# Meta-Complexity Theorems for Bottom-up Logic Programs

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# Introduction



- logic programming of efficient algorithms
- complexity analysis through general meta-complexity theorems
- guaranteed execution time
- logical aspects of fundamental algorithmic paradigms (dynamic programming, union-find, congruence closure, priority queues)
- application to program analysis:
  type inference system = algorithm
- recent papers:
  McAllester [SAS99], Ganzinger/McAllester [IJCAR01]
- related work: efficient fixpoint iteration by Nielson/Seidl [2001]

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# Contents

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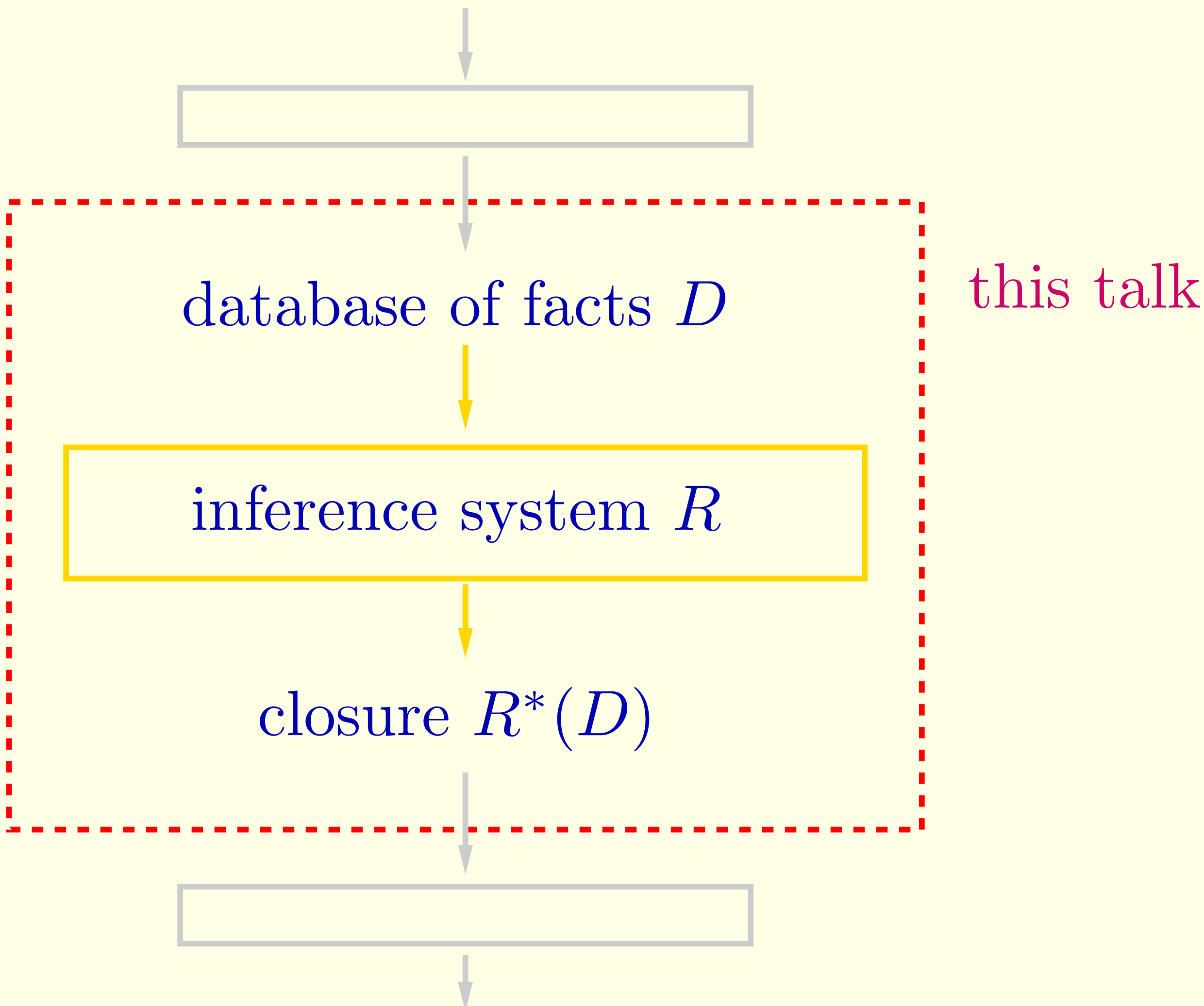
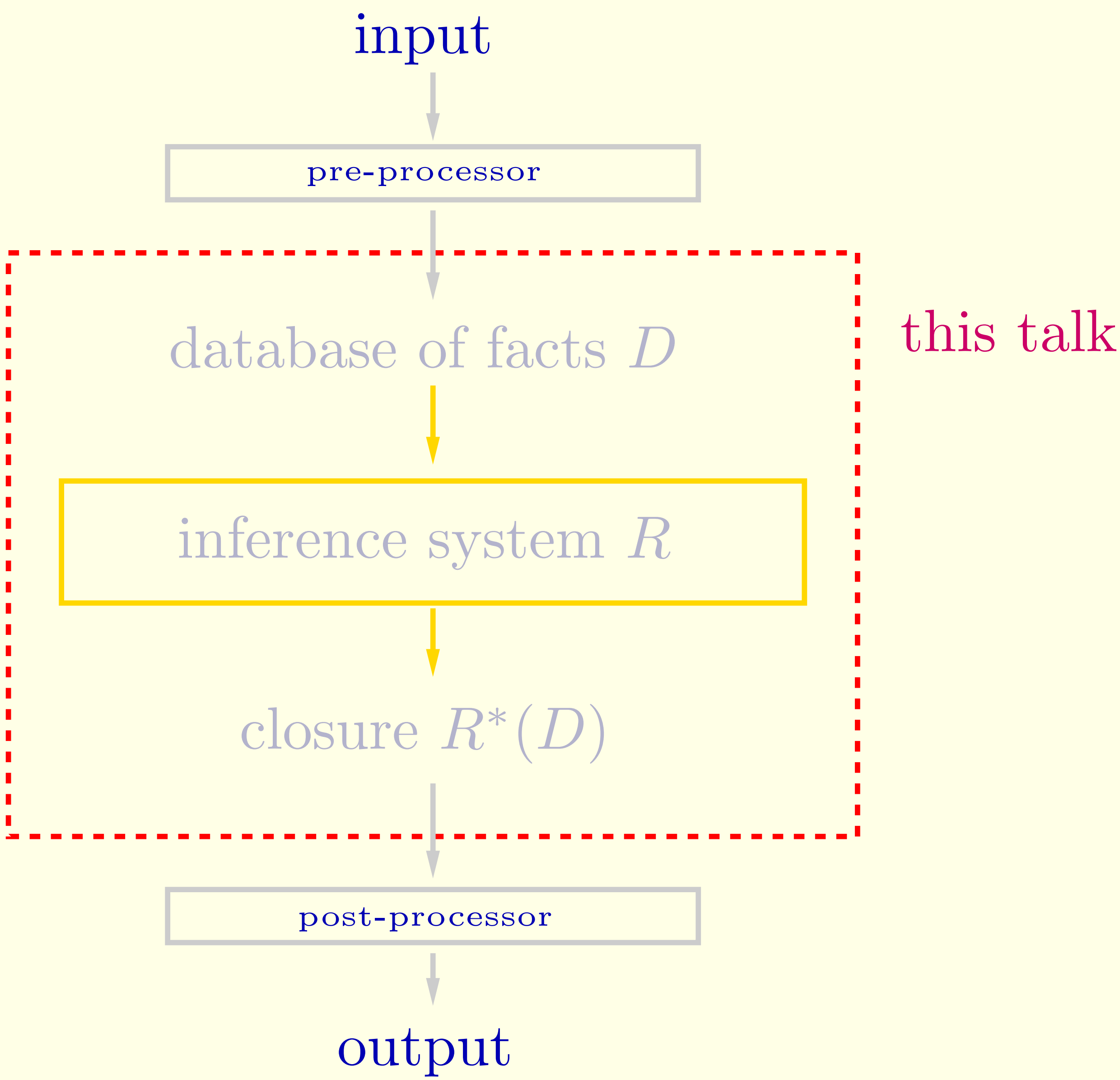
# Paradigm



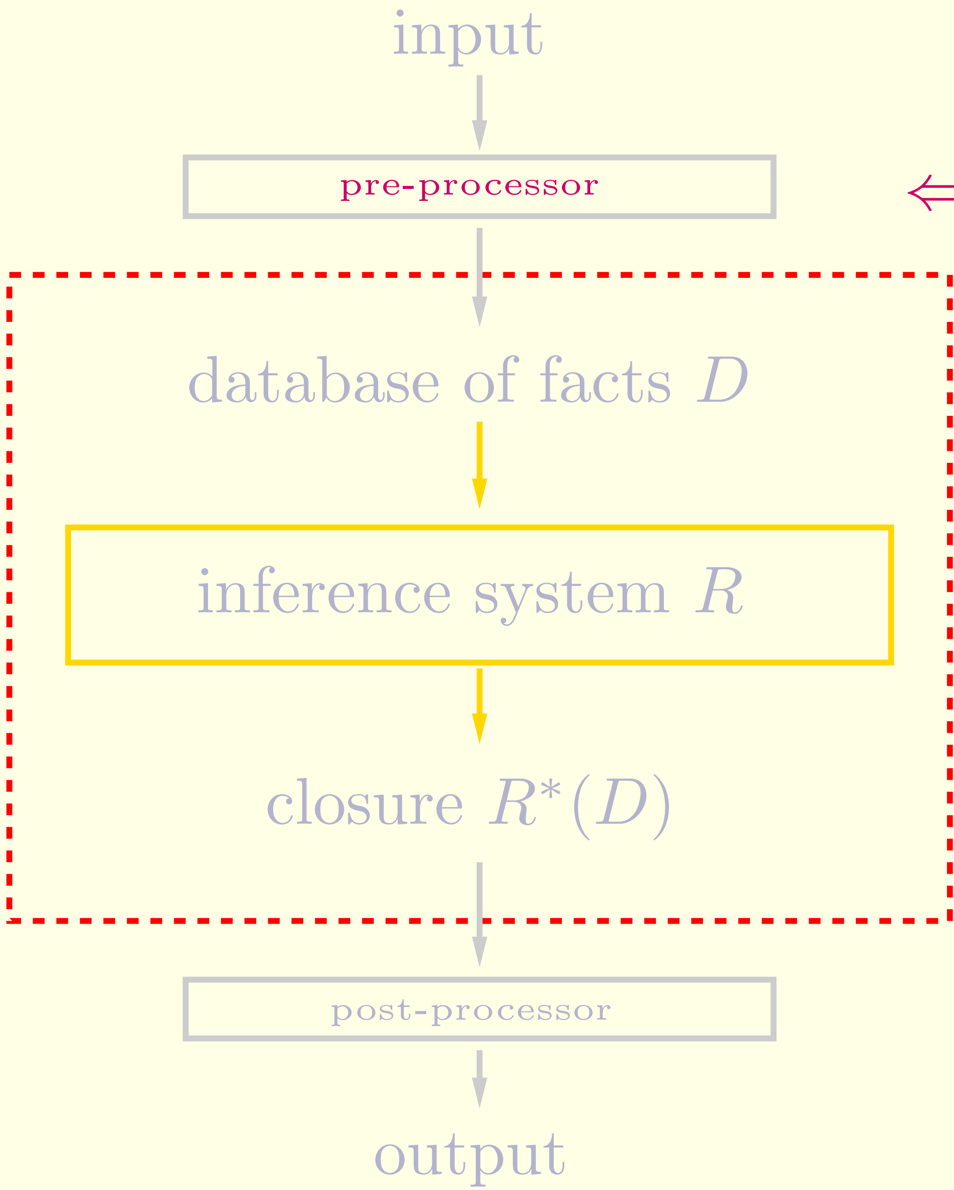
# Paradigm




input
pre-processor
database of facts $D$
this talk
inference system $R$
closure $R^*(D)$
post-processor
output

# Paradigm






$\Longleftarrow$ Paige, Yang 1997

# Reachability in Graphs 

Database:

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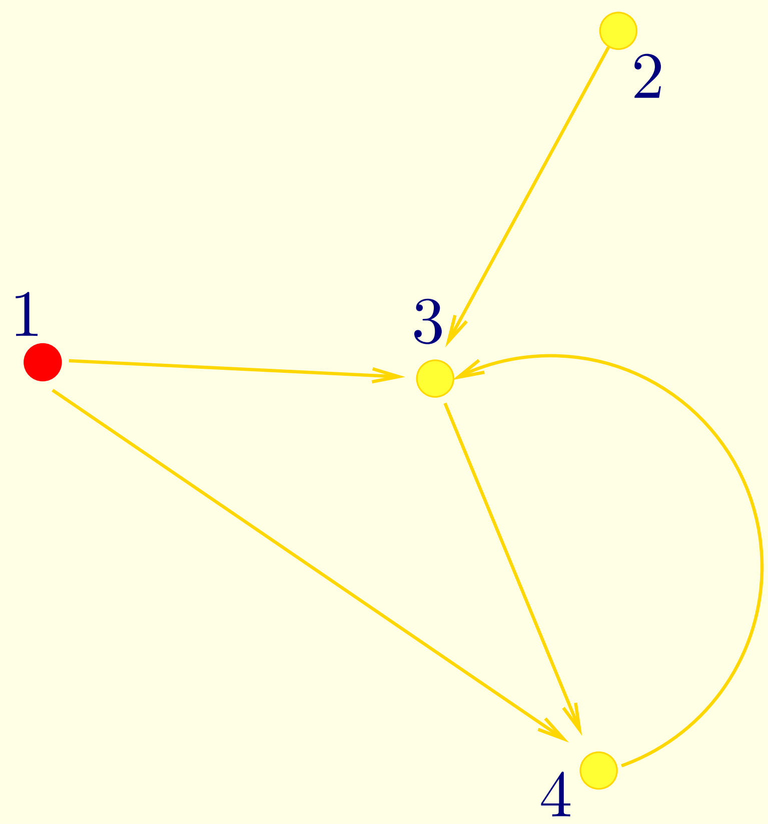
Closure:

$$R^*(D) = D \cup \{r(u) \mid u \text{ reachable from a source}\}$$

# Example



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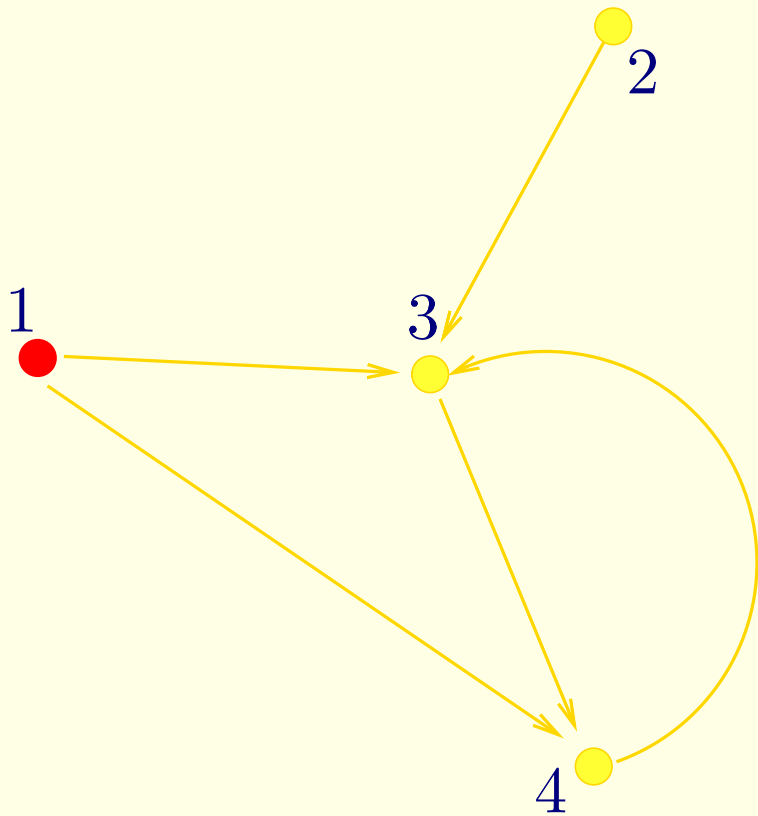


Database

$s(1)$, $e(1,3)$, $e(1,4)$, $e(2,3)$, $e(3,4)$, $e(4,3)$

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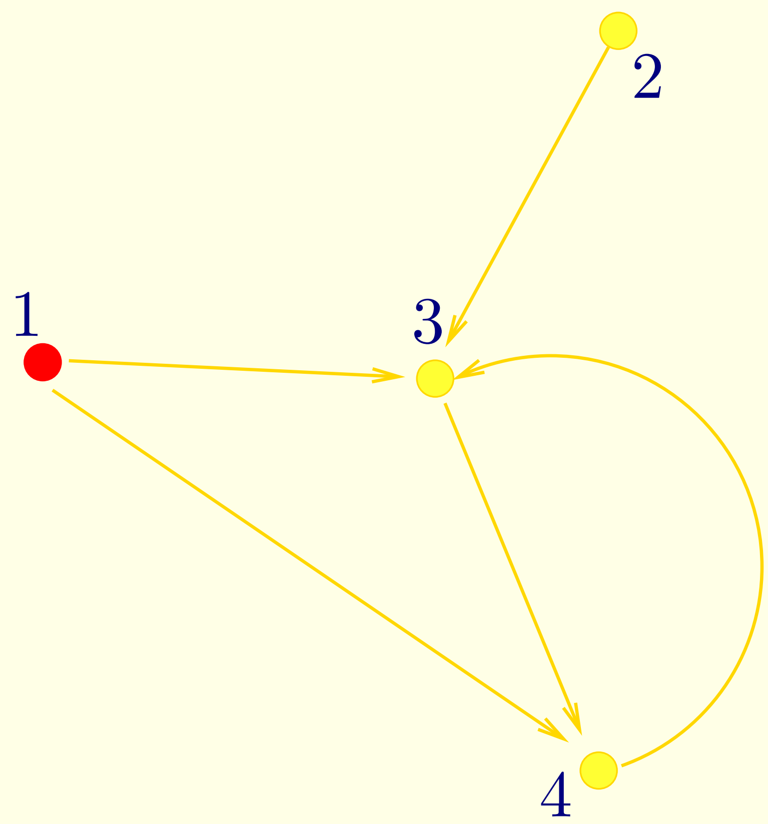


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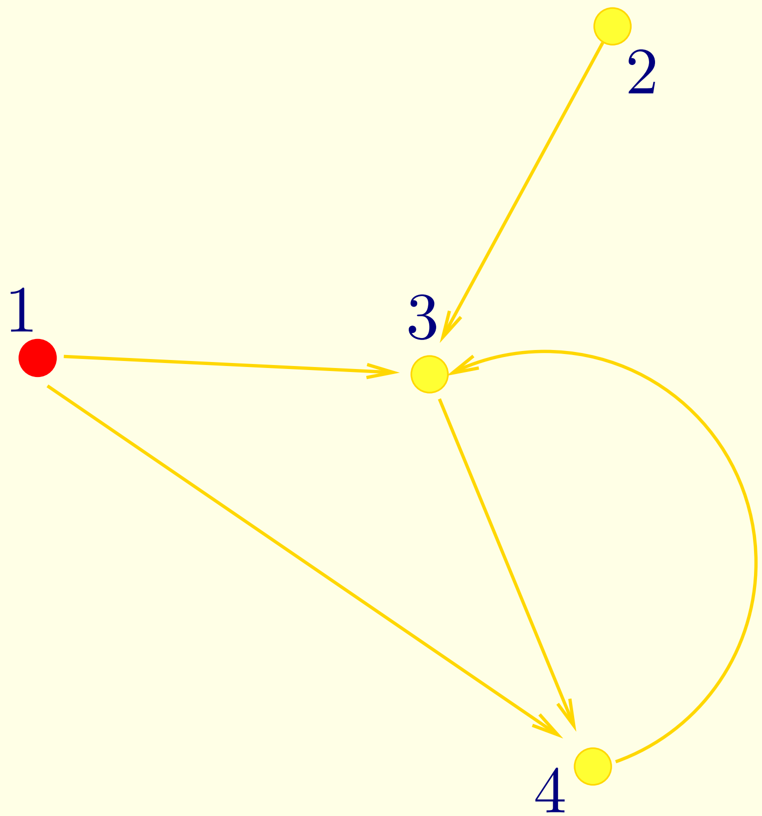


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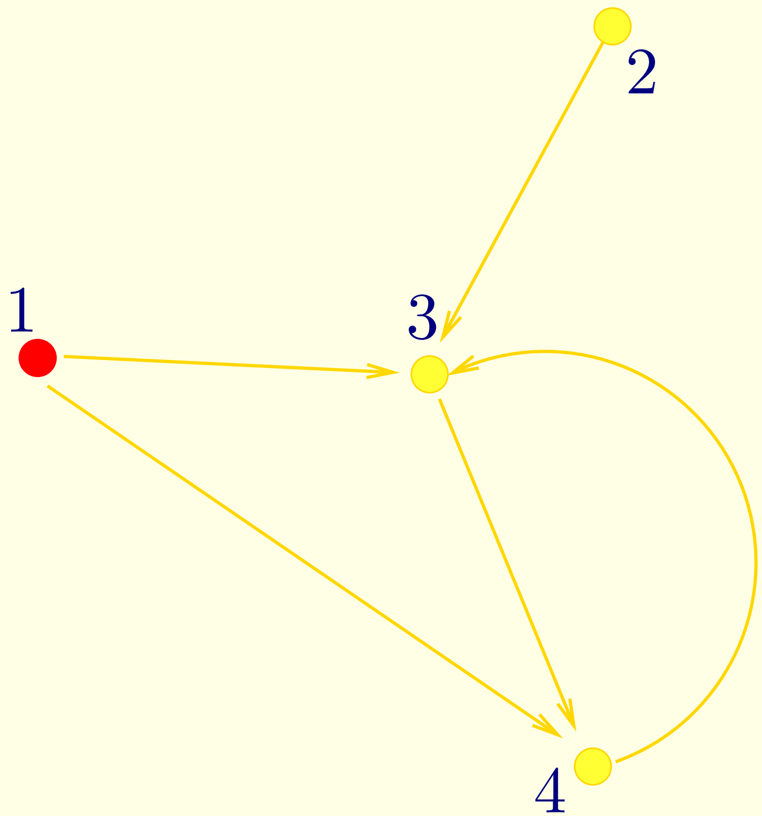


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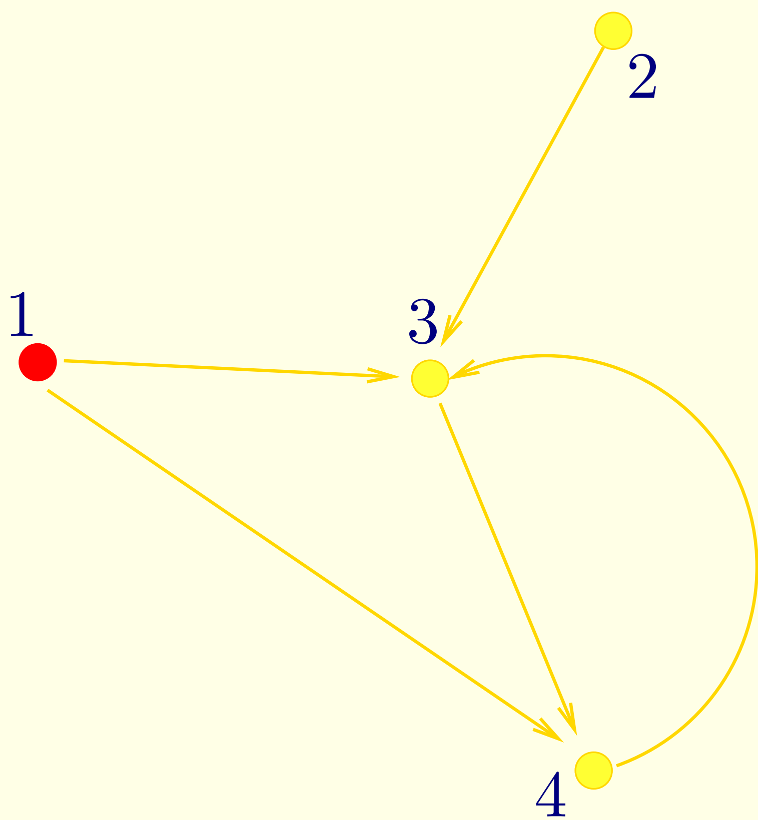


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$\Rightarrow$ saturated.

# First Meta-Complexity Theorem



Bottom-up computation: match prefixes of antecedents against database and fire conclusions

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prefix firings:

$$\pi_R(D) = \left| \{(r\sigma, i) \mid r = A_1 \wedge \ldots \wedge A_i \wedge \ldots \wedge A_n \supset A_0 \in R \right.$$
$$\left. A_j\sigma \in D, \text{ for } 1 \leq j \leq i\} \right|$$

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Extension: constraints for which each solution can be computed in time $O(1)$

$$\frac{s(u)}{r(u)} \qquad\qquad \frac{r(u) \quad e(u,v)}{r(v)}$$

$$r(u) \qquad O(|V|)$$

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$$\frac{s(u)}{r(u)} \quad O(|V|) \qquad\qquad \frac{\begin{array}{c} r(u) \\ e(u,v) \end{array}}{r(v)} \quad \begin{array}{r} O(|V|) \\ +\,O(|E|) \end{array}$$

THEOREM Reachability can be decided in linear time.

# Interprocedural Reachability: Database 

## program

```
1  procedure main
2  begin
3    declare x: int
4    read(x)
5    call p(x)
6  end

7  procedure p(a:int)
8  begin
9    if a>0 then
10     read(g)
11     a:=a-g
12     call p(a)
13     print(a)
14   fi
15 end
```

## facts

```
proc(main,2,6)
next(main,2,5)
call(main,p,5,6)

proc(p,8,15)
next(p,8,12)
call(p,p,12,13)
next(p,13,15)
next(p,8,15)
```

# Interprocedural Reachability: Rules 

Read "$P \Rightarrow L$" as "in procedure $P$ label $L$ can be reached".

$$\frac{proc(P, B_P, E_P)}{P \Rightarrow B_P}$$

$$\frac{\begin{array}{l} next(Q, L, L') \\ Q \Rightarrow L \end{array}}{Q \Rightarrow L'} \qquad \frac{\begin{array}{l} call(Q, P, L_c, R_r) \\ proc(P, B_P, E_P) \\ P \Rightarrow E_P \\ Q \Rightarrow L_c \end{array}}{Q \Rightarrow L_r}$$

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THEOREM $IPR^*(D)$ can be computed in time $O(n)$, with $n = \|D\|$.

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Problem: avoiding $O(|R(D)|^k)$ for rules of length $k$

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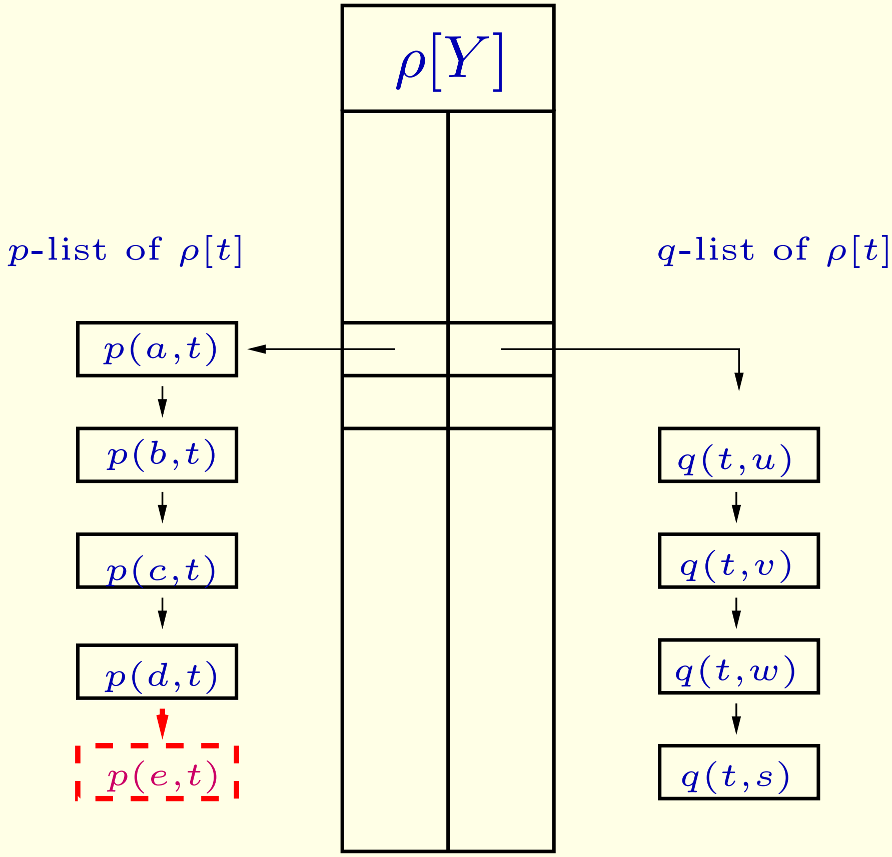

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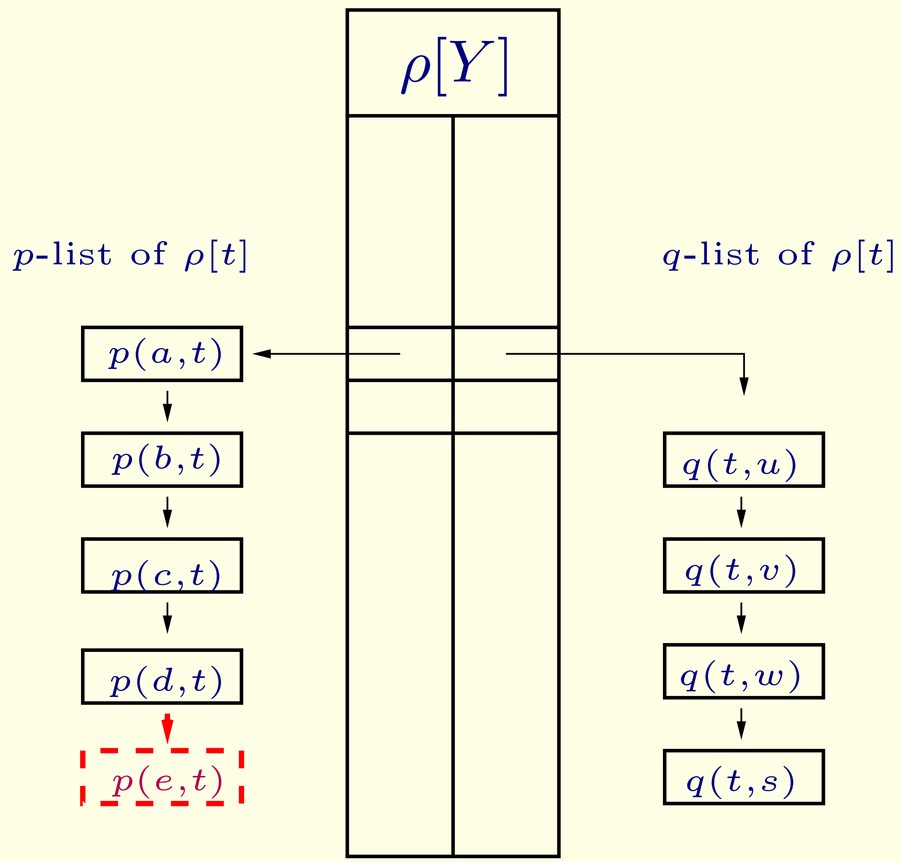


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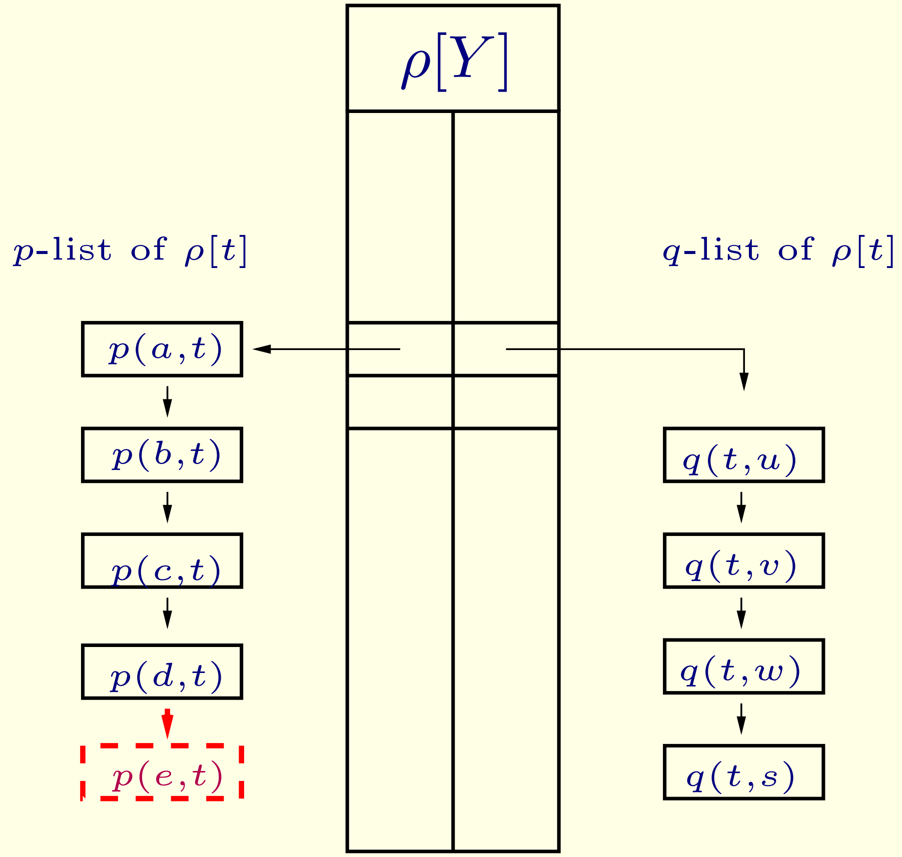


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The inference system can be transformed (maintaining $\pi$) so that it contains only unary rules and binary rules of the form $\rho$.

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  $\Rightarrow$ concept of local inference systems (Givan, McAllester 1993)
- in the presence of transitivity laws, complexity is in $\Omega(n^3)$

# II. Redundancy, Deletion, and Priorities

# Removal of Redundant Information



- redundant information causes inefficiency

$$D = \{\ldots,\ dist(x) \leq d,\ dist(x) \leq d',\ d' < d,\ \ldots\}$$

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- non-determinism don't-care and/or restricted by priorities

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- $S$ visible to a rule if no higher-priority rule is applicable in $S$
- computations are maximal sequences of applications of visible rules
- the final state of a computation starting with $D$ is called an ($R$-) saturation of $D$

# Second Meta-Complexity Theorem 

Let $\mathcal{C} = S_0, S_1, \ldots, S_T$ be a computation.

Prefix firing in $\mathcal{C}$: pair $(r\sigma, i)$ such that for some $0 \leq t < T$:

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Prefix count: $\pi_R(D) = \max\{|\mathrm{p.f.}(\mathcal{C})| \mid \mathcal{C} \text{ a computation from } D\}$

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Let $\mathcal{C} = S_0, S_1, \ldots, S_T$ be a computation.

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Note: again prefix firings count only once; priorities are for free

# Union-Find



$$\texttt{(Refl)}\ \frac{\mathrm{find}(x)}{x \Rightarrow ! \ x}$$

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We are interested in $x \doteq y$ defined as $\exists z(x \Rightarrow! \, z \wedge y \Rightarrow! \, z)$

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$$\texttt{(Refl)}\ \frac{\mathrm{find}(x)}{x \Rightarrow! \ x \quad \mathrm{weight}(x, 1)}$$

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+ symmetric variant of (Orient)

Naive Knuth/Bendix completion

+ normalization (eager path compression) + logarithmic merge

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Logic View: We can (partly) deal with logic programs with equality

Applications: several program analysis algorithms (Steensgaard, Henglein)

# Formal Notion of Redundancy



Let $\succ$ a well-founded ordering on ground atoms.

Definition $A$ is redundant in $S$ (denoted $A \in Red(S)$) whenever

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Corollary Priorities are irrelevant logically $\Rightarrow$ choose them so as to minimize prefix firings

# Deletions based on Redundancy 

Criterion: If

$$r = \ [A_1], \ldots, [A_k], B_1, \ldots, B_m \ \supset \ B$$

and if $S \cup \{A_1\sigma, \ldots, A_k\sigma, B_1\sigma, \ldots, B_m\sigma\}$ is visible to $r$ then

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Note: redundancy should also be efficiently decidable

# III. Instance-based Priorities

$$\textbf{(Init)}\ \frac{}{\mathrm{dist}(\mathrm{src}) \le 0}$$

$$\textbf{(Upd)}\ \frac{\begin{array}{l}[\![\mathrm{dist}(x) \le d]\!]\\ \mathrm{dist}(x) \le d'\\ d' < d\end{array}}{\top}$$

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# Shortest Paths

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Priorities (Dijkstra): always choose an instance of (Add) where $d$ is minimal $\Rightarrow$ allow for instance-based rule priorities

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Prefix firing count: $O(|E|)$, but Dijkstra's algorithm runs in time $O(|E| + |V| \log |V|)$ $\Rightarrow$ one cannot expect a linear-time meta-complexity theorem for instance-based priorities

Basis: Union-find module

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Proof technically involved; uses priority queues with log time operations; memory usage worse

# Further Issues and Questions



- a concept for modules needed (cf. IJCAR paper)
- deletion not always based on redundancy
- “real equality” (on the meta-level)
- how far do we get?
- is deduction without deletion inherently less efficient?
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