Well-Quasi-Ordering Bounded Treewidth Graphs

Michael R. Fellows[1], Danny Hermelin ⋆[2], and Frances A. Rosamond[1]

[1] School of Electrical Engineering and Computer Science,
The University of Newcastle, Calaghan NSW 2308 - Australia
{mike.fellows,frances.rosamond}@cs.newcastle.edu.au
[2] Department of Computer Science, University of Haifa,
Mount Carmel, Haifa 31905 - Israel
danny@cri.haifa.ac.il

Abstract. We show that three subclasses of bounded treewidth graphs are well-quasi-ordered by refinements of the minor order. Specifically, we prove that graphs with bounded feedback-vertex-set are well-quasi-ordered by the topological-minor order, graphs with bounded vertex-covers are well-quasi-ordered by the subgraph order, and graphs with bounded circumference are well-quasi-ordered by the induced-minor order. Our results give an algorithm for recognizing any graph family in these classes which is closed under the corresponding minor order refinement.

1 Introduction

In the framework of parameterized complexity [7, 9], one of the most commonly used parameter is *treewidth* which roughly measures the degree of similarity between the input graph and a tree. The reason for this is that many natural graph problems turn out to be fixed-parameter tractable when the treewidth is taken as a parameter. Indeed, by now many algorithmic methodologies and systematic frameworks are well-established for this parameter [1–4]. The most notable of these is due to Courcelle's Theorem [4] which states that any problem expressible by a monadic second order formula ϕ can be solved in $f(k + |\phi|) \cdot n$ time for graphs of treewidth at most k. Due to the expressibility power of monadic second order logic, Courcelle's Theorem gives a single algorithm to a vast multitude of different combinatorial problems.

Despite the fact that many problems become fixed-parameter tractable when parameterized by treewidth, there are still quite a few problems that are impregnable by any one of its algorithmic methodologies. For instance, vertex layout problems such as *bandwidth*, *cutwidth*, or *linear arrangement* have no known fixed-parameter algorithm when parameterized by treewidth. There is thus room for more methodologies, perhaps by imposing more structure on the input than bounded treewidth. In [8], it was suggested that the *finite forbidden characterization method* can be a very prominent tool towards this aim, and exemplified

⋆ Supported by the Adams Fellowship of the Israel Academy of Sciences and Humanities.

this on the class of bounded *max-leaf* graphs, a subclass of bounded treewidth graphs. In this paper, we further develop this line by applying the finite forbidden characterization method to more subclasses of bounded treewidth graphs.

Before proceeding, let us briefly review the fundamentals behind the finite forbidden characterization method. Probably the best known application of this technique is the astonishing result implied by Robertson and Seymour graph minor project: For any graph family $\mathcal{G}$ closed under minors, there is an $O(n^3)$-time algorithm for deciding whether a given graph G belongs to $\mathcal{G}$, the so-called *recognition problem* for $\mathcal{G}$. There are two main ingredients behind Robertson and Seymour's remarkable result, which are intertwined together in the their massive project:

1. The universe $\mathcal{U}$ of all graphs is *well-quasi-ordered* under minors.
2. Minor *order testing* can be performed in $f(k) \cdot n^3$ time in $\mathcal{U}$.

The first ingredient, along with the minor-closure of $\mathcal{G}$, ensures that $\mathcal{G}$ has a *finite forbidden characterization*, a finite set of graphs $\mathrm{Forb}(\mathcal{G})$ such that any graph G belongs to $\mathcal{G}$ iff no graph in $\mathrm{Forb}(\mathcal{G})$ is a minor of G. The second ingredient says that we can test whether a "small" k-vertex graph is a minor of a "large" n-vertex graph. Thus, assuming both ingredients, an algorithm can have $\mathrm{Forb}(\mathcal{G})$ "hardwired" into it in order to determine whether $G \in \mathcal{G}$, given an input graph G. The running time of such an algorithm will be $O(n^3)$, since the size of $\mathrm{Forb}(\mathcal{G})$, and the sizes of the graphs in $\mathrm{Forb}(\mathcal{G})$, are constant with respect to the size of an input graph G.

The minor order on graphs generalizes graph orders such as the induced-minor, topological-minor, and subgraph orders. However, none these familiar refinements define well-quasi-orders on the universe of all graphs, and not even on the restrictive universe of bounded treewidth graphs [14]. Nevertheless, in bounded treewidth graphs we can perform efficient $f(k) \cdot n$ order testing for all these orders by using Courcelle's Theorem, since order testing for all these is expressible in monadic second order logic. Thus, if any sub-universe in the universe of bounded treewidth graphs turns out to be well-quasi-ordered by one of these orders, we will have a recognition algorithm for any graph family closed under this order. In [8], it was shown that the universe of bounded max-leaf graphs is well-quasi-ordered by topological-minor order, which implies any graph family closed under topological-minors with max-leaf bounded by k can be recognized in $f(k) \cdot n$ time.

In this paper we will show that other sub-universes of bounded treewidth graphs are well quasi ordered by one of the minor order refinements mentioned above. To be more concrete, consider the hierarchal view on some subclasses of bounded treewidth graphs given in Fig. 1, each defined by bounding a specific structural parameter (arrows indicate the direction of inclusion). Thus, **vc** in the figure, for instance, denotes the universe of graphs $\{G : vc(G) \leq k\}$ with vertex-covers of sizes bounded by some natural k, which is included in **circ**, the universe of graphs $\{G : circ(G) \leq k'\}$ with circumference bounded by some possibly different $k' \in \mathbb{N}$ (readers unfamiliar with one of these parameters are referred to [7]). We will show that:

tw = treewidth
circ = circumference
pw = pathwidth
fvs = feedback-vertex-set
mp = max-path
vc = vertex-cover
ml = max-leaf

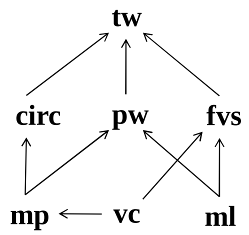

Fig. 1. A hierarchal view of bounded treewidth graphs.

1. **fvs** is well-quasi-ordered by topological minors.
2. **vc** is well-quasi-ordered by subgraphs.
3. **circ** is well-quasi-ordered by induced minors.

We remark that the second item on the list above was already shown indirectly by Ding [6], who proved that **mp** is well-quasi-ordered by induced subgraphs.

Our results are obtained by what we call *reduction theorems*, which allow reducing the question of whether a given universe is well-quasi-ordered by a particular order, to the question of whether a simpler sub-universe is well-quasi-ordered by some refinement on that order. The first of these is suited for universes consisting of graphs which have a small subset of vertices whose removal leaves a very simple structured graph, *e.g.* graphs with bounded vertex-cover or bounded feedback-vertex-set. The second reduction theorem reduces the above question to the question of whether the 2-connected components (*blocks*) of some universe are well-quasi-ordered by induced minors. This can be thought of as an extension to a much more fundamental reduction theorem due to Higman (see Section 2), and requires more involved arguments in comparison to the first reduction theorem. Using our second reduction theorem, we show that graphs of bounded circumference are well-quasi-ordered by induced minors.

The remainder of the paper is devoted to proving these two reduction theorems and discussing their implications. In the next section, we briefly review some preliminary notions that will be used throughout the paper. In particular, we review basic well quasi order terminology, and discuss all graph orders that will be dealt with in the paper. Section 3 is devoted to the first reduction theorem, and Section 4 is devoted to the second. Readers unfamiliar with most of the notions discussed above are encouraged to look in [5, 7], in the relevant chapters.

2 Preliminaries

We next introduce basic notation and terminology that will be used throughout the paper. All graphs in this paper are finite, simple, and undirected. As usual, we denote the vertex-set and edge-set of a given graph G by $V(G)$ and $E(G)$ respectively. For a graph G and a subset of vertices $V \subseteq V(G)$, we let $G - V$

denote the graph obtained by deleting all vertices of V from G. By a universe $\mathcal{U}$, we mean an infinite set of graphs which is closed under vertex deletions, *i.e.* $G \in \mathcal{U} \Longrightarrow G - V \in \mathcal{U}$ for all $V \subseteq V(G)$.

Given a set X, a *quasi-order* $\preceq$ on X is a reflexive transitive subset of $X \times X$. A quasi-order is *well-founded* if it contains no infinite strictly descending sequences, *i.e.* sequences of the form $x_1 \succ x_2 \succ x_3 \cdots$. A good sequence of $\preceq$ is an infinite sequence $x_1, x_2, x_3 \cdots$ which contains a *good pair*, a pair (x_i, x_j) with $x_i \preceq x_j$ and $i < j$. A bad sequence is an infinite sequence which is not good. A *well-quasi-order* (wqo) on X is a well-founded quasi-order with no *bad sequences*. Equivalently, a well-quasi-order is a well-founded quasi-order with no infinite antichain. For a pair of quasi orders $\preceq_1$ and $\preceq_2$ on X, we let $\preceq_1 \cdot \preceq_2$ denote the quasi order on $X \times X$ defined by $(x_1, y_1) \preceq_1 \cdot \preceq_2 (x_2, y_2)$ iff $x_1 \preceq_1 y_1$ and $x_2 \preceq_2 y_2$.

Our interest in well quasi orders on graph universes stems from the fact that they imply that all closed subsets of the universe have finite forbidden characterizations w.r.t. the particular order. By a subset of a universe $\mathcal{U}$ *closed* under a quasi order $\preceq$, we mean a subset $\mathcal{G} \subseteq \mathcal{U}$ with $H \in \mathcal{G}$ whenever $G \in \mathcal{G}$ and $H \preceq G$. A forbidden characterization of a subset $\mathcal{G} \subseteq \mathcal{U}$ w.r.t. $\preceq$ is a set $Forb(\mathcal{G}) \subseteq \mathcal{U}$ where for every graph G we have $G \in \mathcal{G}$ iff $H \not\preceq G$ for all $H \in Forb(\mathcal{G})$. It is easy to see that if $\mathcal{G}$ is closed, then $\mathcal{U} \setminus \mathcal{G}$ is a forbidden characterization of $\mathcal{G}$, where the set of all $\preceq$-minimal graphs of $\mathcal{U} \setminus \mathcal{G}$ being the unique minimal characterization. Now if $\preceq$ is a wqo on $\mathcal{U}$, then this set must be finite since it constitutes an anti-chain w.r.t. $\preceq$.

There are three well-known graph embeddings that induce quasi orders on the universe of all graphs. Given two graphs H and G:

– A *subgraph embedding* of H onto G is an injection $f : V(H) \to V(G)$ with $\{u, v\} \in E(H) \Rightarrow \{f(u), f(v)\} \in E(G)$.
– A *topological-minor embedding* of H onto G is an injection $f : V(H) \to V(G)$ where there exist vertex disjoint paths in G between $f(u)$ and $f(v)$ for every $\{u, v\} \in E(H)$.
– An *induced-minor embedding* of H onto G is a injective mapping $f : V(H) \to 2^{V(G)}$ with $f(v)$ connected in G for all $v \in V(H)$, $f(u) \cap f(v) = \emptyset$ for all $u \neq v \in V(H)$, and $\{u, v\} \in E(H) \iff \exists x \in f(u)$ and $\exists y \in f(v)$ with $\{x, y\} \in E(G)$.

We write $H \subseteq G$ (resp. $H \trianglelefteq G$, $H \sqsubseteq G$) if there exists a subgraph (resp. topological-minor, induced-minor) embedding of H onto G. We will also use $H \cong G$ to denote that H and G are *isomorphic*, *i.e.* there are subgraph embeddings of H onto G and G onto H. Note that $H \subseteq G$ implies $H \trianglelefteq G$, but $H \trianglelefteq G$ does not necessarily imply $H \sqsubseteq G$.

Let $\mathcal{U}$ be some graph universe. A *labeling* of $\mathcal{U}$ is a set $\{\sigma_G : G \in \mathcal{U}\}$, where each σ_G is a *labeling* of the vertices of G by a set of labels Σ_G, *i.e.* $\sigma : V(G) \to \Sigma_G$. The set $\Sigma = \bigcup_{G \in \mathcal{U}} \Sigma_G$ is the set of labels assigned by σ to $\mathcal{U}$. If Σ is wqo by some quasi order $\preceq$, we say that σ is a *wqo labeling w.r.t* $\preceq$. If Σ is finite, then it is wqo by equality, and σ is a wqo labeling w.r.t. $=$.

Finally, for a pair of labelings $\sigma_G^{(1)}$ and $\sigma_G^{(2)}$ of a graph G, we let $\sigma_G^{(1)} \times \sigma_G^{(2)}$ denote the labeling which assigns the label $(\sigma_G^{(1)}(v), \sigma_G^{(2)}(v))$ to each $v \in V(G)$. If $\sigma^{(1)} = \{\sigma_G^{(1)} : G \in \mathcal{U}\}$ and $\sigma^{(2)} = \{\sigma_G^{(2)} : G \in \mathcal{U}\}$ are wqo labelings of a universe $\mathcal{U}$ w.r.t. $\preceq^{(1)}$ and $\preceq^{(2)}$ respectively, then the labeling $\{\sigma_G^{(1)} \times \sigma_G^{(2)} : G \in \mathcal{U}\}$ is a wqo labeling of $\mathcal{U}$ w.r.t. $\preceq^{(1)} \cdot \preceq^{(2)}$ as can be seen by Higman's Lemma (see below).

Well-quasi-ordered labelings of $\mathcal{U}$ allow us to refine the subgraph, topological minor, and induced minor orders on $\mathcal{U}$ in a natural manner. Given a wqo labeling $\sigma = \{\sigma_G : G \in \mathcal{U}\}$ w.r.t $\preceq$, and a pair of graphs H and G in $\mathcal{U}$, we will write $H \subseteq_\sigma G$ (resp. $H \trianglelefteq_\sigma G$) if there is a subgraph (topological-minor) embedding of H onto G with $\sigma_H(v) \preceq \sigma_G(f(v))$ for all $v \in V(H)$. We write $H \cong_\sigma G$ whenever $H \subseteq_\sigma G$ and $H \subseteq_\sigma G$. Also, we extend this definition to the induced-minor order, and write $H \sqsubseteq_\sigma G$ whenever there exists an induced-minor embedding of H onto G where for each $v \in V(H)$ there is some $x \in f(v)$ with $\sigma(v) \preceq \sigma(x)$.

Observe that while $\subseteq$, $\trianglelefteq$, and $\sqsubseteq$ are all quasi orders on the universe of all graphs, none of these is a wqo even on the very restrictive sub-universe of bounded treewidth (or pathwidth) graphs [14]. Nevertheless, there are other even more restrictive sub-universes for which some of these orders become wqo. The following important classical example is due to Kruskal:

Theorem 1 (Kruskal's Labeled Forests Theorem [13]). *The universe of all forests is wqo by $\trianglelefteq_\sigma$ for any wqo labeling σ.*

Kruskal Theorem generalizes an even more fundamental result in the context of well quasi orderings which is known as Higman's Lemma. In graph-theoretic terms, Higman's Lemma can be stated as follows:

Lemma 1 (Higman's Lemma [11]). *If the set of all connected components in some universe $\mathcal{U}$ is wqo by any order $\preceq$, then $\mathcal{U}$ itself is wqo by $\preceq$.*

3 First Reduction Theorem

In this section we state and prove the first of our two reduction theorems. Using this theorem, we will show that the universe of bounded feedback-vertex-set graphs is wqo by topological minors, and that the universe of bounded vertex-cover graphs is wqo by subgraphs. The following notion of k-closures of universes is central in this context:

Definition 1 (k-closure). *Given a natural k, the k-closure of a graph universe $\mathcal{U}$, denoted $\mathcal{U}_k$, is the universe of all graphs G which have a subset of k vertices V with $G - V \in \mathcal{U}$.*

The main idea behind the first reduction theorem is simple: We reduce the question of whether some universe $\mathcal{U}_k$ is wqo by a particular order, to the question of whether the sub-universe $\mathcal{U}$ is wqo under a refinement of that order. This refinement is obtained using wqo labelings on $\mathcal{U}$.

Theorem 2 (First Reduction Theorem). *If a universe $\mathcal{U}$ is wqo by $\subseteq_\sigma$ (resp. $\trianglelefteq_\sigma$) for any finite labeling σ, then $\mathcal{U}_k$ is wqo under $\subseteq$ (resp. $\trianglelefteq$).*

Proof. We prove the theorem for the $\subseteq$ order, the proof for $\trianglelefteq_\sigma$ follows from the same arguments. Let $\{G_i\}_{i=1}^\infty$ be any infinite sequence in $\mathcal{U}_k$. By definition, each graph G_i in this sequence has a subset of k vertices U_i with $G_i - U_i \in \mathcal{U}$. Let V_i denote the subset of vertices $V(G_i) \setminus U_i$. We construct a labeling $\sigma = \{\sigma_i : i \in \mathbb{N}\}$ on $\{G_i : i \in \mathbb{N}\}$ in a way that codifies the adjacency of vertices in U_i with vertices of V_i, for each $i \in \mathbb{N}$. For this, σ_i first assigns each vertex $u \in U_i$ an arbitrary distinct label $\sigma_i(u) \in \{1, \ldots, k\}$, and then it assigns a label in $2^{\{1,\ldots,k\}}$ to each $v \in V_i$ by

$$\sigma_i(v) := \{\, x \quad : \quad \exists u \in U_i \text{ with } \{u, v\} \in E(G_i) \text{ and } \sigma_i(u) = x \,\}$$

(see Fig. 2 for an example). Observe that since the set of labels Σ assigned by σ is finite, it is wqo by equality, and σ is a wqo labeling on $\mathcal{U}$ with respect to $=$.

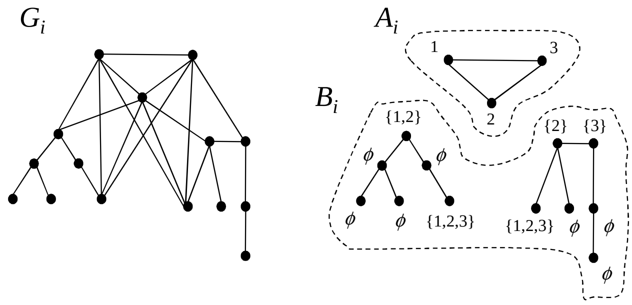

Fig. 2. An example of the labeling used in the proof of the First Reduction Theorem.

Now, for each $i \in \mathbb{N}$, let A_i denote the graph $G_i - V_i$, and let B_i denote $G_i - U_i$. Then $B_i \in \mathcal{U}$ for all $i \in \mathbb{N}$. Since there are only finitely many graphs A_i under isomorphism, and only finitely many ways to label the vertices of these graphs with distinct labels in $\{1, \ldots, k\}$, there must be an infinite subsequence $G_{i_1}, G_{i_2}, \ldots$ in $\{G_i\}_{i=1}^\infty$ with $A_{i_1} \cong_\sigma A_{i_2} \cong_\sigma \cdots$. By our assumption, the family of graphs $\{B_{i_j} : j \in \mathbb{N}\}$ is wqo by $\subseteq_\sigma$, and as this set is infinite, there must be a pair B_{i_x} and B_{i_y}, $x < y$, with $B_{i_x} \subseteq_\sigma B_{i_y}$. Write $i = i_x$ and $j = i_y$. We have the following easy claim:

Claim 2.1. $A_i \cong_\sigma A_j$ and $B_i \subseteq_\sigma B_j$ implies $G_i \subseteq G_j$.

Proof (of Claim 2.1.). Let f_A denote the isomorphic embedding showing that $A_i \cong_\sigma A_j$, and let f_B denote the isomorphic embedding of B_i onto B_j. We argue that the mapping $g = f_A \cup f_B$ is an isomorphic embedding of G_i onto G_j. Clearly, for all edges $\{u, v\} \in E(G_i)$ with either $u, v \in A_i$ or $u, v \in B_i$, we have

$\{g(u), g(v)\} \in E(G_i)$ by our assumptions on f_A and f_B. For $\{u, v\} \in E(G_i)$ with $u \in U_i$ and $v \in v_i$, we have $\sigma_i(u) = \sigma_j(g(u))$ and $\sigma_i(v) = \sigma_j(g(v))$. Thus, by construction of σ, we get $\{u, v\} \in E(G_i) \implies \sigma_i(u) \in \sigma_i(v) \implies \sigma_j(g(u)) \in \sigma_j(g(v)) \implies \{g(u), g(v)\} \in E(G_j)$. □

It follows that $\{G_i\}_{i=1}^{\infty}$ is a good sequence, and as this sequence was chosen arbitrarily, this implies that $\mathcal{U}_k$ does not contain any bad sequences. □

Thus, according to the First Reduction Theorem, in order to determine whether a given universe $\mathcal{U}_k$ is wqo by $\preceq$, where $\preceq \in \{\subseteq, \trianglelefteq\}$, it suffices to determine whether the sub-universe $\mathcal{U}$ is wqo under $\preceq_\sigma$ for any finite labeling σ. We have already seen an example in Kruskal's Labeled Forests Theorem for of a universe that is wqo under $\trianglelefteq_\sigma$ for any finite labeling σ, the universe of all forests. If we let $\mathcal{U}$ denote this universe, then the k-closure of $\mathcal{U}$ is the family of graphs G with $fvs(G) \leq k$. We therefore have:

Corollary 1. *For any $k \in \mathbb{N}$, the universe $\{G : fvs(G) \leq k\}$ is wqo by $\trianglelefteq$.*

Let us now consider graphs of bounded vertex cover, *i.e.* the universe $\{G : vc(G) \leq k\}$ for some natural k. Then if $\mathcal{U}$ is the family of all graphs with no edges, then $\mathcal{U}_k = \{G : vc(G) \leq k\}$. Thus, to show that $\{G : vc(G) \leq k\}$ is wqo by $\subseteq$, we need to show that $\mathcal{U}$ is wqo by $\subseteq_\sigma$ for any wqo labeling σ. But this is immediate from Higman's Lemma. Observe that each connected component of a graph in $\mathcal{U}$ consists of a single vertex. Therefore the set of all connected components of $\mathcal{U}$ is clearly wqo by $\subseteq_\sigma$ for any wqo labeling σ. Hence, by Higman's Lemma we get that $\mathcal{U}$ is wqo by $\subseteq_\sigma$. Applying, the First Reduction Theorem, we thus have:

Corollary 2. *For any $k \in \mathbb{N}$, the universe $\{G : vc(G) \leq k\}$ is wqo by $\subseteq$.*

To exemplify the strength of the First reduction Theorem, we end this section by briefly discussing two more sample applications for it. Consider the following structural parameters for a given graph G:

- $cd(G)$ – the smallest number of vertices necessary to remove from G in order to obtain a *cluster graph*, a graph where each connected component forms a clique (*cf.* [10, 12]). Observe that cluster graphs do not form a subclass of bounded treewidth graphs.
- $cyc(G)$ – the smallest number of vertices necessary to remove from G in order to obtain a 2-regular graph.

Let $\mathcal{U}$ denote the universe of all cluster graphs, and let $\mathcal{V}$ denote the universe of all 2-regular graphs. Then $\mathcal{U}_k = \{G : cd(G) \leq k\}$ and $\mathcal{V}_k = \{G : cyc(G) \leq k\}$. It is not difficult to show using similar arguments above that $\mathcal{U}$ is wqo by $\subseteq_\sigma$ for any wqo labeling of $\mathcal{U}$, and that $\mathcal{V}$ is wqo by $\subseteq_\sigma$ and $\trianglelefteq_\sigma$ for any wqo labeling of $\mathcal{V}$. This involves first considering the universe of complete graphs and the universe of cycles, and then applying Higman's Lemma. Thus:

Corollary 3. *For any $k \in \mathbb{N}$, the universe $\{G : cd(G) \leq k\}$ is wqo by $\subseteq$.*

Corollary 4. *For any $k \in \mathbb{N}$, the universe $\{G : cyc(G) \leq k\}$ is wqo by $\trianglelefteq$.*

4 Second Reduction Theorem

In this section we present our second reduction technique, which we will use to show that the universe $\mathcal{U} = \{G : circ(G) \leq k\}$ is wqo under induced minors, for every natural k. Note that bounded treewidth graphs in general are not wqo by induced minors [14].

Our second reduction theorem is concerned with 2-connected graphs. In general, a connected graph G is called *c-connected* if G has at least c vertices and no removal of less than c vertices leaves G disconnected. Thus, every connected graph is 1-connected in this terminology. A *block* in G is a maximal 2-connected component of G. Thus a block is maximal connected subgraph of G which does not include any *cutvertex*, a vertex whose removal disconnects G. Note that a block A of some graph G may (and will if G is not 2-connected) contain cutvertices of G, but these will not be cutvertices in A. Thus, if C is the set of cutvertices of G which are included in A, then each connected component in $G - (V(A) \setminus C)$ includes exactly one of the vertices in C.

Theorem 3 (Second Reduction Theorem). *If the subset of all 2-connected graphs in some universe $\mathcal{U}$ is wqo by $\sqsubseteq_\sigma$ for any wqo labeling σ, then $\mathcal{U}$ itself is wqo by $\sqsubseteq$.*

Before we prove Theorem 3, we introduce the notions of rooted graphs and rooted closures of graph universes. A *rooted graph* is a pair (G, v) where G is a graph and v is a single distinguished vertex v of G referred to as its *root*. Thus two rooted graphs with the same vertex and edge set, but with different roots, are considered different. Apart from the following definition, we will omit the parentheses notation and simply state that G is a rooted graph with $root(G) = v$.

Definition 2 (Rooted Closure). *The rooted closure of a universe $\mathcal{U}$, denoted $\mathcal{U}_r$, is defined as the universe of rooted graphs $\mathcal{U}_r = \{(G, v) : G \in \mathcal{U}, v \in V(G)\}$.*

We say that a minor-embedding of a rooted graph H onto a rooted graph G *preserves roots* if $root(G) \in f(root(H))$, and we will write $H \sqsubseteq G$ (and say that H is an induced minor of G) only when there exists a root-preserving minor-embedding of H onto G. Our main interest in rooted graphs lies in the above refinement of minor embeddings, and in the obvious fact that $\mathcal{U}$ is wqo under $\sqsubseteq$ whenever $\mathcal{U}_r$ is wqo under $\sqsubseteq$.

As a final prerequisite of the proof of Theorem 3, we need to define the important notion of minimal bad sequences, a notion first introduced in [11]:

Definition 3 (Minimal Bad Sequence). *A bad sequence $G_1, G_2, \ldots$ is* minimal *if for every bad sequence $H_1, H_2, \ldots$, whenever $|V(H_j)| < |V(G_j)|$ for some j, there is always some $i < j$ such that $|V(G_i)| < |V(H_i)|$.*

Thus, a minimal bad sequence can be "constructed" by selecting a graph G_1 that has the smallest number of vertices among all graphs that begin bad sequences. Next, we select a graph G_2 that the smallest number of vertices among all graphs which appear second in bad sequences beginning with graphs of size $|V(G_1)|$, and so forth.

Proof (of Theorem 3). Due to Higman's Lemma, we can simplify our arguments and assume w.l.o.g. that all graphs in $\mathcal{U}$ are connected. The proof is by contradiction. Suppose that all requirements of the theorem hold, and that $\mathcal{U}$ is not wqo by $\sqsubseteq$. We will arrive at a contradiction by showing that $\mathcal{U}_r$, the rooted-closure of $\mathcal{U}$ is wqo by $\sqsubseteq$.

For this, let $G_1, G_2, \ldots$ be a minimal bad sequence in $\mathcal{U}_r$. For each $i \in \mathbb{N}$, select a block A_i in G_i which contains $root(G_i)$, and let C_i denote the set of cutvertices of G_i that are included in A_i. For each cutvertex $c \in C_i$, let B^i_c denote the connected component in $G_i - (V(A_i) \setminus C_i)$ including the vertex c and made into a rooted graph by setting $root(B^i_c) = c$ (see Fig. 3). Observe that for any $c \in C_i$, we have $B^i_c \sqsubseteq G_i$ by the induced-minor root-preserving embedding f that maps every non-root vertex $v \neq c$ of B^i_c to itself, and has $f(c) = A_i \ni root(G_i)$.

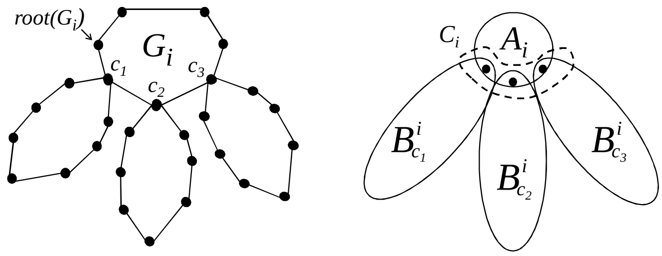

Fig. 3. A graphical example of the notation used in the proof of the Second Reduction Theorem.

Claim 3.1. The family of rooted graphs $\mathcal{B} = \{B^i_c : c \in C_i,\, i \in \mathbb{N}\}$ is wqo by $\sqsubseteq$.

Proof (of Claim 3.1.). Let $\{H_j\}_{j=1}^{\infty}$ be any sequence in $\mathcal{B}$, and for every $j \in \mathbb{N}$, choose an $i(j)$ for which $H_j = B^i_c$ for some $c \in C_i$. Pick a j with smallest $i(j)$, and consider the sequence

$$G_1, \ldots, G_{i(j)-1}, H_j, H_{j+1}, \ldots$$

Then this sequence is good by the minimality of $\{G_i\}_{i=1}^{\infty}$, and by our selection of j, and so it contains a good pair (G, G'). Now, G cannot be among the first $i(j) - 1$ elements of this sequence, since otherwise $G' = H_{j'}$ for some $j' \geq j$, and we will have

$$G \sqsubseteq G' = H_{j'} = B^{i(j')}_c \sqsubseteq G_{i(j')},$$

implying that $(G, G_{i(j')})$ is a good pair in the bad sequence $\{G_i\}_{i=1}^{\infty}$. Thus, (G, G') must be a good pair in $\{H_j\}_{j=1}^{\infty}$, and so $\{H_j\}_{j=1}^{\infty}$ is good. □

We next use the claim above to show that $\{G_i\}_{i=1}^{\infty}$ has a good pair, bringing us to our desired contradiction. For this, we will label the graph family $\mathcal{A} = \{A_i :$

$i \in \mathbb{N}\}$ so that each cutvertex c of a graph A_i gets labeled by their corresponding connected component B^i_c of G_i, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling $\sigma_i = \sigma_i^{(1)} \times \sigma_i^{(2)}$, where $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are two labelings defined by:

– $\sigma_i^{(1)}(v) = 1$ if $v = root(G_i)$, and otherwise $\sigma^{(1)}(v) = 0$.
– $\sigma_i^{(2)}(v) = B^i_v$ if $v \in C_i$, and otherwise $\sigma_i^{(2)}(v) = 0$.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. Now let $\preceq$ denote the quasi order $= \cdot \sqsubseteq$ defined on the set of labels Σ assigned by σ. Then $\preceq$ is a wqo on Σ, and so σ is wqo labeling on $\mathcal{A}$ w.r.t. σ. By the assumptions in the theorem, we know that $\mathcal{A}$ is wqo by $\sqsubseteq_\sigma$. It follows that there is a pair of graphs $A_i, A_j \in \mathcal{A}$ with $A_i \sqsubseteq_\sigma A_j$. To complete the proof we will show that:

Claim 3.2. $A_i \sqsubseteq_\sigma A_j \Rightarrow G_i \sqsubseteq G_j$.

Proof (of Claim 3.2.). Let f be the induced-minor embedding of A_i onto A_j. Then for each cutvertex $c \in C_i$, $f(c)$ contains a vertex $d \in C_j$ with $B^i_c \sqsubseteq B^j_d$. Let f_c denote the induced-minor root-preserving embedding of B^i_c onto B^j_d. We construct an embedding $g : V(G_i) \to 2^{V(G_j)}$ defined by

$$g(v) = \begin{cases} f(v) & : \quad v \text{ is a vertex of } A_i \text{ and } v \notin C_i, \\ f_c(v) & : \quad v \text{ is a vertex of } B^i_c \text{ and } v \neq c, \\ f(v) \cup f_v(v) & : \quad v \in C_i. \end{cases}$$

We argue that g is an induced minor embedding of G_i onto G_j.

To see this, first note that by definitions of f and each f_c, we have $g(u) \cap g(v) = \emptyset$ for any pair of distinct vertices u and v in G_i. Moreover, for any edge $\{u, v\}$ of G_i there is a vertex $x \in g(u)$ and a vertex $y \in g(v)$ with $\{x, y\}$ and edge in G_j. Thus what remains to be shown is that $g(u)$ is connected in G_j for every vertex u of G_i. This is obviously true when $u \notin C_i$, again by the definitions of f and each f_c. If $u \in C_i$, then $f(u)$ contains a vertex $v \in C_j$ for which $B^i_u \sqsubseteq B^j_v$, and v is also contained in $f_v(v)$ since f_v preserves roots. Thus, $g(u)$ is connected also when $u \in C_i$. Noting also that the labeling σ ensures that $root(G_j) \in g(root(G_i))$, we establish that $G_i \sqsubseteq G_j$. □

Thus, the sequence $\{G_i\}_{i=1}^{\infty}$ has a good pair, contradicting the fact that this sequence is a bad sequence in $\mathcal{U}_r$. This is our desired contradiction that completes the proof of the theorem. □

We next deduce from the Second Reduction Theorem that the universe of bounded circumference graphs is wqo by induced minors. For this, first note that any pair of vertices in a 2-connected graph are on some cycle. Thus, all 2-connected graphs with circumference at most k do not contain any paths of length greater than k. We therefore may restrict our attention to connected graphs with no paths of length greater than k, for some fixed $k \in \mathbb{N}$.

We refine our definition of subgraph embedding to induced-subgraph embedding. This is obtained by replacing the one-directional implication in the definition of the former to a bi-directional implication in the latter. That is, a subgraph embedding of a graph H onto a graph G is an injective mapping $f : V(H) \to V(G)$ with $\{u, v\} \in E(H) \iff \{f(u), f(v)\} \in E(G)$. We will write $H \subseteq^* G$ in case there is such an embedding. For a labeling σ, $\subseteq^*_\sigma$ is defined accordingly. Observe that $H \subseteq^*_\sigma G \implies H \sqsubseteq_\sigma G$, for any labeling σ. We have the following lemma due to Ding [6]:

Lemma 2 ([6]). *For any $k \in \mathbb{N}$, the universe $\{G : mp(G) \leq k\}$ is wqo by $\subseteq^*_\sigma$ for any wqo labeling σ.*

Corollary 5. *For any $k \in \mathbb{N}$, the universe $\{G : circ(G) \leq k\}$ is wqo by $\sqsubseteq$.*

References

1. K.R. Abrahamson and M.R. Fellows. Finite automata, bounded treewidth, and well-quasiordering. In *Graph Structure Theory (ed. N. Robertson and P. Seymour)*, pages 539–564, 1993.
2. S. Arnborg, J. Lagergren, and D. Seese. Problems easy for tree-decomposable graphs. *Journal of Algorithms*, 12:308–340, 1991.
3. H.L. Bodlaender. Treewidth: Algorithmic techniques and results. In *Proc. of the 22nd international symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 19–36, 1997.
4. B. Courcelle. The monadic second-order logic of graphs I. Recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.
5. R. Diestel. *Graph Theory*. Springer-Verlag, 2000.
6. G. Ding. Subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 16(5):489–502, 1992.
7. R. Downey and M. Fellows. *Parameterized Complexity*. Springer-Verlag, 1999.
8. M.R. Fellows, D. Lokshtanov, N. Misra, M. Mnich, F. Rosamond, and S. Saurabh. The complexity ecology of parameters: An illustration using bounded max leaf number. In *Proc. of the 3rd conference on Computability in Europe (CiE)*, pages 268–277, 2007.
9. J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer-Verlag, 2006.
10. J. Gramm, J. Guo, F. Hüffner, and R. Niedermeier. Graph-modeled data clustering: Exact algorithms for clique generation. *Theory of Computing Systems*, 38(4):373–392, 2005.
11. G. Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society III*, 2(7):326–336, 1952.
12. F. Hüffner, C. Komusiewicz, H. Moser, and R. Niedermeier. Fixed-parameter algorithms for cluster vertex deletion. In *Proc. of the 8th Latin American Theoretical INformatics (LATIN)*, pages 711–722, 2008.
13. J. B. Kruskal. Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture. *Transactions of the American Mathematical Society*, 95:210–225, 1960.
14. R. Thomas. Well-quasi-ordering infinite graphs with forbidden finite planar minor. *Transactions of the American Mathematical Society*, 312(1):67–76, 1990.