h \in V(H)$, F_0 is incident with h in H' if and only if they are incident in H
- for $j = 1, 2$ and $h \in V(H)$, F'_j is incident with h in H' if and only if $h \in P_j$
- F'_1 and F'_2 are incident with h'
- for all $F_0 \in E(H) \setminus \{F\}$, $\eta'(F_0) = \eta(F_0)$ and $\eta'(F_0, h) = \eta(F_0, h)$ for all $h \in \overline{F_0}$
- for $j = 1, 2$, $\eta'(F'_j) = V_j \setminus Q_j$ and $\eta'(F'_j, h) = \eta(F, h)$ for all $h \in P_j - \{h'\}$
- for $j = 1, 2$, $\eta'(F'_j, h') = A_j \setminus Q_j$.

Observe that since J does not admit a 0-join, $A_j \setminus Q_j \neq \emptyset$ for $j = 1, 2$. It can now be readily verified that (H', η') is a strip-structure of G .

(5) J admits a pseudo-2-join

Apply the procedure as described in [10], 9.1-(3).

Let (H, η) be the resulting strip-structure and let (J, Z) be any stripe of the strip-decomposition. Choose (J', Z') with $|V(J')|$ minimum such that (J, Z) is a thickening of (J', Z') and let X_v ($v \in V(J')$) be the corresponding subsets. Hence no two vertices in $V(J') \setminus Z'$ are twins in J' and there is no W-join (A, B) in J' with $Z' \cap A, Z' \cap B = \emptyset$. Moreover, by (3)-(5), J' does not admit a 0-join, a pseudo-1-join, or a pseudo-2-join. This shows that (J, Z) is a thickening of an almost-unbreakable stripe.

Observe that in applying one of rules (1),(3)-(5), the number of edges of the strip-structure increases by at least one. As it follows from (SD1) that a strip-structure can have at most $|V(G)|$ edges, we need only apply these rules at most $|V(G)|$ times. Rule (2) needs only be possibly applied after Rule (1) has been applied. Furthermore, Rule (1) needs only possibly be applied after one of Rules (3)-(5) have been applied. Hence (1) is applied at most $|V(G)|/2$ times and Rule (2) at most $|V(G)|$ times. Since applying a rule takes polynomial time, it takes polynomial time before no more rule can be applied. \square

C.2 Almost-Unbreakable Stripes

Lemma C.2 *If (J, Z) is (a thickening of) an almost-unbreakable stripe such that $J \setminus Z$ does not admit twins, then either J does not admit twins, or J is a strong clique with $|V(J)| = 2$ and $|Z| = 1$.*

Proof: Suppose that J admits twins u, v . Since $J \setminus Z$ does not admit twins and Z is strongly stable, exactly one of u, v is in Z . But then $N[u] = N[v]$ is a strong clique. As (J, Z) is (a thickening of) an almost-unbreakable stripe, $(\{u, v\}, V(J) \setminus \{u, v\})$ is not a pseudo-1-join of J , and $V(J) \setminus \{u, v\}$ must be strongly stable. Moreover, J does not admit a 0-join. So $|V(J) \setminus \{u, v\}| \leq 1$, and J is a strong clique, and thus $|Z| \leq 1$. But as $J \setminus Z$ does not admit twins, $|V(J)| = 2$ and $|Z| = 1$. \square

Lemma C.3 ([10], 10.4) *Let (J, Z) be (a thickening of) an almost-unbreakable stripe. Then either J is a thickening of an indecomposable member of $\mathcal{S}_0, \dots, \mathcal{S}_7$ or J admits a hex-join.*

We now consider three different cases.

C.3 J is a line graph or a union of two strong cliques

Lemma C.4 *Let (J, Z) be an almost-unbreakable stripe with $|V(J)| > 2$ such that $V(J)$ is the union of two strong cliques. Then $|V(J)| \leq 4$, and $(J, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof: By Lemma C.2, J does not admit twins. Let A, B be disjoint strong cliques in J with $A \cup B = V(J)$ such that A is maximal (i.e. no vertex of B is strongly complete to A). Let W denote the set of vertices of A that are strongly complete to B . Since J does not admit twins and $|V(J)| > 2$, $|W| \leq 1$ and $|A \setminus W|, |B| \geq 1$. Since Z is strongly stable, $|Z \cap A|, |Z \cap B| \leq 1$. Since vertices of Z are strongly simplicial and no member of $A \setminus W$ is strongly complete to B and vice versa, $W \cap Z = \emptyset$. Similarly, each vertex of Z is either strongly anticomplete to $A \setminus W$ or strongly anticomplete to B . Since (A, B) is a homogeneous pair, one of $|A|, |B| > 1$, and since (J, Z) is almost-unbreakable, $Z \neq \emptyset$.

Let $A' = A \setminus W$. Since $(A' \setminus Z, B \setminus Z)$ is a homogeneous pair that is not a W-join and J does not admit twins, $|A' \setminus Z|, |B \setminus Z| \leq 1$. Hence $|V(J)| \leq 5$.

Suppose that $|V(J)| = 5$. Since $|A' \setminus Z| + |B \setminus Z| \leq 2$ and $|Z| \leq 2$, $|A'| + |B| \leq 4$ and thus $|W| = 1$. As (J, Z) is a stripe, no vertex of $V(J)$ is adjacent to more than one vertex of Z . As W

is strongly complete to A' and B , $|Z| = 1$. But then as $|A' \setminus Z| + |B \setminus Z| \leq 2$, $|A'| + |B| \leq 3$, and thus $|V(J)| = |A| + |B| \leq 4$, a contradiction.

Suppose that $|V(J)| = 4$. If say $|A| = 3$, then as J has no 0-join, a vertex of A is adjacent to b , where $B = \{b\}$. If all vertices of A adjacent to B are strongly adjacent to B , or if more than one vertex of A is strongly adjacent to b , then J admits twins, a contradiction. Hence $|W| = 1$, some vertex of A' must be semiadjacent to b , and another vertex of A' is strongly antiadjacent to b . Then $(J, Z) \in \mathcal{Z}_6$. If $|A| = 1$, then no vertex of B is strongly adjacent to a , where $A = \{a\}$, for this would contradict the maximality of A . As only one vertex of B can be semiadjacent to a , B contains twins, a contradiction.

So $|A| = 2$. If $B \cap Z \neq \emptyset$ and $|W| = 1$, then $A \cap Z = \emptyset$ and the vertex in A' must be semiadjacent to the vertex in $B \setminus Z$, as J does not admit twins. Then $(J, Z) \in \mathcal{Z}_6$. If $B \cap Z \neq \emptyset$ and $W = \emptyset$, then as A is neither strongly complete nor strongly anticomplete to $B \setminus Z$, $(A, B \setminus Z)$ is a homogenous pair, and (J, Z) is almost-unbreakable, $A \cap Z \neq \emptyset$. But then J is a four-vertex path and $(J, Z) \in \mathcal{Z}_1$. If $B \cap Z = \emptyset$, then $A \cap Z \neq \emptyset$. As J does not admit twins, $W = \emptyset$. Since $(A \setminus Z, B)$ is a homogenous pair, $A \setminus Z$ is neither strongly complete nor strongly anticomplete to B , and (J, Z) is almost-unbreakable, $B \cap Z \neq \emptyset$, a contradiction.

Finally, suppose that $|V(J)| = 3$. Since $Z \neq \emptyset$ and J does not admit twins, J is not a triangle. But then J is a three-vertex path and (J, Z) is a member of \mathcal{Z}_6 . \square

Corollary C.5 *Let (J, Z) be a thickening of an almost-unbreakable stripe (J', Z') such that J is a union of two strong cliques and $Z \neq \emptyset$. Then (J, Z) is a thickening of a member of $\mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof: Since J is a union of two strong cliques, J' also is a union of two strong cliques. If $|V(J')| = 2$, then J' consists of a strongly adjacent pair of vertices. Then $(J', Z') \in \mathcal{Z}_6$. If $|V(J')| > 2$, then it follows from Lemma C.4 that $(J', Z') \in \mathcal{Z}_1 \cup \mathcal{Z}_6$. The corollary follows. \square

Lemma C.6 *Let G be a thickening of a line trigraph of a graph H such that G admits no 0-join, pseudo-1-join, or pseudo-2-join. Then G admits no W -join and H has no vertex of degree two, or G is a union of two strong cliques.*

Proof: Follows from the proof of [10], 10.3. \square

Lemma C.7 *Let G be a thickening of a line trigraph of a graph H such that G admits no 0-join, pseudo-1-join, pseudo-2-join, or twins, and G is not a union of two strong cliques. Then G admits a biclique. Moreover, a biclique can be found in linear time.*

Proof: Following Lemma C.7, G admits no W -join and H has no vertex of degree two. Hence G is the line graph of H and we can find H from G in linear time [40].

If every edge of H is incident with a vertex of degree one, then H is a star and G is a strong clique, contradicting the assumption of the lemma. Hence there is an edge $uv = e \in E(H)$ such that u, v both have degree at least three. Let P (resp. Q) be the sets of edges of H incident with u but not v (resp. with v but not u). Let V_1 be the singleton set with the vertex of G corresponding to e , V_2 (resp. V_3) the set of vertices corresponding to vertices in P (resp. Q), and let $V_4 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Note that $V_1 \cup V_2$ and $V_1 \cup V_3$ are strong cliques, V_1 is strongly anticomplete to V_4 , and, since $|P| \geq 2$, V_2 is not strongly stable. If $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent, then the corresponding edges share a vertex in H . Hence v_2, v_3 have the same neighbors in V_4 . Finally, as $|P|, |Q| \geq 2$, there exist $p \in P$ and $q \in Q$ with no common endpoint, and thus $V_2 \cup V_3$ is not a strong clique. Then (V_1, V_2, V_3, V_4) is a biclique. Because we know H , this biclique can be found in linear time. \square

Corollary C.8 *Let (J, Z) be an almost-unbreakable stripe with $|V(J)| > 2$. If J is a thickening of a line trigraph, then $(J, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$ or J admits a biclique.*

Proof: By Lemma C.2, J does not admit twins. If J is a union of two strong cliques, then the result follows from Lemma C.4. So assume that J is not a union of two strong cliques. Then Lemma C.7 shows that J admits a biclique. \square

Corollary C.9 *Let G be a thickening of a line trigraph of a graph H such that G admits no 0-join, pseudo-1-join, or pseudo-2-join. Then G admits no W -join and H has no vertex of degree two, or G is a union of two strong cliques. If G is not a union of two strong cliques, then G admits a biclique, and if additionally G has no twins, a biclique can be found in linear time.*

C.4 Dealing with Bicliques If we come across a biclique, we apply another rule to expand the strip-structure. The rule is described in [10], 9.1(4). Afterwards, we again apply the rules of Theorem C.1.

C.5 J is a thickening of an indecomposable member of \mathcal{S}_i $i \in \{1, \dots, 7\}$

Lemma C.10 ([10], 11.1) *Let $J \in \mathcal{S}_i$ for some $i \in \{1, 2, 4, 5, 6, 7\}$ and suppose that J is indecomposable and $V(J)$ is not a union of two strong cliques.*

- *If $z \in V(J)$ is a simplicial vertex, let Z be the set of all simplicial vertices of J . Then $|Z| \leq 2$ and $(J, Z) \in \mathcal{Z}_j$ for some $j \in \{2, 5, 7, 8, 9\}$.*
- *If $z \in V(J)$ is a near-simplicial vertex semiadjacent to z' , let $Z = \{z, z'\}$. Then $(J', Z) \in \mathcal{Z}_2 \cup \mathcal{Z}_5$, where J' is the trigraph obtained from G by making z, z' strongly antiadjacent.*

Lemma C.11 *Let (J, Z) be an almost-unbreakable stripe such that $Z \neq \emptyset$. If J is a thickening of a long circular interval trigraph, then $(J, Z) \in \mathcal{Z}_1 \cup \mathcal{Z}_6$.*

Proof: The proof is essentially the same as [10], 12.1. \square

Lemma C.12 *Let (J, Z) be an almost-unbreakable stripe with $Z \neq \emptyset$. If J is a thickening of an indecomposable member of \mathcal{S}_i , where $i \in \{1, \dots, 7\}$, then J is a union of two strong cliques or $(J, Z) \in \mathcal{Z}_i$, where $i \in \{1, 2, 5, 6, 7, 8, 9\}$.*

Proof: The proof is essentially the same as [10], 12.2. \square

C.6 J admits a hex-join

Lemma C.13 *Let (J, Z) be an almost-unbreakable stripe such that J admits a hex-join. Then $|Z| \leq 2$.*

Proof: Same proof as [10], 13.1. \square

Lemma C.14 *Let (J, Z) be an almost-unbreakable stripe with $|Z| = 2$. Then $V(J)$ is the union of three strong cliques if and only if $V(J) \setminus N[Z]$ is a strong clique.*

Proof: Suppose that $V(J) \setminus N[Z]$ is a strong clique. Since (J, Z) is a stripe, the vertices in Z are strongly simplicial. It follows that $V(J)$ is the union of three strong cliques.

Let $Z = \{z_1, z_2\}$. Suppose that $V(J)$ is the union of three strong cliques A, B, C . Since z_1 and z_2 are strongly antiadjacent, we can assume that $z_1 \in A$ and $z_2 \in B$. But then $V(J) \setminus N[Z] \subseteq C$. \square

Lemma C.15 *Let (J, Z) be a thickening of an almost-unbreakable stripe with $|Z| = 2$, such that $V(J)$ is the union of three strong cliques. Then (J, Z) is a thickening of a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$.*

Proof: Same proof as [10], 13.2. \square

Lemma C.16 *Let (J, Z) be an almost-unbreakable stripe such that J admits a hex-join. Then J is a union of two strong cliques, or $J \setminus N[Z] \neq \emptyset$ and $\alpha(J) \leq 3$.*

Proof: We may assume that J is not a union of two strong cliques. Since the neighborhood of each $z \in Z$ is a strong clique and $|Z| \leq 2$ by Lemma C.13, $J \setminus N[Z] \neq \emptyset$. Moreover, J is a union of three strong cliques by the definition of a hex-join. \square

C.7 Bringing it all together

Lemma C.17 *Let graph G be a thickening of a circular interval trigraph G' . Then G is a circular interval graph or G contains a proper W -join.*

Proof: Suppose that G is not a circular interval graph. Then Eisenbrand et al. ([18], Lemma 2) showed that G admits a W -join (A, B) such that $G[A \cup B]$ contains a C_4 as an induced subgraph. Iteratively remove vertices from A (resp. B) that are complete or anticomplete to B (resp. A) and call the resulting sets A' and B' . As $G[A \cup B]$ has a C_4 as an induced subgraph, $|A'|, |B'| \geq 2$. Hence (A', B') is a proper W -join. \square

Lemma C.18 *Let (J, Z) be a thickening of an almost-unbreakable stripe. Then J admits a proper W -join if and only if there is a proper W -join (A, B) in J such that $Z \cap A, Z \cap B = \emptyset$.*

Proof: Suppose that J admits a proper W -join (A, B) . Suppose w.l.o.g. that $Z \cap A \neq \emptyset$. As Z is strongly stable, we can assume that $Z \cap A = \{z\}$ for some $z \in Z$. Since (A, B) is a proper W -join, $N[z] \cap B \neq \emptyset$. Say $b \in N[z] \cap B$. As A is a strong clique and $z \in A$, $A \subseteq N[z]$. Since z is strongly simplicial, this implies that b is strongly complete to A , contradicting that (A, B) is a proper W -join. The converse is trivial. \square

Theorem C.19 *Let G be a connected claw-free graph, such that G does not admit twins or proper W -joins and $\alpha(G) > 3$. Then either*

- G is a thickening of a member of \mathcal{S}_2 ,
- $G \in \mathcal{S}_3$, or
- G admits a strip-structure such that for each strip (J, Z) either
 - (J, Z) is a spot, or
 - (J, Z) is a stripe with $1 \leq |Z| \leq 2$ for which either
 - * J is a circular interval graph,
 - * (J, Z) is a thickening of a member of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$, or
 - * $|Z| = 1$, $\alpha(J) \leq 3$, and $J \setminus N[Z] \neq \emptyset$.

Moreover, we can distinguish the cases and find the strip-structure in polynomial time.

Proof: By Theorem C.1, G has a purified strip-structure (H, η) of nullity zero such that all its strips are spots or thickenings of almost-unbreakable stripes.

Suppose first that $|E(H)| = 1$. Then the unique strip of (H, η) is (G, \emptyset) , which is a spot or a thickening of an almost-unbreakable stripe. Since spots have $|Z| = 2$, G is a thickening of an almost-unbreakable stripe. Hence G does not admit a 0-join, a 1-join, or a generalized 2-join. Since $\alpha(G) > 3$, G is not a union of three cliques, and G does not admit a hex-join. Choose G' with $|V(G')|$ minimum such that G is a thickening of G' and let X_v ($v \in V(G')$) denote the corresponding sets. Then $G' \in \mathcal{S}_0 \cup \dots \cup \mathcal{S}_7$ following Lemma C.3. We consider the various cases:

\mathcal{S}_7 : Suppose that G has a strong stable set I of size at least four. Then I corresponds to an independent set in G'' , where G'' is G' with all semiadjacent edges removed. We call this independent set I as well. Since for any $X \subseteq V(G')$ with $|X| = 4$, at least two pairs of vertices in X are strongly adjacent, the same holds with respect to G'' . Applying this to any subset of I of size four, we obtain a contradiction to the assumption that I is stable. Hence $G' \notin \mathcal{S}_7$.

\mathcal{S}_6 : Recall that each member of \mathcal{S}_6 is a union of three strong cliques. Then G is also a union of three strong cliques and $\alpha(G) \leq 3$, a contradiction.

\mathcal{S}_5 : Suppose that G has a strong stable set I of size at least four. Then I corresponds to an independent set in G'' , where G'' is G' with all semiadjacent edges removed. We call this independent set I as well. Let $A, B, C, d_1, \dots, d_5, X$ be as in the definition of \mathcal{S}_5 . Since A, B, C are strong cliques, $|I \cap (A \cup B \cup C)| \leq 3$. Suppose that $|I \cap (A \cup B \cup C)| = 3$, then I has precisely one vertex from each of A, B, C , say a_i, b_j, c_k respectively. Since c_k is strongly adjacent to b_j if and only if $j \neq k$, $j = k$. Similarly, $i = k$. But then $i = j$, thus a_i and b_j are adjacent in G' . But then a_i and b_j must be semiadjacent in G' and $c_k \in X$. Hence $c_k \notin V(G'')$, a contradiction. Hence $|I \cap (A \cup B \cup C)| \leq 2$.

Note that d_3, d_4, d_5 are strongly adjacent, so $|I \cap \{d_3, d_4, d_5\}| \leq 1$. Therefore $d_1 \in I$ or $d_2 \in I$. As d_1 is strongly complete to $A \cup B \cup C$, $d_1 \in I$ implies that $|I| \leq 3$. Hence $d_1 \notin I$ and $d_2 \in I$. As d_2 is strongly $A \cup B$ complete, $|I \cap (A \cup B \cup C)| \leq 1$, implying that $|I| \leq 3$, a contradiction.

\mathcal{S}_4 : Suppose again that G has a strong stable set I of size at least four. Then we can again consider G'' , obtained from G' by removing semiadjacent edges. Then I is an independent set in G'' as well. Let H, h_1, \dots, h_7 be as in the definition of \mathcal{S}_4 . Let E_6 denote the set of edges of H incident with h_6 and let x be the vertex added to $L(H)$ to obtain G' . Note that x is strongly adjacent to precisely the edges $E(C)$ of the induced cycle $C = \{h_1, \dots, h_5\}$. Observe that $V(G'') = E_6 \cup E(C) \cup \{x\}$. Since E_6 is a strong clique in G'' , $|I \cap E_6| \leq 1$. Hence if $x \in I$, then $|I| \leq 2$, a contradiction. Otherwise, as $E(C)$ induces a 5-cycle, $|I \cap E(C)| \leq 2$ and thus $|I| \leq 3$, a contradiction.

\mathcal{S}_1 : It suffices to show that $\alpha(G) \leq 3$ if $G' = G_0$. So let v_1, \dots, v_{12} be as in the definition of \mathcal{S}_1 and let I and G'' be defined as above. If $v_{11}, v_{12} \in I$, then $|I| = 2$, a contradiction. Let $I' = I \cap \{v_1, \dots, v_{10}\}$ and let J denote the set of indices of the vertices in I' . If J contains only even or only odd integers, then $|I'| \leq 2$ as v_i is adjacent to v_{i+2} (indices modulo 10) for $1 \leq i \leq 10$. Hence $|I| \leq 3$, a contradiction. But then J contains both odd and even integers and thus $v_{11}, v_{12} \notin I$. If $j \in J$, then $j - 2, j - 1, j + 1, j + 2 \notin J$ (integers modulo 10). Hence $|I| = |J| \leq \lfloor 10/3 \rfloor = 3$, a contradiction.

It follows that G is a thickening of an indecomposable member of $\mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

Suppose that G is a thickening of a member of \mathcal{S}_0 . Since G is not a union of two strong cliques and G does not admit twins, G admits a biclique by Lemma C.7. Moreover, it can be found in linear time. We then apply the Rule for bicliques and recurse.

Suppose that G is a thickening of a member of \mathcal{S}_3 . Since G does not admit a proper W-join, it follows from Lemma C.17 that G is circular interval graph. Hence we can just check if G is a circular interval graph to determine whether G is a thickening of a member of \mathcal{S}_3 . Recognizing a circular interval graph takes linear time [14].

If neither of the above two cases hold, then G must be a thickening of a member of \mathcal{S}_2 .

Now consider the case when $|E(H)| > 1$. Let (J, Z) be a strip under consideration. We can check in constant time whether or not (J, Z) is a spot. So assume that (J, Z) is not a spot. By attempting to bicolor the complement of J , we can check in polynomial time whether J is a union of two cliques. If so, then (J, Z) is a thickening of a member of $\mathcal{Z}_1 \cup \mathcal{Z}_6$ by Corollary C.5. We describe later how to deal with this.

Suppose now that J is a thickening of a line trigraph. We know that (J, Z) is not a union of two strong cliques. But then, by Lemma C.2 and C.6, J does not admit a W-join or twins. Hence J is a line graph $L(H)$ for some graph H . Hence to determine whether J is a thickening of a line trigraph, it suffices to determine whether J is a line graph $L(H)$. Recognition of line graphs and finding H can be done in linear time [40]. Since J is not the union of two strong cliques, it follows from Lemma C.7 that J admits a biclique, which can be found in linear time. We apply the Rule for bicliques and recurse.

Suppose that $|Z| = 2$. If $V(J) \setminus N[Z]$ is a strong clique (which can be checked for in linear time), then $V(J)$ is the union of three strong cliques by Lemma C.14. Then it follows from Lemma C.15 that (J, Z) is a thickening of a member of \mathcal{Z}_i , where $i \in \{1, 2, 3, 4\}$. We show later how to recognize the case when (J, Z) is a thickening of a member of \mathcal{Z}_1 .

Suppose that $|Z| = 1$, $\alpha(J) \leq 3$, and $J \setminus N[Z] \neq \emptyset$. This can be checked for in polynomial time. Moreover, this is exactly one of the cases in the theorem statement.

Observe that if $V(J)$ is the union of three nonempty strong cliques, we are exactly in one of the above two cases by Lemma C.14.

Suppose that J is not a thickening of a line trigraph, $V(J)$ is not the union of two or three strong cliques, and (J, Z) is not a spot. Recall that (J, Z) is a thickening of some almost-unbreakable stripe (J', Z') . Following Lemma C.3, this implies that J' is a thickening of an indecomposable member of $\mathcal{S}_0, \dots, \mathcal{S}_7$ or J' admits a hex-join. If J' admits a hex-join, then J' is a union of three strong cliques and so is J , a contradiction. If J' is a thickening of an indecomposable member of \mathcal{S}_0 , then so is J , a contradiction. Hence J' is a thickening of an indecomposable member of $\mathcal{S}_1, \dots, \mathcal{S}_7$. It follows from Lemma C.12 that (J', Z') is a member of \mathcal{Z}_i , where $i \in \{1, 2, 5, 6, 7, 8, 9\}$, or $V(J')$ is a union of two strong cliques. In the latter case, $V(J)$ is also a union of two strong cliques, a contradiction.

If (J', Z') is a member of \mathcal{Z}_i , where $i \in \{7, 8, 9\}$, then we can show that $\alpha(J) \leq 3$ and $J \setminus N[Z] \neq \emptyset$. Suppose that (J', Z') is a member of

\mathcal{Z}_7 : Since $J' \in \mathcal{S}_4$, we have already seen that $\alpha(J) \leq 3$. Since the edge h_6h_7 is not incident with any edges of the cycle $h_1 \dots h_5$, $J \setminus N[Z] \neq \emptyset$.

\mathcal{Z}_8 : Since $J' \in \mathcal{S}_5$, we have already seen that $\alpha(J) \leq 3$. Since d_5 is not incident with the nonempty strong cliques A, B, C , it follows that $J \setminus N[Z] \neq \emptyset$.

\mathcal{Z}_9 : Since $J' \in \mathcal{S}_7$ (J' is antiprismatic), we have already seen that $\alpha(J) \leq 3$. Moreover, z is strongly antiadjacent to the nonempty strong cliques A, B, C , so $J \setminus N[Z] \neq \emptyset$.

It follows that $(J', Z') \in \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_5 \cup \mathcal{Z}_6$ and thus that (J, Z) is a thickening of a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_5 \cup \mathcal{Z}_6$.

It remains to recognize whether (J, Z) is a thickening of an almost-unbreakable member of $\mathcal{Z}_1 \cup \mathcal{Z}_6$. So assume that this is indeed the case. Suppose that J contains a proper W-join (A, B) . If $Z \cap A, Z \cap B = \emptyset$, then (A, B) is also a proper W-join in G , a contradiction. But then J does not admit a proper W-join by Lemma C.18, and J is a circular interval graph by Lemma C.17. So to check if (J, Z) is a thickening of a member of $\mathcal{Z}_1 \cup \mathcal{Z}_6$, it suffices to check if J is a circular interval graph, which can be done in linear time [14].

Note that the proof relies on recursion whenever we find a biclique. However, every time we find a biclique and apply the Rule for bicliques, $|E(H)|$ increases by at least one. As $|E(H)| \leq |V(G)|$ by definition, at most $|V(G)|$ recursive steps are needed. The theorem follows. \square

Observe that Theorem 2.1 is an immediate consequence of Theorem C.19.

Theorem C.20 *Let G be a connected claw-free graph, such that G does not admit twins or proper W -joins and $\alpha(G) > 3$. Then either*

- G is a thickening of a member of \mathcal{S}_2 ,
- $G \in \mathcal{S}_3$, or
- G admits a strip-structure such that for each strip (J, Z) either
 - (J, Z) is a spot, or
 - (J, Z) is a stripe with $1 \leq |Z| \leq 2$ for which either
 - * J is a circular interval graph,
 - * (J, Z) is a thickening of a member of \mathcal{Z}_5 , or
 - * $\alpha(J) \leq 3$ and $J \setminus N[Z] \neq \emptyset$.

Moreover, we can distinguish the cases and find the strip-structure in polynomial time.

Proof: We follow the proof of Theorem C.19. We claim that any stripe (J, Z) that is a thickening of a member (J', Z') of $\mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4$ has $\alpha(J) \leq 3$ and $J \setminus N[Z] \neq \emptyset$.

\mathcal{Z}_2 : Since $J' \in \mathcal{S}_6$, we know that $\alpha(J) \leq 3$. Since the strong clique C in the definition of \mathcal{S}_6 has $|C \setminus X| \geq 2$ and both a_0 and b_0 are strongly antiadjacent to C , it follows that $J \setminus N[Z] \neq \emptyset$.

\mathcal{Z}_3 : Note that $V(J)$ is the union of three nonempty strong cliques. Hence $\alpha(J) \leq 3$ and $J \setminus N[Z] \neq \emptyset$.

\mathcal{Z}_4 : Note that $V(J)$ is the union of three nonempty strong cliques. Hence $\alpha(J) \leq 3$ and $J \setminus N[Z] \neq \emptyset$.

The theorem follows immediately. \square

D Proofs for Section 5

Proof of Lemma 5.2: Note that any member of \mathcal{Z}_2 has at most three semiadjacent pairs of vertices. The first goal is to find these pairs (if they exist).

Observe that the definition of \mathcal{Z}_2 is symmetric for the choice of a_0, b_0 . That is, if $Z = \{z_1, z_2\}$, we can choose $a_0 = z_1$ and $b_0 = z_2$, or vice versa. Hence we may assume that we know a_0, b_0 . But then let $A = N(a_0)$, $B = N(b_0)$, and $C = V(J) \setminus (A \cup B)$. By definition, A , B , and C must be strong cliques. It is also clear that $A, B \neq \emptyset$ and that $|C| \geq 2$.

- (1) If $a \in A$ and $b \in B$ are adjacent, they have the same set of neighbors in C . Moreover, all neighbors of a and b in C are strong neighbors.

Suppose that $c \in C$ is adjacent to a , but antiadjacent to b . Then $\{a, a_0, b, c\}$ is a claw (with center a).

Let A^* denote the set of vertices in A that are strongly complete to C . Let B^* denote the set of vertices in B that are strongly complete to C .

- (2) (A^*, B^*) is a W-join, or $|A^*|, |B^*| \leq 1$

Suppose that $|A^*| = 0$ and $|B^*| > 1$. Then by (1), B^* is strongly anticomplete to A . But then any two vertices in B^* are twins. This contradicts the assumption that J does not admit twins. Similarly, $|B^*| = 0$ implies that $|A^*| \leq 1$. So suppose that $|A^*|, |B^*| \geq 1$ and w.l.o.g. that $|A^*| \geq 2$. It follows from the definition of A^* and B^* and from (1) that A^* is strongly anticomplete to $B \setminus B^*$ and that B^* is strongly anticomplete to $A \setminus A^*$. If A^* is strongly complete or strongly anticomplete to B^* , then any two vertices of A^* form twins, a contradiction. Hence A^* is neither strongly complete nor strongly anticomplete to B^* and thus (A^*, B^*) is a W-join.

- (3) If a_i, b_i are semiadjacent in J' , for some $(J', Z') \in \mathcal{Z}_2$ for which (J, Z) is a thickening of (J', Z') , then $X_{a_i} \subseteq A^*$ and $X_{b_i} \subseteq B^*$.

This follows immediately from the fact that if a_i, b_i are semiadjacent, then $c_i \in X$ by the definition of \mathcal{Z}_2 , and thus a_i and b_i are complete to $C \setminus X$ in J' .

Let A' denote the set of all vertices in A that are strongly anticomplete to B . Similarly, let B' denote the set of all vertices in B that are strongly anticomplete to A . Let Q be the set of vertices in C complete to $A \setminus A'$ and to $B \setminus B'$. Let $P = C \setminus Q$.

- (4) P is strongly complete to A' and B'

By definition, any vertex of $A \setminus A'$ is adjacent to some vertex of $B \setminus B'$. By (1) and the definition of P , any $p \in P$ is antiadjacent to at least one vertex in $A \setminus A'$ and to at least one vertex in $B \setminus B'$. But then it follows from the definition of \mathcal{Z}_2 that P is strongly complete to A' and B' .

Now let X_{q_1} denote the set of vertices in Q that are antiadjacent to at least one vertex of A' . Let $N_{A'}$ denote the set of vertices in A' that some $x \in X_{q_1}$ is antiadjacent to.

- (5) $(X_{q_1}, N_{A'})$ is a W-join, or $|X_{q_1}|, |N_{A'}| \leq 1$.

We claim that B is strongly complete to X_{q_1} . For suppose that $c \in X_{q_1}$ and $b \in B$ are antiadjacent. Then $c \in X_{c_i}$ and $b \in X_{b_i}$ in any thickening of a member of \mathcal{Z}_2 that results in (J, Z) . However, c is

antiadjacent to some $a \in A'$ by the definition of Q . It follows that $a \in X_{a_i}$ in this thickening. But then a and b are adjacent in J , contradicting that $a \in A'$.

Since $X_{q_1} \subseteq Q$, X_{q_1} is strongly complete to $A \setminus A'$. By definition of X_{q_1} , X_{q_1} is strongly complete to $A' \setminus N_{A'}$. By (4), $N_{A'}$ is strongly complete to P . Moreover, by the definition of X_{q_1} , $Q \setminus X_{q_1}$ is complete to A' , and thus to $N_{A'}$ as well. Hence if $|X_{q_1}|, |N_{A'}| \geq 1$ and at least one of $|X_{q_1}|, |N_{A'}| \geq 2$, $(X_{q_1}, N_{A'})$ is a W-join.

So suppose that $|X_{q_1}| = 0$ and $|N_{A'}| \geq 2$. Then observe that any two members of $N_{A'}$ are twins, a contradiction. Similarly, if $|N_{A'}| = 0$, then $|N_{A'}| \leq 1$.

Let X_{q_2} denote the set of vertices in Q that are antiadjacent to at least one vertex of B' . Let $N_{B'}$ denote the set of vertices in B' that some $x \in X_{q_2}$ is antiadjacent to. Then $(X_{q_1}, N_{B'})$ is a W-join, or $|X_{q_1}|, |N_{B'}| \leq 1$, using similar arguments as in the proof of (5).

- (6) If a_i, c_i are semiadjacent in J' , where (J, Z) is a thickening of $(J', Z') \in \mathcal{Z}_2$, then $X_{a_i} \subseteq N_{A'}$ and $X_{c_i} \subseteq X_{q_1}$. Similarly, if b_i, c_i are semiadjacent, then $X_{b_i} \subseteq N_{B'}$ and $X_{c_i} \subseteq X_{q_2}$.

This follows immediately from the definition of \mathcal{Z}_2 .

It is also easy to see that the sets $A^*, B^*, X_{q_1}, N_{A'}, X_{q_2}, N_{B'}$ are pairwise disjoint. Hence any of $(A^*, B^*), (X_{q_1}, N_{A'}), (X_{q_2}, N_{B'})$ that form a W-join can be replaced by a semiadjacent pair of vertices, with the same index i . The indices are different for each pair however. We use indices from $\{1, 2, 3\}$ for this.

Now let $\bar{A}, \bar{B}, \bar{C}$ denote the vertices remaining in A, B, C respectively after removing vertices involved in any one of the three possible W-joins we just found. Now consider the graph G' , where $V(G') = \bar{A} \cup \bar{B} \cup \bar{C}$ and

- $a \in \bar{A}, b \in \bar{B}$ are adjacent if and only if they are strongly adjacent in J ,
- $a \in \bar{A}, c \in \bar{C}$ are adjacent if and only if they are strongly antiadjacent in J ,
- $a \in \bar{B}, c \in \bar{C}$ are adjacent if and only if they are strongly antiadjacent in J .

Then from the definition of \mathcal{Z}_2 , G' can only contain vertices, edges, and triangles as connected components. We label the vertices in each connected component with a unique label from $\{4, 5, \dots\}$. This concludes the recognition algorithm. \square

Proof of Lemma 5.4: Let $Z = \{z_1, z_2\}$. Since $V(J)$ is not a union of two strong cliques, $V(J) \setminus N[Z] \neq \emptyset$. Consider vertices in $V(J) \setminus N[Z]$. If there is only one such vertex v , then $X_{h_2h_3}$ is the set of strong neighbors of v in $N(z_1)$ and $X_{h_3h_4}$ is the set of strong neighbors of v in $N(z_2)$. Note that $|V(J) \setminus N[Z]| = 1$ implies that all edges of H are incident with h_2 or h_4 , except the one corresponding to v , which is incident with h_3 . If v is semiadjacent to some vertex of J , then v corresponds to an edge incident to a vertex of degree two in H . Then according to Figure 2, the choice of $X_{h_2h_3}$ and $X_{h_3h_4}$ made above corresponds to some valid choice of H . So assume that v is strongly complete to all its neighbors. There are several non-isomorphic forms that H can take now (see Figure 3). However, these are indistinguishable if we just know J . It is easy to see however that the choice of $X_{h_2h_3}$ and $X_{h_3h_4}$ made above corresponds to some valid choice of H .

If there are at least two such vertices u, v in $V(J) \setminus N[Z]$, then $X_{h_2h_3}$ is the intersection of the strong neighborhoods of u, v , and z_1 and $X_{h_3h_4}$ is the intersection of the strong neighborhoods of u, v , and z_2 (see for example Figure 4).

We can now identify the other parts of H . First, $h_1h_2 = z_1$ and $h_4h_5 = z_2$. Let L_2, L_3 , and L_4 respectively denote the set of vertices in $V(J) \setminus (X_{h_2h_3} \cup X_{h_3h_4} \cup Z)$ adjacent and complete to $\{h_1h_2\} \cup X_{h_2h_3}$, $X_{h_2h_3} \cup X_{h_3h_4}$, and $X_{h_3h_4} \cup \{h_4h_5\}$. If $v \in L_2$ (L_3 respectively L_4) is not adjacent

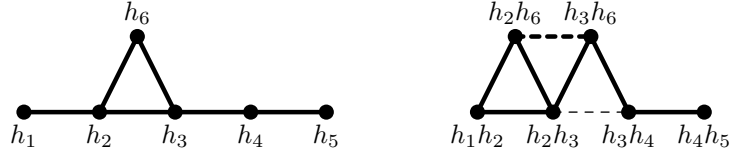


Figure 2: A graph H with exactly one edge incident to h_3 that can lead to a semiadjacent pair of vertices when used in the definition of \mathcal{Z}_3 .

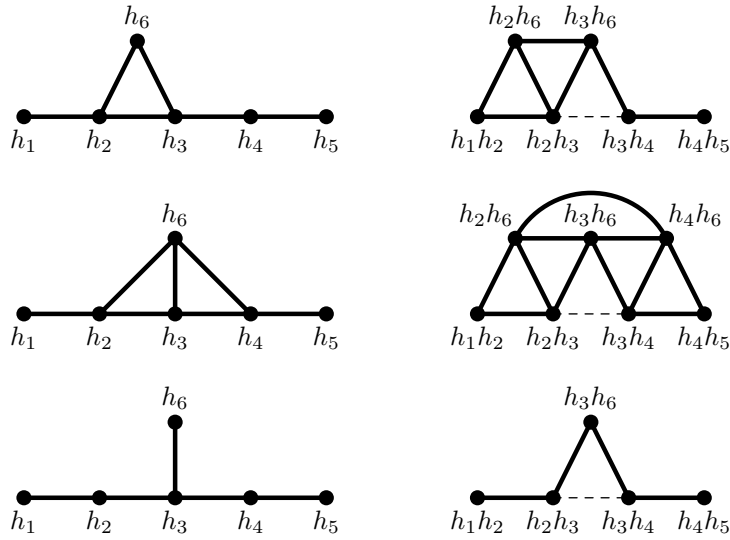


Figure 3: Three examples of graphs H with exactly one edge incident to h_3 and the stripes they induce when used in the definition of \mathcal{Z}_3 . Observe that the top two graphs in the right column can be seen as thickenings of an appropriately chosen version of the bottom one.

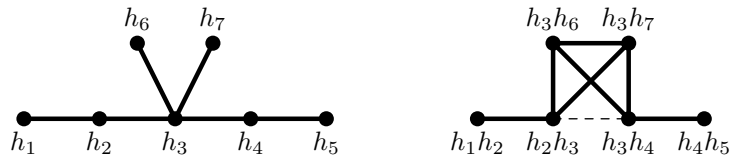


Figure 4: A graph H with more than one edge incident to h_3 and the induced stripe when used in the definition of \mathcal{Z}_3 .

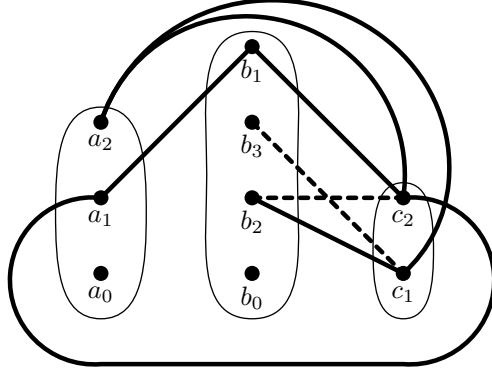


Figure 5: The stripe \mathcal{Z}_4 .

to any vertices in $L_3 \cup L_4$ ($L_2 \cup L_4$ respectively $L_2 \cup L_3$), then v is an edge incident with a pendant vertex of H . If $v \in L_2$ is adjacent to a vertex $u \in L_3$ and a vertex $w \in L_4$, then u and w must be adjacent, and u, v, w correspond to edges incident with a vertex of degree three.

Now consider all vertices of L_2, L_3, L_4 not characterized so far. We can partition each set L_i into disjoint cliques as follows. For each $1 \leq i < j \leq 3$, construct a bipartite graph G_{ij} , where $V(G_{ij})$ consists of the vertices of L_i and L_j and $E(G_{ij})$ consists of all edges between L_i and L_j in J . For each connected component of G_{23} and G_{24} , group the vertices of L_2 in this connected component. This induces a partition of L_2 . Do the same for L_3 and L_4 . We say that two groups $C \in L_i, C' \in L_j$ with $i \neq j$ are related if a vertex of C is adjacent to a vertex of C' . By the construction of the groups, each group will be related to exactly one other group. These pairs correspond exactly to edges of H incident with a vertex of degree two. \square

Proof of Lemma 5.6: Recall that $Z = \{a_0, b_0\}$. Moreover, $|N(a_0)| = 2$, while $|N(b_0)| \geq 3$, as J does not admit twins. Hence we can assume we know a_0 and b_0 . Let u, v denote the neighbors of a_0 . Then $u = a_1$ and $v = a_2$, or vice versa. By definition, $X_{c_2} = (N(u) \cap N(v)) \setminus \{a_0\}$. Similarly, $\{b_1\} = N(b_0) \cap (N(u) \cup N(v))$ and $\{a_1\} = N(b_1) \setminus (N[b_0] \cup X_{c_2})$. Then we have also identified a_2 . It follows that $X_{c_1} = N(a_2) \setminus (\{a_0, a_1\} \cup X_{c_2})$. Now let X_{b_2} be the subset of the remaining vertices that are complete to X_{c_1} and put the other vertices in X_{b_3} . If X_{b_3} happens to be strongly anticomplete to X_{c_1} , there must be a vertex we put into X_{b_2} that is strongly anticomplete to X_{c_2} . Add this vertex to the set X_{b_3} . See Figure 5.

If the sets we identified are not pairwise disjoint, or not all nonempty, or not all cliques, then (J, Z) is not a thickening of a member of \mathcal{Z}_4 . Otherwise, it is easy checked if (J, Z) is a thickening of a member of \mathcal{Z}_4 . \square

Proof of Lemma 5.8: Suppose that (J, Z) is indeed a thickening of $(J', Z') \in \mathcal{Z}_5$ and let v_1, \dots, v_{13}, X be as in the definition of \mathcal{Z}_5 . Note that v_{13} is (strongly) adjacent to both v_7 and v_8 . Since $Z' = \{v_7, v_8\}$ and no vertex of a stripe can be adjacent to more than one vertex of Z' , $v_{13} \in X$. There is also quite a bit of symmetry in the labeling of $V(J')$. In fact, we can freely swap the labels of v_7 and v_8 or of v_{11} and v_{12} , by appropriately relabeling the other vertices of J' (i.e. relabeling those vertices with label $1, \dots, 6, 9, 10$). See Figure 6.

It follows that if $Z = \{z_1, z_2\}$, we can assume w.l.o.g. that $v_7 = z_1$ and that $v_8 = z_2$. Then $N(v_7) = \{v_1, v_2\}$ and $N(v_8) = \{v_4, v_5\}$. Observe that v_9 is the only vertex adjacent to both v_1 and v_2 . Hence $X_{v_9} = (N(v_1) \cap N(v_2)) \setminus \{v_7\}$. Note that X_{v_9} can be found in this way even though we do not yet know exactly which vertex of $N(v_7)$ corresponds to v_1 and which to v_2 . Similarly,

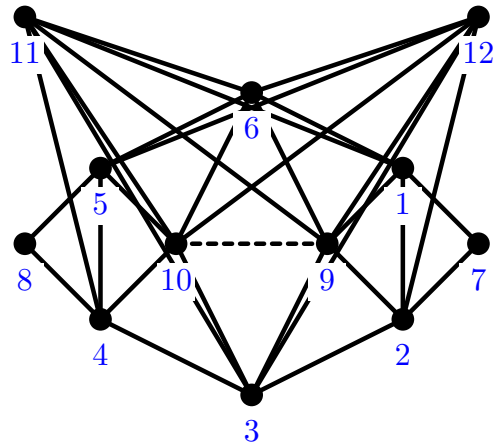
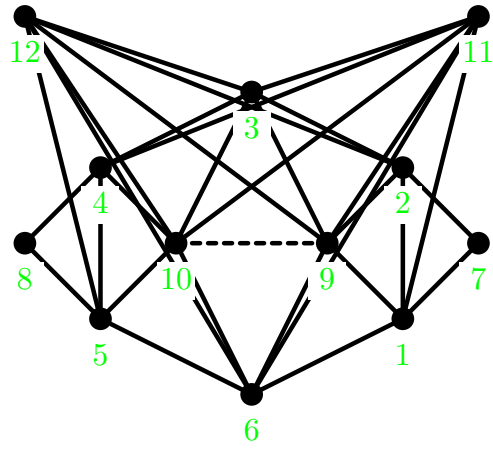
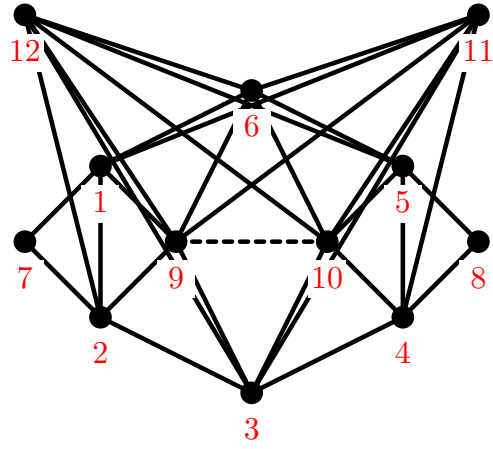


Figure 6: The stripe in \mathcal{Z}_5 with $|Z| = 2$ with three possible labelings.

$X_{v_{10}} = (N(v_4) \cap N(v_5)) \setminus \{v_8\}$. Then either $|X_{v_9}| = |X_{v_{10}}| = 1$ and X_{v_9} and $X_{v_{10}}$ are strongly anticomplete to each other, or $|X_{v_9}|, |X_{v_{10}}| \geq 1$ and X_{v_9} and $X_{v_{10}}$ are neither strongly complete nor strongly anticomplete to each other.

Now X_{v_9} and $X_{v_{10}}$ share exactly two neighbors, namely v_3 and v_6 , i.e. $\{v_3, v_6\} = (N(X_{v_9}) \setminus X_{v_{10}}) \cap (N(X_{v_{10}}) \setminus X_{v_9})$. Then J has at most two vertices that were not previously considered. These are v_{11} and v_{12} . We assign these labels arbitrarily. Following our earlier observation, we can now fix the labels of v_1, \dots, v_6 . \square