Extremal Subgraphs of Random Graphs

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Abstract

Let $\mathcal{K}_\ell$ denote the complete graph on ℓ vertices. We prove that there is a constant $c = c(\ell)$, such that whenever $p \geq n^{-c}$, with probability tending to 1 when n goes to infinity, *every* maximum $\mathcal{K}_\ell$-free subgraph of the binomial random graph $G_{n,p}$ is $(\ell-1)$-partite. This answers a question of Babai, Simonovits and Spencer [BSS90].

The proof is based on a tool of independent interest: we show, for instance, that the *maximum cut* of almost all graphs with M edges, where $M \gg n$, is *nearly unique*. More precisely, given a maximum cut C of $G_{n,M}$, we can obtain all maximum cuts by moving at most $\mathcal{O}(\sqrt{n^3/M})$ vertices between the parts of C.

1 Introduction

It is well-known that, in many different contexts, large triangle-free graphs are bipartite. For example, Mantel [Man07] proved that the maximum triangle-free subgraph of a complete graph on n vertices is a complete bipartite graph with $\lfloor n/2 \rfloor$ vertices in one class and $\lceil n/2 \rceil$ vertices in the other class. Erdős, Kleitman, and Rothschild [EKR76] proved that such a statement is also true in a probabilistic sense. More precisely, they showed that if T_n denotes a graph drawn uniformly at random from the set of all triangle-free graphs on n labeled vertices, then the probability that T_n is bipartite tends to 1 for n tending to infinity. Recently, this result was generalized independently by Steger [Ste05] and Osthus, Prömel and Taraz [OPT03] to the case that, in addition to the number of vertices, also the number of edges is prescribed. The following result is from [OPT03].

Theorem 1.1. *Let $T_{n,m}$ denote a graph drawn uniformly at random from the set of all triangle-free graphs on n labeled vertices and m edges. Then for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \Pr\left[T_{n,m} \text{ is bipartite}\right] = \begin{cases} 1, & \text{if } m = o(n) \\ 0, & \text{if } \frac{n}{2} \leq m \leq (1-\varepsilon)\frac{\sqrt{3}}{4} n^{\frac{3}{2}}\sqrt{\log n} \\ 1, & \text{if } m \geq (1+\varepsilon)\frac{\sqrt{3}}{4} n^{\frac{3}{2}}\sqrt{\log n}. \end{cases}$$

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For a graph G, let $t(G)$ denote the maximum number of edges in a triangle-free subgraph (not necessarily induced) of G, and let $b(G)$ be the maximum number of edges in a bipartite subgraph of G. So $b(G)$ is just the maximum size of a *cut* in G. Of course, we always have $t(G) \geq b(G)$. Our general intuition – guided by the above results – suggests that, for dense enough graphs, these two parameters will typically be equal.

In 1990, Babai, Simonovits and Spencer [BSS90] studied these parameters for the binomial random graph $G_{n,p}$, which was introduced by Erdős and Renyi in [ER60]. They proved, among others, the following result.

Theorem 1.2. *There is a positive constant* δ *such that, for* $p \geq \frac{1}{2} - \delta$,

$$\lim_{n\to\infty} \Pr\left[t(G_{n,p}) = b(G_{n,p})\right] = 1.$$

It seems unlikely that the property "$t(G_{n,p}) = b(G_{n,p})$" has a threshold for constant p; indeed, Babai et. al. asked in [BSS90] whether this result could be extended to cover edge probabilities p of the form n^{-c}, for some positive constant c.

As far as we know, Theorem 1.2 could hold whenever $p = p(n) \geq n^{-1/2+\varepsilon}$, for arbitrary $\varepsilon > 0$. The property does not hold for example when $p_0(n) = \frac{1}{10}(\log n)^{1/2} n^{-1/2}$, as an easy calculation shows that the random graph G_{n,p_0} asymptotically almost surely has an induced 5-cycle H such that no other vertex has more than one neighbour in H: any maximum-size triangle-free subgraph then includes all the edges of H, and is not bipartite.

In this paper, we answer affirmatively the question of Babai, Simonovits and Spencer: we prove that Theorem 1.2 holds whenever $p = p(n) \geq n^{-c}$, for some fixed $c > 0$. In fact, we prove the following stronger result.

Theorem 1.3. *There is a positive constant* c *such that, if* $p = p(n) \geq n^{-c}$, *then*

$$\lim_{n\to\infty} \Pr\left[\textit{every maximum triangle-free subgraph of } G_{n,p} \textit{ is bipartite}\right] = 1.$$

It should be noted that Theorem 1.1 cannot be used directly to prove a theorem of this type. For given p, the result does imply that there is an $m = m(n)$ such that the expected number of non-bipartite triangle-free subgraphs of $G_{n,p}$ with m edges is $o(1)$, while the expected number of bipartite subgraphs with m edges tends to infinity. However, the events that particular bipartite subgraphs exist in the graph are very far from being independent, so this certainly does not prove that there is asymptotically almost surely a bipartite subgraph of $G_{n,p}$ with this number m of edges.

Let us indicate our general strategy for proving Theorem 1.3, and explain the main points of difficulty. We need a little notation first. Let $[n] := \{1, \ldots, n\}$ and $p = p(n) \geq n^{-c}$, where $c > 0$ is some fixed and small constant. For a bipartition $\Pi = (A, B)$ of $[n]$, and a graph G with vertex set $[n]$, we let $E(G; \Pi)$ denote the set of edges of G with one endpoint in each part. The edges of $E(G; \Pi)$ are said to go *across* Π; the other edges of G are said to be *inside* the (parts of the) partition. A *d-perturbation* of $E(G; \Pi)$ is a triangle-free subgraph of G obtained by adding at most d edges, which are inside Π, and removing any number of edges from $E(G; \Pi)$.

An adaptation of the proof of Babai et. al. [BSS90] enables us to restrict our attention to triangle-free subgraphs of the $G_{n,p}$ that are "almost bipartite", specifically that are p^{-c}-perturbations of some bipartite subgraph, for some positive constant c. One of the new ingredients here is that we have to use the sparse regularity lemma and a probabilistic embedding lemma (see Section 2) in order to cover cases where $p = o(1)$.

If we now *fix* a partition Π with the additional constraint that the two classes A and B are roughly equal, it is not too hard to show that, with reasonably high probability, no p^{-c}-perturbation of $E(G_{n,p}; \Pi)$ has more edges than $E(G_{n,p}; \Pi)$. However, for fixed $G = G_{n,p}$, this is certainly not true simultaneously for *all* partitions Π: for instance, if $\{x, y\}$ is an edge of G, and x, y and all their common neighbours are in A, then $E(G; \Pi)$ could be enlarged by adding the edge $\{x, y\}$, keeping the graph triangle-free.

On the other hand, we only need to consider partitions Π in which $E(G; \Pi)$ is *optimal*, i.e., has the maximum number of edges among all bipartite subgraphs, or nearly so. By definition, such partitions have more edges going across them than typical partitions do, so it seems plausible that a fixed near-optimal $E(G; \Pi)$ is still unlikely to have a p^{-c}-perturbation with more edges. We are able to explicitly confirm this intuition.

However, the calculations we are making will not work if there are *too many* near-optimal partitions Π, as then it becomes too likely that *one* of them could be improved by a p^{-c}-perturbation. The final ingredient of our proof is to show that this is unlikely to be the case: in the range we consider, a random graph typically has relatively *few* bipartitions that are optimal or near-optimal.

Before making this statement more precise, we need some more notation. The *distance* of two bipartitions/cuts $\Pi = (A, B)$ and $\Pi' = (A', B')$ of $[n]$ is defined as the number of vertices in which they differ, i.e.

$$\mathrm{dist}(\Pi, \Pi') := \min\left\{|A' \cap A| + |B' \cap B|, |A' \cap B| + |B' \cap A|\right\}. \tag{1.1}$$

Observe that due to $n = (|A' \cap A| + |B' \cap B|) + (|A' \cap B| + |B' \cap A|)$ we have for all pairs Π and Π' that $\mathrm{dist}(\Pi, \Pi') \le \frac{n}{2}$. We denote by $\mathrm{dist}(G; \Pi)$ the *minimum* distance of Π to an optimal bipartition of G, if Π is not optimal; otherwise $\mathrm{dist}(G; \Pi)$ denotes the *maximum* distance of Π to a different optimal bipartition. Furthermore, we say that two cuts have *gap* g, if the difference of their sizes is precisely g, i.e., if

$$\mathrm{gap}(G; \Pi, \Pi') := |E(G; \Pi)| - |E(G; \Pi')| = g. \tag{1.2}$$

Finally, we say that Π has gap g if its number of edges differs from an optimal bipartition by exactly g, that is we let $\mathrm{gap}(G; \Pi) := b(G) - |E(G; \Pi)|$. Our result for near-optimal bipartitions is as follows. We state it for the uniform random graph $G_{n,M}$.

Theorem 1.4. *There is a constant $C > 1$ such that the following is true for sufficiently large n. Let $n^{-1} \ll p = p(n) \le \frac{1}{2}$ and $M = M(n) := p\binom{n}{2}$. Furthermore, let $r = r(n) \ge 1$ satisfy $r \ll (pn)^{1/8}$ and $\omega = \omega(n) \gg 1$, and define*

$$s_0 := C \cdot \omega \cdot r^4 \cdot \sqrt{np^{-1}}.$$

Then

$$\Pr\left[\exists \Pi : \mathrm{gap}(G_{n,M}; \Pi) = r - 1 \;\; \text{and} \;\; \mathrm{dist}(G_{n,M}; \Pi) \ge 2s_0\right] \;\le\; \omega^{-1}.$$

In other words, for most graphs, the maximum cut is *unique* up to movements of a small number of vertices.

Although our main focus is on the most appealing case of triangle-free graphs, our methods extend easily to more general settings. Let $\mathcal{K}_\ell$ be the complete graph on ℓ vertices. We have the following result, replacing the triangle by an arbitrary complete graph.

Theorem 1.5. *Let $\ell \geq 3$. There is a $c = c(\ell) > 0$ such that, whenever $p = p(n) \geq n^{-c}$,*

$$\lim_{n\to\infty} \Pr\left[\textit{every maximum } \mathcal{K}_\ell\textit{-free subgraph of } G_{n,p} \textit{ is } (\ell\text{-}1)\textit{-partite}\right] = 1.$$

We believe that a similar result is true not only for complete graphs, but also for many other graphs as well. In a graph H with chromatic number $\chi := \chi(H)$, a *colour-critical edge* is an edge e such that the graph with edge set $E(H) \setminus \{e\}$ has chromatic number $\chi - 1$. It is known that, if H has a colour-critical edge, then the maximum number of edges in an H-free graph is the Turán number, i.e., the largest H-free graph is the same as the largest χ-partite graph. If H does not have a colour-critical edge, then this fails, as adding one edge to the Turán graph does not create a copy of H.

We expect that Theorem 1.5 is true of any fixed H that has at least one color-critical edge. On the other hand, such a result automatically fails for any graph H without a colour-critical edge. Babai et. al. [BSS90] discuss what can be proved for graphs without colour-critical edges. We treat neither case here.

Outline of the Paper. The paper is structured as follows. In Section 2 we introduce some notation and state a few facts from the theory of random graphs. Let $\mathcal{T}(G)$ denote the set of maximum triangle-free subgraphs of a given graph G. In Section 3 we prove that for every $T \in \mathcal{T}(G_{n,p})$ there exists a.a.s. a bipartition Π_T such that T is a $p^{-12}\log^2 n$-perturbation of Π_T. Next, in Sections 4 and 5 we present the proofs of Theorems 1.4 and 1.3. Finally, Section 6 demonstrates how we can adapt our proofs to prove Theorem 1.5.

2 Preliminaries & Notation

In this section we will present some basic facts from the theory of random graphs and from probability theory, which we will use frequently in the remainder of the paper. Without further reference we will often use the following estimates for the tail of the binomial distribution, which can be found for instance in [JŁR00].

Lemma 2.1. *Let X be a random variable that is binomially distributed with parameters n and p, and set $\lambda := \mathbb{E}[X] = np$. For any $t \geq 0$*

$$\Pr[X \geq \lambda + t] \leq e^{-\frac{t^2}{2(\lambda + t/3)}} \quad \textit{and} \quad \Pr[X \leq \lambda - t] \leq e^{-\frac{t^2}{2\lambda}}.$$

Let us introduce some additional notation. We denote by $\mathcal{G}_n$ the set of all graphs with vertex set $[n]$. Furthermore, let $G \in \mathcal{G}_n$ and $X, Y \subseteq [n]$. In the remainder we will denote by $E(G)$ the edge set of G, by $E(G;\, X)$ the set of edges between vertices in X and by $E(G;\, X, Y)$ the set of edges of G joining a vertex of X and a vertex of Y. Furthermore, $e(G;\, X) := |E(G;\, X)|$ and $e(G;\, X, Y) := E(G;\, X, Y)$, where edges inside $X \cap Y$ are counted only once.

Applying the above tail bounds to edge sets in random graphs, we easily obtain the following statement. Unless stated otherwise, logarithms are always to the base e.

Proposition 2.2. *Let $p \gg \frac{\log n}{n}$ and define*

$$\mathcal{B}_{n,p} := \Big\{ G \in \mathcal{G}_n \mid \exists X, Y \subseteq [n] \textit{ such that } X \cap Y = \emptyset, |X| \geq |Y| \geq 10p^{-1}\log n \textit{ and } \big|e(G; X, Y) - p|X||Y|\big| \geq \frac{1}{2}p|X||Y| \Big\}.$$

Then $\Pr[G_{n,p} \in \mathcal{B}_{n,p}] = o(1)$.

The proposition above is not best possible, but it suffices for our purposes. In the subsequent proofs we will often exploit the "equivalence" of the binomial random graph model $G_{n,p}$ and the uniform random graph model $G_{n,M}$, when $p = M/\binom{n}{2}$. More precisely, we will use *Pittel's inequality* (see e.g. [JŁR00]) which states that for any property $\mathcal{Q}$ of graphs

$$\Pr\left[G_{n,M} \notin \mathcal{Q}\right] \leq 3\sqrt{M}\Pr\left[G_{n,p} \notin \mathcal{Q}\right], \quad \text{where } p = M/\tbinom{n}{2}. \tag{2.1}$$

Now let us turn our attention to optimal bipartitions of the uniform random graph. Recall that for a given graph G, we denote by $b(G)$ the number of edges in an optimal bipartition of G; the following proposition provides bounds for $b(G_{n,M})$, which hold with high probability.

Proposition 2.3. *Let $M \gg n$. For sufficiently large n*

$$\Pr\left[\frac{M}{2} \leq b(G_{n,M}) \leq \frac{M}{2} + \sqrt{4nM}\right] \geq 1 - e^{-n}.$$

Proof. The inequality $b(G_{n,M}) \geq M/2$ is a well-known fact that holds for all graphs with M edges. In order to show the upper bound, let $p := M/\binom{n}{2}$, $L := \frac{p}{2}\binom{n}{2}$ and $\Delta := \sqrt{2pn^2(n-1)}$. In the sequel we will show

$$\Pr\left[b(G_{n,p}) \geq L + \Delta\right] \ll n^{-1}e^{-n}, \tag{2.2}$$

which, together with (2.1), proves the proposition. To see (2.2), define for every partition $\Pi = (A, B)$ of the vertex set of $G_{n,p}$ the random variable

$$X_\Pi = \begin{cases} 1, & e(G_{n,p}; \Pi) \geq L + \Delta \\ 0, & \text{otherwise.} \end{cases}$$

The number of edges of $G_{n,p}$ across Π is binomially distributed with parameters $|A||B|$ and p; with Lemma 2.1 we obtain for sufficiently large n

$$\mathbb{E}\left[X_\Pi\right] = \Pr\left[\text{Bin}\left(|A||B|, p\right) \geq L + \Delta\right] \leq e^{-\frac{\Delta^2}{2(p|A||B| + \frac{\Delta}{3})}} \leq e^{-\frac{\Delta^2}{2(L + \frac{\Delta}{3})}} \leq e^{-2n}.$$

Therefore, if we let $X = \sum_\Pi X_\Pi$, we readily obtain $\Pr\left[X = 0\right] \leq 2^n e^{-2n} \ll n^{-1}e^{-n}$. ■

Recall that we say that a bipartition has gap g if the number of the edges joining vertices in different parts differs from the size of an optimal partition by exactly g. The next proposition states that a.a.s. all bipartitions of $G_{n,M}$ with small gap are "balanced".

Proposition 2.4. *Let $M \gg n$, $p := M/\binom{n}{2}$ and $\lambda = \lambda(n) \geq 0$. Furthermore, let*

$$\begin{aligned} \mathcal{B}_{n,M} \;:=\; \Big\{ & G \in \mathcal{G}_{n,M} \mid \forall \Pi = (A, B) \text{ such that } \operatorname{gap}(G; \Pi) \leq \lambda \text{ it holds} \\ & \left||A| - \frac{n}{2}\right| \leq 3n^{\frac{3}{4}}p^{-\frac{1}{4}} + \lambda^{\frac{1}{2}}p^{-\frac{1}{2}} \text{ and } \left||B| - \frac{n}{2}\right| \leq 3n^{\frac{3}{4}}p^{-\frac{1}{4}} + \lambda^{\frac{1}{2}}p^{-\frac{1}{2}} \Big\}. \end{aligned}$$

For sufficiently large n we have that $\Pr\left[G_{n,M} \in \mathcal{B}_{n,M}\right] \geq 1 - e^{-n}$.

Proof. We show the analogous result for the binomial random graph $G_{n,p}$ and use inequality (2.1) to prove the statement. Let $\Pi = (A, B)$ be a partition of the vertex set and write $|A| = \frac{n}{2} + d$ and $|B| = \frac{n}{2} - d$. Now assume that $|d| > 3n^{3/4}p^{-1/4} + \lambda^{1/2}p^{-1/2}$. The number of possible edges across Π is

$$|A| \cdot |B| = \left(\frac{n}{2} + d\right)\left(\frac{n}{2} - d\right) = \frac{n^2}{4} - d^2 \le \frac{n^2}{4} - (9n^{3/2}p^{-1/2} + \lambda p^{-1}).$$

The number C_Π of edges across Π is binomially distributed with parameters $|A||B|$ and p. Let us assume that Π is a bipartition of $G_{n,p}$ with gap at most λ. With Lemma 2.1 we obtain that, whenever n is sufficiently large, with probability larger than $1 - e^{-\frac{3}{2}n}$, the number of edges in $G_{n,p}$ is at least $p\binom{n}{2} - \sqrt{2}n^{3/2}p^{1/2}$. This implies with Proposition 2.3 that every optimal bipartition of $G_{n,p}$ contains for sufficiently large n at least $\frac{pn^2}{4} - n^{3/2}p^{1/2}$ edges. Hence, for sufficiently large n, the probability that Π has gap less than λ is at most

$$\begin{aligned}
&\Pr\left[C_\Pi \ge \frac{pn^2}{4} - n^{3/2}p^{1/2} - \lambda\right] \\
&\le \Pr\left[C_\Pi \ge \mathbb{E}\left[C_\Pi\right] + (9n^{3/2}p^{-1/2} + \lambda p^{-1}) \cdot p - (n^{3/2}p^{1/2} + \lambda)\right] \\
&\le \Pr\left[C_\Pi \ge \mathbb{E}\left[C_\Pi\right] + 8n^{3/2}p^{1/2}\right] \le e^{-2n},
\end{aligned}$$

where the last step is again due to Lemma 2.1. Therefore,

$$\Pr\left[G_{n,p} \notin \mathcal{B}_{n,p}\right] \le e^{-\frac{3}{2}n} + 2^n \cdot e^{-2n} \ll n^{-1}e^{-n},$$

which completes the proof with Pittel's inequality (2.1). ∎

Finally, we state bounds for the number of *non*-edges across any optimal bipartition of the random graph $G_{n,M}$. The following corollary is a straightforward consequence of Propositions 2.3 and 2.4.

Corollary 2.5. *Let $M \gg n$ and set*

$$\bar{b}(G) = \min\left\{|A||B| - e(G; \Pi) \;\middle|\; \Pi = (A, B) \textit{ is an optimal bipartition of } G\right\}.$$

There is a constant $C > 0$ such that for sufficiently large n

$$\Pr\left[\bar{b}(G_{n,M}) \ge \frac{1}{2}\left(\binom{n}{2} - M\right) - \sqrt{\frac{Cn^5}{M}}\right] \ge 1 - 2e^{-n}.$$

Proof. Set $p := M/\binom{n}{2}$. First we apply Proposition 2.3 to $G_{n,M}$ to obtain that with probability at least $1 - e^{-n}$, every maximum bipartition of $G_{n,M}$ has size smaller than $\frac{M}{2} + \sqrt{4nM}$. Furthermore, we apply Proposition 2.4 with $\lambda = 0$ to obtain that with probability larger than $1 - e^{-n}$ all maximum cuts $\Pi = (A, B)$ of $G_{n,M}$ satisfy $|A|, |B| \ge \frac{n}{2} - 3n^{\frac{3}{4}}p^{-\frac{1}{4}}$. We deduce that with probability at least $1 - 2e^{-n}$, the minimum number of non-edges across any optimal bipartition is at least

$$\left(\frac{n}{2} - 3n^{\frac{3}{4}}p^{-\frac{1}{4}}\right)\left(\frac{n}{2} + 3n^{\frac{3}{4}}p^{-\frac{1}{4}}\right) - \frac{M}{2} - \sqrt{4nM}$$

and the claim follows from $p = M/\binom{n}{2}$ and $M \le n^2$. ∎

3 Finding a Near-Optimal Bipartition

Suppose we have $p = p(n) \geq n^{-c}$ for some positive (small) constant c. For a graph G, we denote by $\mathcal{T}(G)$ the set of maximum triangle-free subgraphs of G. In this section we will prove that a.a.s. every $T \in \mathcal{T}(G_{n,p})$ is "almost" bipartite. More precisely, our proof consists of two parts:

- In Lemma 3.8 we mimic the proof of [BSS90] to show that there is a bipartition $\Pi = \Pi_T = (A, B)$ with the property that at most $o(pn^2)$ edges of T do not go across Π, i.e., connect vertices in A or in B. The new ingredient here is an application of the sparse version of Szemerédi's regularity lemma and a probabilistic embedding lemma, see below.
- Second, in Lemma 3.11 we show that in fact there is a bipartition Π' with the property that at most $p^{-12} \log^2 n$ edges of T do not go across Π'. This proof uses similar ideas as in [BSS90], but differs from the original proof in most details.

Before we continue with our proof let us first introduce a variant of *Szemérédi's regularity lemma* which can be meaningfully applied to *sparse* graphs. Before we state it formally, we need a few technical definitions.

Definition 3.1. A bipartite graph $B = (V_1 \cup V_2, E)$ is called (ε, p)-*regular* if for all $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ with $|V_1'| \geq \varepsilon |V_1|$ and $|V_2'| \geq \varepsilon |V_2|$,

$$\left| \frac{e(B;\, V_1', V_2')}{|V_1'||V_2'|} - \frac{|E|}{|V_1||V_2|} \right| \leq \varepsilon p.$$

Definition 3.2. Let $G = (V, E)$ be a graph and $\varepsilon > 0$. A partition $(C_i)_{i=0}^k$ of V is called an *equitable partition with exceptional class* C_0 if $|C_1| = |C_2| = \ldots = |C_k|$ and $|C_0| \leq \varepsilon |C_1|$. An (ε, p)-*regular partition* is an equitable partition $(C_i)_{i=0}^k$ such that with the exception of at most εk^2 pairs, the pairs (C_i, C_j), $1 \leq i \leq j \leq k$, are (ε, p)-regular.

Definition 3.3. Let $G = (V, E)$ be a graph and let $0 < \eta \leq 1$, $0 < p \leq 1$ and $b \geq 1$. We say that G is (η, b, p)-*upper-uniform* if, for all disjoint sets X and Y with $|X|, |Y| \geq \eta |V|$,

$$\frac{e(G;\, X, Y)}{|X||Y|} \leq bp.$$

We now state the sparse variant of Szemerédi's regularity lemma; see [Koh97] and [KR03].

Theorem 3.4. *For any $0 < \varepsilon < 1/2$ and $b, m_0 \geq 1$, there are constants $\eta = \eta(\varepsilon, b, m_0) > 0$ and $M_0 = M_0(\varepsilon, m_0) \geq m_0$ such that for any $p > 0$, any (η, b, p)-upper-uniform graph with at least m_0 vertices has an (ε, p)-regular partition $(C_i)_{i=0}^k$ such that $m_0 \leq k \leq M_0$.*

A further tool which we will need in our proofs is an *embedding lemma*, which essentially states that almost every graph that can be partitioned so that all pairs of classes are suitably dense and (ε, p)-regular contains a copy of any fixed graph H. We need one further definition before we make this result precise.

Definition 3.5. For a graph $H = (V_H, E_H)$ with vertex set V_H and edge set E_H let $\mathcal{G}(H, n, m, \varepsilon)$ be the class of graphs on vertex set $V = \bigcup_{x \in V_H} V_x$, where the V_x are pairwise disjoint sets of size n, and edge set $E = \bigcup_{\{x,y\} \in E_H} E_{xy}$, where E_{xy} is the edge set of an $(\varepsilon, m/n^2)$-regular bipartite graph with m edges between V_x and V_y.

Unfortunately, it can be shown that *not all* graphs in $\mathcal{G}(H,n,m,\varepsilon)$ contain a copy of H. On the other hand, if m is sufficiently large and ε is sufficiently small, we can hope that only a *tiny fraction* of the graphs in $\mathcal{G}(H,n,m,\varepsilon)$ do not contain a copy of H. This was conjectured by Kohayakawa, Łuczak and Rödl in [KŁR97].

Conjecture 3.1. *Let H be a fixed graph. For any $\beta > 0$, there exist constants $\varepsilon_0 > 0$, $C > 0$, $n_0 > 0$ such that for all $m \geq Cn^{2-1/d_2(H)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$ it holds*

$$|\{G \in \mathcal{G}(H,n,m,\varepsilon) : H \textit{ is not a subgraph of } G\}| \leq \beta^m \binom{n^2}{m}^{e(H)}.$$

Here $d_2(H) := \max\left\{\frac{e_F-1}{|V(F)|-2} \mid F \subseteq H, |V(F)| \geq 3\right\}$ denotes the 2-density of a graph.

In this work we only need a weaker version of the above conjecture, which holds if H is a complete graph and if the number of edges m is slightly larger. The theorem below was proved by Gerke, Marciniszyn and Steger in [GMS05].

Theorem 3.6. *Let $\ell \geq 3$. For all $\beta > 0$, there exist constants $n_0 \in \mathbb{N}$, $C > 0$, and $\varepsilon_0 > 0$ such that*

$$|\{G \in \mathcal{G}(\mathcal{K}_\ell,n,m,\varepsilon) : \mathcal{K}_\ell \textit{ is not a subgraph of } G\}| \leq \beta^m \binom{n^2}{m}^{\binom{\ell}{2}},$$

provided that $m \geq Cn^{2-1/(\ell-1)}$, $n \geq n_0$, and $0 < \varepsilon \leq \varepsilon_0$.

In fact, in [GMS05] a much stronger *counting* version of the above theorem was proved. We do not need this strengthening here. Note also that the above theorem implies Conjecture 3.1 for $H = \mathcal{K}_3$, which was proved already by Kohayakawa, Łuczak and Rödl in [KŁR96].

A final ingredient in our proofs is the following lemma from [KRS04], which states that (ε,p)-regular graphs, whose edge number is only specified within bounds, contain a $(3\varepsilon,p)$-regular spanning subgraph with a given number of edges.

Lemma 3.7. *Let $p \gg n^{-1}$. For every $\varepsilon > 0$, $\alpha > 0$, and $C > 1$ there exists an n_0 such that the following holds. If $B = (V_1 \cup V_2, E)$ is an (ε,p)-regular graph satisfying $|V_1|, |V_2| \geq n_0$ and $\alpha p|V_1||V_2| \leq e(B;V_1,V_2) \leq Cp|V_1||V_2|$, then there exists an $(3\varepsilon,p)$-regular graph $B' = (V_1 \cup V_2, E')$ with $E' \subseteq E$ and $|E| = \alpha p|V_1||V_2|$.*

Now we proceed with our results. Recall that $\mathcal{T}(G)$ denotes the set of maximum triangle-free subgraphs of the graph G.

Lemma 3.8. *Let $\varepsilon > 0$. There exists $C > 0$ such that, for $p \geq Cn^{-1/2}$, a random graph $G_{n,p}$ a.a.s. has the following property. For all $T \in \mathcal{T}(G_{n,p})$ there is a partition $\Pi_T = \Pi = (A,B)$ of the vertex set such that all but at most εpn^2 edges of T go across Π. Furthermore, $\frac{n}{2} - \varepsilon n \leq |A|, |B| \leq \frac{n}{2} + \varepsilon n$.*

Proof. The proof is similar to the proof of the analogous result in [BSS90] for constant density p. The new ingredients here are the sparse version of Szemerédi's regularity lemma (Theorem 3.4) and the probabilistic embedding lemma (Theorem 3.6).

First we collect some properties of random graphs. Using Chernoff's inequality it is easy to verify that, for every $c, \varepsilon \in (0,1]$, a.a.s. every subset U of the vertices of $G_{n,p}$ with $|U| \geq cn$ spans more than $(1-\varepsilon)\frac{1}{2}p|U|^2$ and less than $(1+\varepsilon)\frac{1}{2}p|U|^2$ edges. Similarly, we have that, for every $\xi, \varepsilon > 0$, a random graph $G_{n,p}$ a.a.s. is such that whenever X and Y are two disjoint subsets of the vertices with $|X|, |Y| \geq \xi n$ we have $|e(G_{n,p};\, X, Y) - p|X||Y|| \leq \varepsilon p|X||Y|$. In particular, this implies that $G_{n,p}$ is a.a.s. $(\mu, (1+\varepsilon), p)$-upper-uniform, for all fixed $\mu > 0$. Hence, a.a.s. Theorem 3.4 applies to $G_{n,p}$ and all its spanning subgraphs.

Next we show how to choose the constant C. To do this we need some careful preparations. Let

$$\mathcal{F}(n, m, \alpha) := \big\{G \in \mathcal{G}(\mathcal{K}_3, n, m, \alpha) \ :\ \mathcal{K}_3 \text{ is not a subgraph of } G\big\}.$$

We apply Theorem 3.6 with $\beta := \frac{\varepsilon^3}{e^6}$ to obtain the constants n_ε, C_ε and ε', which may depend on ε. Next we let $\varepsilon'' := \frac{1}{3}\min\{\varepsilon, \varepsilon'\}$, $b := 1 + \varepsilon$, $m_0 := \varepsilon^{-1}$, and apply Theorem 3.4 for ε'', b and m_0 to obtain constants η and M_0. Finally, we let $\mu := \min\{\eta, \frac{1-\varepsilon''}{2M_0}\}$, and $C := \frac{C_\varepsilon}{\varepsilon\mu^2}$.

We claim that for all $p \geq Cn^{-1/2}$ the random graph $G_{n,p}$ a.a.s. does not contain a graph from $\bigcup_{\tilde{n} \geq \mu n} \mathcal{F}(\tilde{n}, \varepsilon p\tilde{n}^2, \varepsilon')$, where . To see this, let X denote the number of such copies; we prove the claim by showing $\mathbb{E}[X] = o(1)$. Let $M(\tilde{n}) := \varepsilon p\tilde{n}^2$ and observe that

$$\mathbb{E}[X] \leq \sum_{\tilde{n} \geq \mu n} n^{3\tilde{n}} \cdot |\mathcal{F}(\tilde{n}, M(\tilde{n}), \varepsilon')| \cdot p^{3M(\tilde{n})}. \tag{3.1}$$

Now we recall that $n_\varepsilon, C_\varepsilon$, and ε' were chosen in such a way that we can apply Theorem 3.6 with $\beta := \frac{\varepsilon^3}{e^6}$ to obtain the bound $|\mathcal{F}(\tilde{n}, m, \varepsilon')| \leq \beta^m \binom{\tilde{n}^2}{m}^3$ for all $m \geq C_\varepsilon \tilde{n}^{3/2}$. We need to check that $M(\tilde{n})$ satisfies $M(\tilde{n}) \geq C_\varepsilon \tilde{n}^{3/2}$. This follows from our choice of $C = \frac{C_\varepsilon}{\varepsilon\mu^2}$ and the assumption $p \geq Cn^{-1/2}$:

$$M(\tilde{n}) = \varepsilon p\tilde{n}^2 \geq \varepsilon p(\mu n)^2 \geq \varepsilon C\mu^2 n^{3/2} \geq C_\varepsilon n^{3/2}.$$

Together with the inequality $\binom{n}{k} \leq (\frac{en}{k})^k$ we thus obtain from (3.1) that

$$\mathbb{E}[X] \leq \sum_{\tilde{n} \geq \mu n} n^{3\tilde{n}} \beta^{M(\tilde{n})} \left(\frac{e}{\varepsilon p}\right)^{3M(\tilde{n})} p^{3M(\tilde{n})} = \sum_{\tilde{n} \geq \mu n} n^{3\tilde{n}} e^{-3M(\tilde{n})} = o(1),$$

where the last two equalities follow from the choice of β and the fact that $M(\tilde{n}) = \Omega(n^{3/2})$. This completes the proof of the claim.

Now consider a random graph $G_{n,p}$ for $p \geq Cn^{-1/2}$. The above discussion shows that $G_{n,p}$ is a.a.s. a $(\mu, 1+\varepsilon, p)$-upper-uniform graph, and that it does not contain a graph from the set $\mathcal{F}(\tilde{n}, \varepsilon p\tilde{n}^2, \varepsilon')$, for all $\tilde{n} \geq \mu n$. Furthermore, a.a.s. every subset U of the vertices of $G_{n,p}$ with $|U| \geq \mu n$ has the property $\big|e(G_{n,p};\, U) - \frac{1}{2}p|U|^2\big| \leq \varepsilon p|U|^2$, and for every two disjoint subsets X and Y of size at least μn it holds that $\big|e(G_{n,p};\, X, Y) - p|X||Y|\big| \leq \varepsilon p|X||Y|$.

In the remainder of the proof we assume that $G_{n,p}$ has all these properties. Let $T \in \mathcal{T}(G_{n,p})$ denote any maximum triangle-free subgraph of $G_{n,p}$. We apply Theorem 3.4 to T with ε'', $b = 1 + \varepsilon$ and $m_0 = \varepsilon^{-1}$ to obtain an (ε'', p)-regular partition $(C_i)_{i=0}^k$, where $m_0 \leq k \leq M_0$. Next we define the reduced graph R consisting of k labeled vertices corresponding to the classes $C_1, \ldots, C_k$, and an edge between two vertices whenever the corresponding partition classes form an (ε'', p)-regular bipartite graph with at least $\varepsilon p|C_1|^2$ edges. Now we show that if R contained a triangle, then T would contain a triangle. To see this, observe first that we have $|C_i| \geq \frac{(1-\varepsilon)n}{M_0} \geq \mu n$, for all $1 \leq i \leq k$. Additionally, if R contained a triangle, by definition, there would exist three sets $C_{i_1}, C_{i_2}, C_{i_3}$ that would induce three bipartite graphs that are (ε'', p)-regular and contain at least $\varepsilon p|C_1|^2$ edges. With Lemma 3.7

we deduce that these bipartite graphs have (spanning) subgraphs with *exactly* $\varepsilon p|C_1|^2$ edges, which are $(3\varepsilon'', p)$-regular and therefore also (ε', p)-regular. That is, T contains a graph from $\mathcal{G}(\mathcal{K}_3, |C_1|, \varepsilon p|C_1|^2, \varepsilon')$. As $G_{n,p}$ and hence also $T \subseteq G_{n,p}$ does not contain a graph from $\mathcal{F}(|C_1|, \varepsilon p|C_1|^2, \varepsilon')$ this implies that T contains a triangle, contradicting the fact that T is triangle-free. We conclude that R contains no triangle.

The remainder of the proof is essentially the same as the proof of the *Main Lemma* in [BSS90] – we only sketch roughly the details and refer the reader to [BSS90] for a more detailed proof. Since R contains no triangle, Turán's theorem yields $e(R) \leq k^2/4$. On the other hand, we can show $e(R) \geq (1 - 40\varepsilon)\frac{k^2}{4}$. To see this, observe that the number of edges of T which join vertices of the same C_i, or vertices of C_0 to some other vertex, or correspond to a "low-density" or non-regular pair (C_i, C_j) is at most $7\varepsilon pn^2$. Furthermore, the number of edges of T in a "high-density" regular pair (C_i, C_j) is at most $(1+\varepsilon)p\left(\frac{n}{k}\right)^2$. Therefore we obtain

$$e(T) \leq e(R) \cdot p\left(\frac{n}{k}\right)^2 + 8\varepsilon pn^2.$$

But since the $G_{n,p}$ has the property that any two disjoint sets X, Y of size $n/2$ satisfy a.a.s. $e(G_{n,p}; X, Y) \geq (1-\varepsilon)p\frac{n^2}{4}$ we know that $e(T) \geq (1-\varepsilon)p\frac{n^2}{4}$, from which the claimed lower bound for $e(R)$ follows easily. Now, due to the *stability lemma* in [Sim68], there is a function $\gamma \to 0$ (when $\varepsilon \to 0$), such that we can find a bipartition (A_R, B_R) of R with the property that at most γk^2 edges do not go across the parts, and $|A_R|, |B_R| \leq \frac{k}{2} + \gamma k$. This completes the proof of the lemma, as it can easily be seen that this implies the existence of a bipartition of T with the claimed properties. (Note that the bound $\pm\gamma k$ suffices to obtain the claim of the lemma if we start with a sufficiently small $\varepsilon > 0$. We omit the details.) ▪

Before we proceed with showing that we can find a much better bipartition than stated in the above lemma, we need two auxiliary tools, which will be used extensively in the sequel.

Lemma 3.9. *Let $k \geq 1$ be an integer, $p \geq n^{-\frac{1}{3k}}$ and $c \in (0,1)$. Then the random graph on the vertex set $[n]$ has a.a.s. the following property: For every subset U of the vertices of size $|U| > cn$ there exists a set Q_U of $\mathcal{O}\left(p^{-k}\right)$ vertices, such that* every *k-tuple of $[n] \setminus Q_U$ is completely joined to at least $(1-c)p^k|U|$ and at most $(1+c)p^k|U|$ vertices in U.*

Proof. The proof is very similar to the proof of the *randomness lemma* in [BSS90] and we omit some of the details. The important difference here is that the statement also holds for k-tuples which might have a non-empty intersection with U, and also that an upper bound on the size of the common neighborhood is given.

We call a set of k vertices $\{v_1, \ldots, v_k\}$ *violating*, if the number of their common neighbors in U is smaller that $(1-c)p^k|U|$ or larger than $(1+c)p^k|U|$. Assume there exist at least $t := Cp^{-k}$ pairwise disjoint sets of k vertices that are violating, where C will be chosen later. Let X denote the union of these sets and let $U' := U \setminus X$. Observe that $|X| = kt = Ckp^{-k} = \mathcal{O}\left(n^{1/3}\right) \ll p^k|U| = \Omega(n^{2/3})$. For each of these t sets, the number of joint neighbors in U' is either smaller than $(1-c)p^k|U| \leq (1-c/2)p^k|U'|$ or larger than $(1+c)p^k|U| - |X| \geq (1+c/2)p^k|U'|$. For each k-tuple, the probability that the number of joint neighbors in U' is so small or so big is by Chernoff's theorem less than $e^{-C'p^kn}$, for an appropriate constant C'. As all these events are independent, we obtain that the probability that there exist t violating sets can be bounded from above by 2^n (the number of ways to choose U) times n^{kt} (the number of ways to choose the t sets of k vertices each) times $e^{-C'tp^kn}$; hence the probability that $G_{n,p}$

does not satisfy the claim of the lemma at most

$$2^n \cdot n^{kt} \cdot e^{-C' t p^k n} = o(1),$$

if C is chosen appropriately. ∎

Observe that the above lemma captures only cases where the set U has linear size. As at some points in our subsequent proofs we will need to consider also subsets U which have *sublinear* size, we also state the following lemma. The proof is completely analogous as above.

Lemma 3.10. *Let $c \in (0,1)$ and let $k \geq 1$ be an integer. Then there is a $C = C(c,k) > 0$ such that $G_{n,p}$ has a.a.s. the following property. For every subset U of the vertices with $p^k|U| > C \log n$ there exists a set Q_U of $\mathcal{O}\left(p^{-k} \log n\right)$ vertices, such that* every *k-tuple of $[n] \setminus Q_U$ is completely joined to at least $(1-c)p^k\,|U|$ and at most $(1+c)p^k\,|U|$ vertices in U.*

With these preparations we can now prove the main result of this section.

Lemma 3.11. *For $p \geq n^{-1/15}$ the following holds a.a.s.: for every $T \in \mathcal{T}(G_{n,p})$ there is a partition $\Pi_T = \Pi = (A,B)$ of the vertex set such that all but $p^{-12}\log^2 n$ edges of T go across Π. Furthermore, $|A| = \frac{n}{2} + o(n)$ and $|B| = \frac{n}{2} + o(n)$.*

Proof. We restrict our proof to the case $p = o(1)$, because for the remaining cases the statement follows directly from [BSS90]. Note that the assumption $p = o(1)$ implies that, for n sufficiently large, p^{-1} is larger than any given constant. Within the proof we will often use this fact in order to keep the formulas simpler. A more careful handling of the inequalities would easily lead to an improvement on the bound $p \geq n^{-1/15}$ in the statement of the lemma. As however our bound in the second part of the proof (Sections 4 and 5) is even weaker, we put emphasis on readability of the proof instead of optimizing the constant.

Let us first collect some properties of $G_{n,p}$. We apply Lemma 3.9 with $k = 2$ and $c = \frac{1}{4}$, which yields that $G_{n,p}$ has a.a.s. the property that for every set U of size at least $\frac{n}{4}$, all pairs of vertices have at least $\frac{3}{4}p^2|U|$ and at most $\frac{5}{4}p^2|U|$ common neighbors in U, except $\mathcal{O}\left(p^{-2}\right)$ vertices (which may depend on U). Furthermore, we apply Lemma 3.10 with $c = \frac{1}{4}$ and obtain that $G_{n,p}$ a.a.s. fulfills the condition that for every sufficiently large subset U of the vertex set the number of vertices having more than $\frac{5}{4}p|U|$ or less than $\frac{3}{4}p|U|$ neighbors in U is at most $\mathcal{O}\left(p^{-1}\log n\right)$. Finally, we apply Proposition 2.2 and obtain that $G_{n,p}$ has a.a.s. the property that for any two sufficiently large disjoint subsets of the vertex set, the number of edges between them is approximately the expected number of edges between them.

Let $0 < \varepsilon < \frac{1}{1000}$. We apply Lemma 3.8 using ε^5 in place of ε. It follows that for sufficiently large n there is a.a.s. a partition $\Pi_T = (A_T, B_T)$ of any $T \in \mathcal{T}(G_{n,p})$ such that

$$e(T;\, A_T) + e(T;\, B_T) \leq \varepsilon^5 p n^2,$$

and $\frac{n}{2} - \varepsilon^5 n \leq |A_T|, |B_T| \leq \frac{n}{2} + \varepsilon^5 n$. Moreover, we apply Proposition 2.4 with $\lambda = \varepsilon^5 p n^2$, which yields that every bipartition $\Pi' = (A', B')$ with gap at most $\varepsilon^5 p n^2$ satisfies $|A'|, |B'| \geq \frac{n}{2} - n^{3/4}p^{-1/4} - \lambda^{1/2}p^{-1/2} \geq (1-\varepsilon)\frac{n}{2}$, and similarly, $|A'|, |B'| \leq (1+\varepsilon)\frac{n}{2}$. In the remainder of the proof we assume that $G_{n,p}$ has all the above properties.

Let $T \in \mathcal{T}(G_{n,p})$. We shall call a partition $\Pi = (A,B)$ *optimal with respect to* T, if $e(T;\, \Pi)$ attains its maximum over all possible partitions. Recall that $b(G)$ denotes the size of an optimal bipartition of G. From our assumptions we know that if Π was optimal (with respect to T) we would have

$$e(T;\, \Pi) \geq e(T) - \varepsilon^5 pn^2 \geq b(G_{n,p}) - \varepsilon^5 pn^2,$$

which implies that Π has gap at most $\varepsilon^5 pn^2$. We deduce that all optimal bipartitions $\Pi = (A, B)$ of T have the properties $(1-\varepsilon)\frac{n}{2} \leq |A|, |B| \leq (1+\varepsilon)\frac{n}{2}$ and $e(T;\, A) + e(T;\, B) \leq \varepsilon^5 pn^2$. In the remainder we fix such an optimal bipartition $\Pi = (A, B)$.

Before we continue, let us introduce some notation. For a graph G, a vertex v and a subset S of the vertices of G let

$$d(G;\, v, S) := |\Gamma(G; v) \cap S|\,,$$

where $\Gamma(G; v)$ denotes the set of neighbors of v in G. We call an edge *horizontal* if it is in T and joins two vertices in A or two vertices in B. The *horizontal degree* of a vertex $v \in A$ is given by

$$d_H(v) := d(T; v, A).$$

We call an edge *missing*, if it is in $G_{n,p}$ joining two vertices in A and B, *but is not* in T. The number of missing edges at a vertex $v \in A$ (with respect to the partition Π) is thus

$$d_M(v) := d(G_{n,p}; v, B) - d(T; v, B).$$

In the remainder of the proof we will repeatedly use the following strategy. We will assume that the horizontal edges satisfy some property. We then will use the assumptions on $G_{n,p}$, and the assumption that Π is optimal with respect to T, in order to expose more missing edges than horizontal edges. However, this will clearly contradict the maximality of T, as we could delete all horizontal edges from T, and add all missing edges to it in order to obtain a larger triangle-free graph.

In order to formalize the idea, let us first define some sets of *exceptional* vertices and discuss a few of their properties. For this, let for a subset U of the vertex set of $G_{n,p}$

$$\mathcal{B}_1(U) := \left\{ v \in V(G_{n,p}) : \big||\Gamma(G_{n,p}; v) \cap U| - p|U|\big| \geq \frac{p|U|}{4} \right\}.$$

and

$$\mathcal{B}_2(U) := \left\{ v \in V(G_{n,p}) : \exists u \in V(G_{n,p}) : \Big||\Gamma(G_{n,p}; v) \cap \Gamma(G_{n,p}; u) \cap U| - p^2|U|\Big| \geq \frac{p^2|U|}{4} \right\}.$$

With the above definition, let

$$\begin{aligned} X_1^A &= \big(\mathcal{B}_1(A) \cup \mathcal{B}_1(B) \ \cup\ \mathcal{B}_2(A) \cup \mathcal{B}_2(B)\big) \ \cap\ A, && (3.2)\\ X_2^A &= \big\{v \in A \mid d_H(v) \geq \varepsilon pn\big\} \setminus X_1^A, && (3.3)\\ X_3^A &= \big\{v \in A \mid d_M(v) \geq d_H(v) + 5\varepsilon pn\big\} \setminus (X_1^A \cup X_2^A). && (3.4) \end{aligned}$$

Furthermore, let $A_0 = A$ and $A_i = A_{i-1} \setminus X_i^A$ for $i \in \{1, 2, 3\}$ and define similarly the above sets with A replaced by B. Finally, let $X_i = X_i^A \cup X_i^B$ for $i \in \{1, 2, 3\}$.

Observe that due to our assumptions we have $|X_1| = \mathcal{O}\left(p^{-2}\right)$. Moreover, we can estimate the number of vertices in X_3 as follows. The number m of missing edges in T incident to at least one vertex in X_3 is

$$m \geq \frac{1}{2} \sum_{v \in X_3} d_M(v) \geq \frac{1}{2} |X_3| \cdot 5\varepsilon pn.$$

But m is no greater than $\varepsilon^5 pn^2$– we deduce $|X_3| \leq \varepsilon n$, with room to spare.

With the above assumptions we now show the following statements, which gather "self-improving" information on the number of horizontal edges.

(i) X_2 is small, i.e., $|X_2| \leq \varepsilon p^{-2}$.
(ii) Set $H_3 := E(T;\, A_3) \cup E(T;\, B_3)$. For large n we have $|H_3| \leq p^{-2} n \log n$.
(iii) $|X_3| \leq p^{-4} \log n$ (i.e. the actual number of exceptional vertices in X_3 is much smaller than the above calculation allows).
(iv) There are no vertices in A_3 or B_3 with horizontal degree in H_3 greater than $p^{-6} \log n$.
(v) We improve the bound on $|H_3|$: $|H_3| \leq \frac{1}{2} p^{-12} \log^2 n$.
(vi) Finally, $e(T; A) + e(T; B) \leq p^{-12} \log^2 n$.

First we prove (i). Let $X_2^A = \{v_1, \ldots, v_t\}$, where $t = |X_2^A|$, and observe that for every $v \in X_2^A$ we have $d_H(v) \leq d(T; v, B)$, as otherwise Π would not be optimal. Furthermore, note that *all* edges between the sets $\Gamma(T; v) \cap A$ and $\Gamma(T; v) \cap B$ in $G_{n,p}$ must be missing, as otherwise T would contain a triangle. In the following we show a lower bound for the total number of missing edges m, which will immediately translate into an upper bound for t.

We write

$$\mathcal{F}(v_i) := E\big(G_{n,p};\ \Gamma(T; v_i) \cap A,\ \Gamma(T; v_i) \cap B\big),$$

i.e., $\mathcal{F}(v_i)$ is the set of edges in $G_{n,p}$ which are missing "due to" v_i. Observe that we can estimate $f_i := |\mathcal{F}(v_i)| \geq \frac{p}{2} \cdot (\varepsilon pn)^2$, as we assumed that the $G_{n,p}$ satisfies Proposition 2.2. Let $t_0 = \left(\frac{\varepsilon}{2p}\right)^2$ and suppose that t satisfies $t \geq t_0$. Now define the quantity $m_0 := |\bigcup_{i=1}^{t_0} \mathcal{F}(v_i)|$ and observe that $m \geq m_0$. In order to bound m_0, we apply the inclusion-exclusion principle:

$$m_0 \geq \sum_{i=1}^{t_0} f_i - \sum_{1 \leq i < j \leq t_0} |\mathcal{F}(v_i) \cap \mathcal{F}(v_j)| \geq t_0 \cdot \frac{p}{2} \cdot (\varepsilon pn)^2 - \binom{t_0}{2} \max_{i<j} |\mathcal{F}(v_i) \cap \mathcal{F}(v_j)|. \quad (3.5)$$

Recall that the size of the common neighborhood of any two vertices $v, w \in X_2^A$ in A as well as in B is at most $p^2 n$, by the definition of X_1. So $|\mathcal{F}(v) \cap \mathcal{F}(w)|$ is at most the maximum size of the set of edges of $G_{n,p}$ between two disjoint vertex sets of this size. Hence the assumption that $G_{n,p}$ satisfies Proposition 2.2 implies

$$|\mathcal{F}(v) \cap \mathcal{F}(w)| \leq 2p \cdot (p^2 n)^2.$$

From (3.5) we obtain $m_0 \geq \frac{\varepsilon^4}{16} pn^2$ due to the definition of t_0. Therefore, whenever $|X_2^A| \geq t_0$ we achieve a contradiction, as m_0 is at most $\varepsilon^5 pn^2$.

Now we prove (ii). Recall that A_3 and B_3 denote the sets of vertices which are not exceptional. We may assume $e(T;\, A_3) \geq e(T;\, B_3)$, as otherwise we could interchange the roles of A and B. Our objective is to derive upper and lower bounds for the number N of instances of a configuration called a "chord"; these bounds will immediately imply a bound on $|H_3|$ that

Figure 1: A chord in T.

will show (ii). A chord consists of three vertices $x, y \in A_3$ and $z \in B$ with the property that x and y are connected by an edge in H_3, y and z are connected in $G_{n,p}$ and the edge $\{x, z\}$ is missing (i.e., it is in $G_{n,p}$ but not in T).

Consider an edge $\{x, y\} \in E(T;\, A_3)$. Then every vertex z in the common neighborhood of x and y in B forms a triangle in $G_{n,p}$. Hence, one of the edges $\{x, z\}$ or $\{y, z\}$ *must* be a missing edge (as otherwise there would be a triangle in T), which means that the three vertices x, y and z are a chord. Recall that due the definition of X_1 we know that the common neighborhood of any two vertices in A_3 in B is at least $\frac{3}{4}p^2|B|$. Therefore, a lower bound for the number of chords is

$$N \;\geq\; e(T; A_3) \cdot \frac{3}{4}p^2|B| \;\geq\; \frac{1}{2}|H_3| \cdot \frac{p^2 n}{4} \;=\; |H_3|\frac{p^2 n}{8}. \tag{3.6}$$

In the sequel we derive an upper bound for N. We first bound the number of chords that contain a vertex $x \in A_3$ such that $d_{H_3}(x) \leq Cp^{-1}\log n$, where C is the constant from Lemma 3.10 for $c = \frac{1}{4}$. We bound the number of such chords containing x by their horizontal degree times an upper bound on the size of a common neighbourhood of any two vertices in A_3 in $G_{n,p}$. Using again the definition of X_1^A we deduce that the common neighborhood of any two vertices in A_3 in B is of size at most $\frac{5}{4}p^2|B|$. Hence,

$$\begin{aligned} &\sum_{\substack{x \in A_3:\\ d_{H_3}(x) \leq Cp^{-1}\log n}} \;\sum_{w \in \Gamma(T;\, x) \cap A_3} |\Gamma(G_{n,p};\, x) \cap \Gamma(G_{n,p};\, w) \cap B| \\ &\qquad \leq |A_3| \cdot Cp^{-1}\log n \cdot \frac{5}{4}p^2|B| \;\leq\; Cp\, n^2 \log n. \end{aligned} \tag{3.7}$$

Now consider the vertices $x \in A_3$ such that $d_{H_3}(x) \geq Cp^{-1}\log n$. The number of chords containing such a vertex can be bounded as follows. Consider the set of missing neighbors of x, i.e., $S := (\Gamma(G_{n,p};\, x) \cap B) \setminus \Gamma(T;\, x)$, and note that $|S| = d_M(x)$. As we assumed that $G_{n,p}$ satisfies Lemma 3.10 with $c = \frac{1}{4}$, all but $C'p^{-1}\log n$ vertices in S have in $G_{n,p}$ at most $\frac{5}{4}pd_{H_3}(x)$ neighbors in the set $\Gamma(T;\, x) \cap A_3$. Here C' is an appropriately chosen constant. Furthermore, note that $d_M(x) \leq d_{H_3}(x) + 5\varepsilon pn \leq 6\varepsilon pn$, as $x \in A_3$. We deduce that the number of chords containing a vertex of high horizontal degree can be bounded by

$$\begin{aligned} &\sum_{x : d_{H_3}(x) \geq Cp^{-1}\log n} \left(d_M(x) \cdot \frac{5}{4}pd_{H_3}(x) + C'p^{-1}\log n \cdot d_{H_3}(x) \right) \\ \leq &\sum_{\substack{x \in A_3:\\ d_{H_3}(x) \geq Cp^{-1}\log n}} \left(\frac{30}{4}\varepsilon p^2 n + C'p^{-1}\log n \right) \cdot d_{H_3}(x) \leq 16\varepsilon p^2 n \cdot |H_3|. \end{aligned} \tag{3.8}$$

Now by combining (3.6), (3.7) and (3.8) we obtain for sufficiently large n

$$N \leq |H_3| \cdot p^2 n \left(\frac{1}{8} - 16\varepsilon \right) \leq 2Cp\, n^2 \log n,$$

from which the bound on $|H_3|$ follows with room to spare from our assumptions on p and ε.

We continue by proving (iii). Let $H = E(T;\, A) \cup E(T;\, B)$ and recall that H_3 is the subset of H restricted to A_3 and B_3. Our strategy is as follows. We estimate $|H|$ from above and the number of missing edges from below. Comparing the two bounds will yield a contradiction, unless $|X_3| \le p^{-4} \log n$. First observe that due to (i) and (ii) the number of horizontal edges is at most

$$\sum_{v \in X_1 \cup X_2 \cup X_3} d_H(v) + |H_3| \le |X_1 \cup X_2| \cdot n + \sum_{v \in X_3} d_H(v) + |H_3| \le 2p^{-2} n \log n + \sum_{v \in X_3} d_H(v). \quad (3.9)$$

On the other hand, observe that for all vertices $v \in X_3$ we have $d_H(v) \le \varepsilon pn$, due to the definition of X_2. Therefore, the number of missing edges is at least

$$\begin{aligned} \frac{1}{2} \sum_{v \in X_3} d_M(v) &\ge \frac{1}{2} \sum_{v \in X_3} (d_H(v) + 5\varepsilon pn) \\ &\ge \frac{1}{2} \sum_{v \in X_3} (2d_H(v) + 4\varepsilon pn) \ge \sum_{v \in X_3} d_H(v) + 2|X_3|\varepsilon pn. \end{aligned} \quad (3.10)$$

Now replace all horizontal edges from T with all missing edges in order to obtain a different triangle-free graph. Comparing (3.9) and (3.10) yields that to avoid a contradiction X_3 must satisfy

$$2p^{-2} n \log n \ge 2|X_3| \varepsilon pn \;\Rightarrow\; |X_3| \le \varepsilon^{-1} p^{-3} \log n \le p^{-4} \log n,$$

whenever n is sufficiently large.

Next we show (iv). Let v be a vertex in A_3 with $d_{H_3}(v) \ge p^{-6} \log n$ – we handle vertices in B_3 analogously. Recall that $A_3 = A \setminus (X_1 \cup X_2 \cup X_3)$. The definitions of the sets X_i thus imply together with the fact that $|B| \ge (1-\varepsilon)\frac{n}{2}$ that the number of neighbours of v in B is at least

$$d(T; v, B) = d(G_{n,p}; v, B) - d_M(v) \ge \frac{pn}{8} - 6\varepsilon pn \ge \frac{pn}{9}.$$

Furthermore, note that the edges between the vertex sets $\Gamma(T; v) \cap B$ and $\Gamma(T; v) \cap A_3$ would form triangles in T. We deduce with Proposition 2.2 that the number of missing edges in T is at least

$$e(G_{n,p};\; \Gamma(T; v) \cap B,\; \Gamma(T; v) \cap A_3) \ge \frac{p}{2} \cdot \frac{pn}{9} \cdot p^{-6} \log n \ge \frac{1}{20} p^{-4} n \log n. \quad (3.11)$$

Finally observe that the number of horizontal edges in T is due to (i)-(iii) at most

$$|H_3| + |X_1| \cdot n + |X_2 \cup X_3| \cdot \frac{5}{4} pn \le 2p^{-3} n \log n. \quad (3.12)$$

But this contradicts with (3.11) the maximality of T – we conclude that there are no vertices in $A_3 \cup B_3$ with horizontal degree at least $p^{-6} \log n$, i.e., (iv) is shown.

Now we show (v). For this, let R be a *matching* of maximum cardinality in H_3, and let m be the number of missing edges in T. As in the previous proofs, the central idea is to derive a lower bound on m which contradicts the maximality of T.

Let us assume assume w.l.o.g. that $e(T;\, A_3) \ge \frac{1}{2}|H_3|$. Using (iv) we readily obtain that $|R| \ge \frac{1}{4}|H_3|p^6(\log n)^{-1}$, as we can construct a matching by greedily removing edges from H_3. In order to estimate the number of missing edges, let $e = \{u, v\} \in R$, where $u, v \in A_3$. As $u, v \notin X_1$ they have at least $\frac{3}{4}p^2|B| \ge \frac{p^2 n}{4}$ common vertices in B. Hence, for every $e \in R$ there are at least $\frac{p^2 n}{4}$ missing edges. As these edges are *distinct* for different edges in R we easily deduce that

$$m \ge |R| \cdot \frac{p^2 n}{4} \;\ge\; |H_3| \frac{p^6}{4 \log n} \cdot \frac{p^2 n}{4} \;\ge\; |H_3| \cdot \frac{p^8}{16 \log n}\, n.$$

As in (3.12) we obtain that the number of horizontal edges is at most $2p^{-3} n \log n$. In order to avoid a contradiction $|H_3|$ must hence satisfy

$$|H_3| \cdot \frac{p^8}{16 \log n}\, n \le 2p^{-3} n \log n,$$

from which the claimed bound on $|H_3|$ follows for sufficiently large n.

Finally we show (vi). Let $X = X_1 \cup X_2 \cup X_3$. From our assumptions and (i)-(v) we know that $|X| \le 2p^{-4} \log n$, if n is sufficiently large, and the number of horizontal edges in $(A \cup B) \setminus X$ in T is less than $\frac{1}{2}p^{-12} \log^2 n$.

To prove the statement, we replicate the argument from (iv). Let d be the maximum horizontal degree of a vertex in X, and suppose $d \ge p^{-5} \log n$. Furthermore, suppose that the maximum is attained at a vertex $v \in A$. Observe that v has at least d neighbors in B, as otherwise Π would not have been maximal. It follows that the number of missing edges is at least $\frac{pd^2}{4}$, because we assumed that the $G_{n,p}$ satisfies Proposition 2.2. On the other hand, the number of horizontal edges is at most $|X| \cdot d + \frac{1}{2}p^{-12} \log^2 n$. Therefore, d must satisfy

$$\frac{pd^2}{4} \le 2p^{-4} \log n \cdot d + \frac{1}{2}p^{-12} \log^2 n$$

which can only hold if $d \le 3p^{-13/2} \log n$. This concludes the proof. ∎

4 On Properties of (Near-)Optimal Bipartitions

Before we proceed with the proof of Theorem 1.4 we introduce two auxiliary tools. The first lemma is a statement about the number of non-edges between sufficiently large parts of the vertex set of $G_{n,M}$. More precisely, for an ordered partition Π of the vertex set in two pairs of (sufficiently large) parts, we want to bound the probability that the number of *non*-edges between the parts of the pairs is not near its expected value. The result is not best possible, but it suffices for our purposes and keeps the calculations short.

Lemma 4.1. *Let $n^{1/2} \le s \le \frac{n}{2}$ and $\Pi = (V_1, W_1, V_2, W_2)$ be a partition of $[n]$ such that $|V_1| + |V_2| = s$ and $|W_1|, |W_2| \ge n/7$. Furthermore, for a graph G with vertex set $[n]$ let*

$$\overline{e}(G;\, \Pi) := |\Pi| - e(G;\, V_1, W_1) - e(G;\, V_2, W_2), \text{ where } |\Pi| := |V_1||W_1| + |V_2||W_2|. \tag{4.1}$$

Let $n \ll M \le \frac{1}{2}\binom{n}{2}$ and $p := M/\binom{n}{2}$. Then there is a constant $C > 0$ such that

$$\Pr\Big[\exists \Pi : \big|\overline{e}(G_{n,M}; \Pi) - (1-p)|\Pi|\big| \ge C \cdot s^{-1/2} \cdot (1-p)|\Pi|\Big] \le e^{-n}.$$

Proof. We show the analogous result for the $G_{n,p}$ and then use Pittel's inequality (2.1) to complete the proof. For a partition Π with the above properties define the event

$$\mathcal{E}_\Pi \ := \ \left|\bar{e}(G_{n,p};\Pi) - (1-p)|\Pi|\right| \geq C \cdot s^{-1/2} \cdot (1-p)|\Pi|.$$

Observe that according to our assumptions we have $|\Pi| \geq \frac{sn}{7}$. A straightforward application of Lemma 2.1 then yields that we can choose C such that $\Pr[\mathcal{E}_\Pi] \leq 20^{-n}$. As there are at most 4^n ways for choosing Π, the desired probability can be bounded from above by $4^n \cdot 20^{-n} \ll n^{-1}e^{-n}$. ∎

The next proposition is a technical estimate for the probability that a *tri*nomially distributed random variable of a special form deviates from its expectation. The bound is tight up to the determined constant.

Proposition 4.2. *Let* $\alpha < \frac{1}{2}$, *and* $d \leq \min\{\sqrt{\alpha N}, \sqrt{(1-2\alpha)N}\}$ *such that* $2\alpha N + d < N$. *There is a constant* $C > 0$ *such that*

$$\binom{N}{\alpha N, \alpha N + d} \alpha^{2\alpha N + d}(1-2\alpha)^{(1-2\alpha)N-d} \geq \frac{C}{\alpha N}. \tag{4.2}$$

Proof. We obtain with Stirling's formula $1 \leq x!/(x^x e^{-x}\sqrt{2\pi x}) \leq 2$ that there is a constant $C' > 0$ such that, if $d \leq \sqrt{\alpha N}$,

$$\begin{aligned}
&\binom{N}{\alpha N, \alpha N + d} = \frac{N!}{(\alpha N)!(\alpha N + d)!((1-2\alpha)N-d)!} \\
&\geq \frac{1}{16\pi} \cdot \frac{\sqrt{N}}{\sqrt{\alpha N(\alpha N + d)(N - 2\alpha N - d)}} \cdot \frac{N^N}{(\alpha N)^{\alpha N}(\alpha N + d)^{\alpha N + d}((1-2\alpha)N-d)^{(1-2\alpha)N-d}} \\
&\geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d}\left(1 + \frac{d}{\alpha N}\right)^{-\alpha N - d} \cdot (1-2\alpha)^{-(1-2\alpha)N+d}\left(1 - \frac{d}{(1-2\alpha)N}\right)^{-(1-2\alpha)N+d}.
\end{aligned}$$

Recall that $d \leq \min\{\sqrt{\alpha N}, \sqrt{(1-2\alpha)N}\}$. The inequality $1 + x \leq e^x$ implies with this fact that we can choose $C > 0$ such that

$$\begin{aligned}
\binom{N}{\alpha N, \alpha N + d} &\geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d}(1-2\alpha)^{-(1-2\alpha)N+d} \cdot e^{-\frac{(d+\alpha N)d}{\alpha N}}\ e^{-\frac{d}{(1-2\alpha)N}(-(1-2\alpha)N+d)} \\
&\geq \frac{C'}{\alpha N} \cdot \alpha^{-2\alpha N - d}(1-2\alpha)^{-(1-2\alpha)N+d} \cdot e^{-d-\frac{d^2}{\alpha N}}\ e^{d-\frac{d^2}{(1-2\alpha)N}} \\
&\geq \frac{C}{\alpha N} \cdot \alpha^{-2\alpha N - d}(1-2\alpha)^{-(1-2\alpha)N+d}.
\end{aligned}$$

Substituting this bound in the left-hand side of (4.2) yields immediately the statement. ∎

With the above tools we are ready to prove the main result of this section.

Proof of Theorem 1.4. Recall that for two bipartitions Π and Π' of the vertex set of a graph G we denote by $\mathrm{dist}(\Pi, \Pi')$ the number of *vertices* in which Π and Π' differ, and we say that Π has *gap* g (i.e. $\mathrm{gap}(G;\Pi) = g$), if the number of edges across the parts of Π is exactly g less than the number of edges across an optimal bipartition. Note that for every pair of partitions Π and Π' it holds $\mathrm{dist}(\Pi, \Pi') \leq \frac{n}{2}$.

The central idea in our proof is to consider how the random variable $b(G_{n,M})$ changes, as the uniform random graph $G_{n,M}$ evolves. More precisely, let $t = t(n) > 0$ and consider the *expected change*

$$\mathbb{E}\left[b(G_{n,M+t}) - b(G_{n,M})\right]. \tag{4.3}$$

In order to obtain bounds for the above expression, we look at it from two different points of view: either removing t edges uniformly at random from $G_{n,M+t}$ *or* adding t edges uniformly at random to $G_{n,M}$.

Suppose that we delete t edges uniformly at random from $G_{n,M+t}$ in order to obtain a graph with M edges, and let Π^* be any optimal bipartition of $G_{n,M+t}$. Observe that the size of the optimal bipartition decreases by *at most* the number of edges among the t deleted that go across Π^* (as the size of an optimal bipartition decreases only if edges are removed from *all* optimal bipartitions). Hence, the expected decrease of the size of an optimal bipartition can be bounded from above with the proportion of edges of $G_{n,M+t}$ across Π^*. Now we apply Proposition 2.3, which yields that $G_{n,M+t}$ has with probability $1 - e^{-n}$ the property that all its optimal bipartitions have size at most $\frac{1}{2}(M+t) + \sqrt{4n(M+t)}$; we obtain

$$\mathbb{E}\left[b(G_{n,M+t}) - b(G_{n,M})\right] \leq t \cdot \frac{\frac{1}{2}(M+t) + \sqrt{4n(M+t)}}{M+t} + e^{-n} \leq t \cdot \left(\frac{1}{2} + \sqrt{\frac{5n}{M}}\right). \tag{4.4}$$

On the other hand, let G be a graph and $\Pi^*(G)$ denote a *canonical* optimal bipartition of G, which is is uniquely determined by G (note that we can always induce a canonical ordering on the set of all partitions; Π^* is then simply the first optimal partition with respect to this ordering). Furthermore, let $i^* := i^*(n)$ be the minimum integer such that $2^{i^*}s_0 \geq \frac{n}{4}$. For $0 \leq i \leq i^*$ define the event

$$\mathcal{P}_i \ := \ \exists \Pi : \operatorname{gap}(G_{n,M};\, \Pi) = r - 1 \ \text{ and } \ 2^i s_0 \leq \operatorname{dist}\bigl(\Pi,\, \Pi^*(G_{n,M})\bigr) \leq 2^{i+1}s_0,$$

and set $p_i := \Pr\left[\mathcal{P}_i\right]$. Note that the union of the events $\mathcal{P}_i$ is equivalent to the event

$$\mathcal{P} \ := \ \exists \Pi : \operatorname{gap}(G_{n,M};\, \Pi) = r - 1 \ \text{ and } \ \operatorname{dist}\bigl(\Pi,\, \Pi^*(G_{n,M})\bigr) \geq s_0.$$

Now, if $r \geq 2$, then for every partition Π we obviously have that $\operatorname{dist}(G;\, \Pi) \leq \operatorname{dist}\bigl(\Pi,\, \Pi^*(G)\bigr)$. Hence, whenever $r \geq 2$ we deduce that the desired probability p_{r,s_0} can be bounded from above by the probability of the union of the events $\mathcal{P}_i$. On the other hand, if $r = 1$, then we have that $\operatorname{dist}(G;\, \Pi) \geq \operatorname{dist}\bigl(\Pi,\, \Pi^*(G)\bigr)$ and the above argument is not applicable. However, the complementary event $\overline{\mathcal{P}}$ of $\mathcal{P}$ implies that all optimal bipartitions have distance at most s_0 from $\Pi^*(G_{n,M})$. But then the distance of any two optimal bipartitions of $G_{n,M}$ is at most $2s_0$, and therefore also in this case the probability of the union of the $\mathcal{P}_i$'s is an upper bound for p_{r,s_0}.

In the main part of the proof we will show that for sufficiently large n we have $p_i \leq \frac{1}{2^{i+1}\omega}$ for all $0 \leq i \leq i^*$. Then the statement of the theorem follows with the above considerations immediately:

$$p_{r,s_0} \ \leq \ \sum_{i=0}^{i^*(n)} p_i \ \leq \ \frac{1}{2\omega}\sum_{i\geq 0} 2^{-i} \ \leq \ \frac{1}{\omega}.$$

In the remainder we assume that $0 \leq i \leq i^*$ is fixed. Set $s_i := 2^i s_0$ and $t_i := r^2 \frac{n(n-1)}{s_i(n-s_i)}$. If we add t_i edges uniformly at random to $G_{n,M}$, one of the following two events can occur:

(i) $\Pi^* = \Pi^*(G_{n,M}) = (A^*, B^*)$ remains one of the optimal bipartitions

(ii) a bipartition different from Π^* "overtakes" Π^*, i.e., its size is larger than the size of Π^* after having added t_i edges to $G_{n,M}$.

Denote by $\mathcal{I}_{t_i}$ the increase of the size of Π^*, and by $\mathbb{1}_{\mathcal{O}}$ the indicator variable of the event that a bipartition Π different from Π^* (if it exists in $G_{n,M}$) overtakes Π^*. Due to linearity of expectation, we obtain

$$\mathbb{E}\left[b(G_{n,M+t_i}) - b(G_{n,M})\right] \geq \mathbb{E}\left[\mathcal{I}_{t_i}\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{O}}\right]. \tag{4.5}$$

We first bound $\mathbb{E}\left[\mathcal{I}_{t_i}\right]$. For this, observe that Π^* increases by the number of edges among the t_i added that go across Π^*; therefore, the expected increase of $e(G_{n,M};\ \Pi^*)$ is t_i times the proportion of *non*-edges across Π^* in $G_{n,M}$. Let $C' > 0$ be the constant that is guaranteed by Corollary 2.5. We apply Corollary 2.5 to obtain bounds for the minimum number of non-edges across any maximum bipartition of $G_{n,M}$, which hold with exponentially high probability; furthermore, due to our assumption $M \leq \frac{1}{2}\binom{n}{2}$ we obtain that $\binom{n}{2} - M \geq \frac{1}{4}n^2$. These facts imply that, for sufficiently large n, it holds

$$\mathbb{E}\left[\mathcal{I}_{t_i}\right] \ \geq\ (1 - 2e^{-n}) \cdot t_i \cdot \frac{\frac{1}{2}\left(\binom{n}{2} - M\right) - \sqrt{\frac{C'n^5}{M}}}{\binom{n}{2} - M} \ \geq\ t_i \cdot \left(\frac{1}{2} - \sqrt{\frac{20C'n}{M}}\right). \tag{4.6}$$

Next we bound $\mathbb{E}\left[\mathbb{1}_{\mathcal{O}}\right]$ from below. Suppose that there is a bipartition $\Pi = (A, B)$ in $G_{n,M}$ with the properties $\operatorname{gap}(G_{n,M};\ \Pi) = r - 1$ and $\operatorname{dist}(\Pi, \Pi^*) =: s$ with $s_i \leq s \leq 2s_i$. Denote by $\mathcal{E}_{\mathcal{B}al}$ the event that

$$|A^*|, |B^*|, |A|, |B| \geq \frac{n}{2} - 4n^{\frac{3}{4}}p^{-\frac{1}{4}} \tag{4.7}$$

and observe that Proposition 2.4 implies $\Pr\left[G_{n,M} \in \mathcal{E}_{\mathcal{B}al}\right] \geq 1 - e^{-n}$, with room to spare. In order to avoid ambiguities, we chose the notation without loss of generality such that $|A^*| \geq |B^*|$ and

$$s = |A \cap B^*| + |A^* \cap B| \leq |A \cap A^*| + |B \cap B^*|.$$

Note that if we add t_i edges uniformly at random to $G_{n,M}$, edges added between $A^* \cap B$ and $A^* \cap A$, and between $B^* \cap A$ and $B^* \cap B$, contribute to $e(G_{n,M};\ \Pi)$, while edges added between $A^* \cap B$ and $B^* \cap B$, and between $B^* \cap A$ and $A^* \cap A$, contribute to $e(G_{n,M};\ \Pi^*)$. All other added edges contribute simultaneously to both Π^* and Π, or to none of them. This motivates the definition of two ordered partitions

$$\Pi_{\mathcal{X}} := (A^* \cap B, A^* \cap A, B^* \cap A, B^* \cap B) \ \text{ and } \ \Pi_{\mathcal{Y}} := (A^* \cap B, B^* \cap B, B^* \cap A, A^* \cap A).$$

Additionally, let $|\Pi_{\mathcal{X}}|$, $\overline{e}(G_{n,M}; \Pi_{\mathcal{X}})$, and similarly $|\Pi_{\mathcal{Y}}|$ and $\overline{e}(G_{n,M}; \Pi_{\mathcal{Y}})$, be defined as in Lemma 4.1, see (4.1). As discussed above, the motivation for this definition is that the event $\mathcal{O}$ will occur, if out of the t_i edges added to $G_{n,M}$, the number of edges added among those counted by $\overline{e}(G_{n,M}; \Pi_{\mathcal{X}})$ is *at least* r plus the number of edges added among those counted by $\overline{e}(G_{n,M}; \Pi_{\mathcal{Y}})$. We shall denote this event by $\mathcal{E}_{\mathcal{X}\mathcal{Y}}$.

In subsequent steps of the proof we will need bounds for the number of non-edges in $\Pi_{\mathcal{X}}$ and $\Pi_{\mathcal{Y}}$, i.e., for the quantities $\overline{e}(G_{n,M};\ \Pi_{\mathcal{X}})$ and $\overline{e}(G_{n,M};\ \Pi_{\mathcal{Y}})$. Let $\tilde{C}$ be the constant guaranteed by Lemma 4.1 and define the event

$$\mathcal{E}_{edges} := \bigcap_{\mathcal{S} \in \{\mathcal{X}, \mathcal{Y}\}} \left\{ G \in \mathcal{G}_{n,M} : \left|\overline{e}(G;\ \Pi_{\mathcal{S}}) - (1-p)|\Pi_{\mathcal{S}}|\right| \leq \tilde{C}s^{-1/2}(1-p)|\Pi_{\mathcal{S}}| \right\}. \tag{4.8}$$

Observe that due to our assumptions we have

$$|A^* \cap B| + |B^* \cap A| = s \geq s_0 \geq r^4\sqrt{np^{-1}} \geq \sqrt{n}. \tag{4.9}$$

Furthermore, due to (4.7) we have for sufficiently large n that

$$|A^* \cap B|, |A \cap B^*| \leq \frac{s}{2} + 4n^{3/4}p^{-1/4} \overset{(s\leq n/2)}{\leq} \frac{n}{4} + 4n^{3/4}p^{-1/4} \leq \frac{n}{3}. \tag{4.10}$$

Hence, whenever n is sufficiently large, we obtain that

$$|A \cap A^*| = |A| - |A \cap B^*| \geq \frac{n}{2} - 4n^{3/4}p^{-1/4} - \frac{n}{3} \geq \frac{n}{7},$$

and similarly, $|B \cap B^*| \geq \frac{n}{7}$. This implies with (4.9) that $|\Pi_{\mathcal{X}}|$ and $|\Pi_{\mathcal{Y}}|$ fulfill the conditions of Lemma 4.1. Therefore $\Pr\left[G_{n,M} \in \mathcal{E}_{edges}\right] \geq 1 - e^{-n}$.

Recall that p_i denotes the probability that there is a bipartition $\Pi \neq \Pi^*$ with gap $r-1$ and distance between $2^i s_0$ and $2^{i+1}s_0$ (but at most $\frac{n}{2}$) from Π^*, and denote by $\mathcal{E}_i$ the corresponding event in $G_{n,M}$. The above discussion yields

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{O}}\right] \geq \Pr\left[\mathcal{E}_i \wedge \mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}\right] \cdot \Pr\left[\mathcal{E}_{\mathcal{X}\mathcal{Y}} \mid \mathcal{E}_i \wedge \mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}\right]. \tag{4.11}$$

Note that due to Proposition 2.4 and Lemma 4.1, as discussed above, we can estimate

$$\Pr\left[\mathcal{E}_i \wedge \mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}\right] = p_i - \Pr\left[\mathcal{E}_i \setminus (\mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges})\right] \geq p_i - \Pr\left[\overline{\mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}}\right] \geq p_i - 2e^{-n}.$$

Our aim is to show that there is a constant $c_1 > 0$ such that

$$\Pr\left[\mathcal{E}_{\mathcal{X}\mathcal{Y}} \mid \mathcal{E}_i \wedge \mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}\right] \;\geq\; c_1 \cdot r^{-2}. \tag{4.12}$$

This will complete the proof of the theorem, as from (4.4), where we set $t = t_i$, (4.5), (4.6), (4.11) and the above estimate we obtain

$$t_i \cdot \left(\frac{1}{2} - \sqrt{\frac{20C'n}{M}}\right) + (p_i - 2e^{-n}) \cdot \frac{c_1}{r^2} \;\leq\; t_i \cdot \left(\frac{1}{2} + \sqrt{\frac{5n}{M}}\right),$$

which implies with $s \leq \frac{n}{2}$ and hence $t_i = r^2\frac{n(n-1)}{s_i(n-s_i)} \leq \frac{2r^2n}{s_i}$ that we can choose a constant c_2 such that for sufficiently large n

$$p_i \;\leq\; c_2 \cdot t_i \cdot \sqrt{\frac{n}{M}} \cdot r^2 \;\leq\; 4c_2 \cdot \frac{r^4 n}{s_i}\sqrt{\frac{1}{pn}} \;\leq\; \frac{8c_2 \cdot r^4}{2^{i+1} \cdot C \cdot \omega \cdot r^4 \cdot \sqrt{np^{-1}}}\sqrt{\frac{n}{p}} \;\leq\; \frac{1}{2^{i+1}\omega},$$

if we set $C := 8c_2$.

Let $q := \Pr\left[\mathcal{E}_{\mathcal{X}\mathcal{Y}} \mid \mathcal{E}_i \wedge \mathcal{E}_{\mathcal{B}al} \wedge \mathcal{E}_{edges}\right]$; to complete the proof we estimate q so as to obtain (4.12). Let for brevity $\overline{e}_{\mathcal{X}} := \overline{e}(G_{n,M}; \Pi_{\mathcal{X}})$ and $\overline{e}_{\mathcal{Y}} := \overline{e}(G_{n,M}; \Pi_{\mathcal{Y}})$. Furthermore, let $\overline{E}_{\mathcal{X}}$ denote the *set* of edges counted in $\overline{e}_{\mathcal{X}}$, and define similarly $\overline{E}_{\mathcal{Y}}$. In order to obtain a lower bound for q, it suffices to consider the event that precisely $x_i := \frac{s(n-s)}{n(n-1)}t_i$ edges out of $\overline{E}_{\mathcal{Y}}$ are added to $G_{n,M}$ and $x + r$ out of $\overline{E}_{\mathcal{X}}$. Suppose that we add the t_i edges one after the other – then there are $\binom{t_i}{x_i+r+1,x_i}$ ways to choose the points in time at which those edges are taken of $\overline{E}_{\mathcal{X}}$ and $\overline{E}_{\mathcal{Y}}$. Furthermore, the probability that an edge out of $\overline{E}_{\mathcal{X}}$ is added to $G_{n,M}$ is at least $\frac{\overline{e}_{\mathcal{X}}-t_i}{\binom{n}{2}-M}$, and similarly, the probability that an added edge is contained in $\overline{E}_{\mathcal{Y}}$ can be bounded from below by $\frac{\overline{e}_{\mathcal{Y}}-t_i}{\binom{n}{2}-M}$. Moreover, the probability that an added edge belongs neither to $\overline{E}_{\mathcal{X}}$ nor $\overline{E}_{\mathcal{Y}}$ is at least $1-\frac{\overline{e}_{\mathcal{X}}+\overline{e}_{\mathcal{Y}}}{\binom{n}{2}-M-t_i}$. Putting all together yields

$$q \geq \binom{t_i}{x_i, x_i+r} \cdot \left(\frac{\overline{e}_{\mathcal{X}}-t_i}{\binom{n}{2}-M}\right)^{x_i+r} \left(\frac{\overline{e}_{\mathcal{Y}}-t_i}{\binom{n}{2}-M}\right)^{x_i} \left(1-\frac{\overline{e}_{\mathcal{X}}+\overline{e}_{\mathcal{Y}}}{\binom{n}{2}-M-t_i}\right)^{t_i-2x_i-r}. \tag{4.13}$$

We now simplify the above expression so as to be able to apply Proposition 4.2 to obtain (4.12). As a first preparation, we derive tight bounds for the quantities $|\Pi_{\mathcal{X}}|$ and $|\Pi_{\mathcal{Y}}|$. Let $s^{(1)} := |A^* \cap B|$, $s^{(2)} := |B^* \cap A|$ and note that $s = s^{(1)} + s^{(2)}$. Furthermore, define the quantity $\delta := s^{(1)} - \frac{s}{2}$. Observe that

$$|\Pi_{\mathcal{Y}}| = s^{(1)}(|B^*| - s^{(2)}) + s^{(2)}(|A^*| - s^{(1)}) = \frac{s(n-s)}{2} - \underbrace{\delta(|A^*| - |B^*| - 2\delta)}_{:=\Delta}.$$

Due to the event $\mathcal{E}_{\mathcal{B}al}$ we have $\big||A^*| - |B^*|\big| \leq 8n^{3/4}p^{-1/4}$ and $|s^{(1)} - s^{(2)}| \leq 8n^{3/4}p^{-1/4}$. This implies with the above definitions

$$|\delta| \leq \min\left\{\frac{s}{2}, 4n^{3/4}p^{-1/4}\right\} \quad \text{and hence} \quad |\Delta| \leq 32 \min\left\{sn^{3/4}p^{-1/4}, n^{3/2}p^{-1/2}\right\}.$$

Due to $s \geq s_0 = C\omega r^4 \sqrt{np^{-1}}$ we can deduce

$$|\Pi_{\mathcal{Y}}| = \frac{s(n-s)}{2}\Big(1 + \Theta\big(\min\left\{(pn)^{-1/4}, \omega^{-1}r^{-4}\right\}\big)\Big) \overset{(r \ll (pn)^{1/8})}{=} \frac{s(n-s)}{2}\big(1 + o(r^{-2})\big). \tag{4.14}$$

Now note that $|\Pi_{\mathcal{X}}| + |\Pi_{\mathcal{Y}}| = s(n-s)$ implies $|\Pi_{\mathcal{X}}| = \frac{s(n-s)}{2} + \Delta$; hence the above statement is also valid for $|\Pi_{\mathcal{X}}|$.

Before we proceed let us make some auxiliary calculations, which we will need in the remainder. First,

$$x_i = \frac{s(n-s)}{n(n-1)} \cdot t_i = \frac{s(n-s)}{n(n-1)} \cdot r^2 \frac{n(n-1)}{s_i(n-s_i)} \overset{(s_i \leq s \leq 2s_i)}{\leq} 2r^2.$$

Due to (4.14) we may assume for sufficiently large n $|\Pi_{\mathcal{X}}|, |\Pi_{\mathcal{Y}}| \geq \frac{sn}{5}$, which implies with the event $\mathcal{E}_{edges}$ that we have $\overline{e}_{\mathcal{X}} \geq (1 - \tilde{C}s^{-1/2})(1-p)|\Pi_{\mathcal{X}}| \geq \frac{sn}{11}$. With $s_i \leq s \leq \frac{n}{2}$ and $s_i \geq s_0 \geq \omega r^4 \sqrt{np^{-1}}$ we can deduce if n is large enough

$$\frac{t_i x_i}{\overline{e}_{\mathcal{Y}}} \leq \frac{11}{sn} \cdot \frac{r^2 n(n-1)}{s_i(n-s_i)} \cdot 2r^2 \leq \frac{22r^4}{s_i^2} = o(1) \quad \text{and} \quad s^{-\frac{1}{2}} x_i \leq \frac{2r^2}{(\omega r^4 \sqrt{np^{-1}})^{\frac{1}{2}}} \leq n^{-\frac{1}{4}}. \tag{4.15}$$

Now we proceed with simplifying (4.13). Using the inequality $1 - y \geq e^{-2y}$, which is valid for e.g. $0 \leq y \leq \frac{1}{2}$, the third term in right-hand side of (4.13) can be estimated for sufficiently large n with

$$\begin{aligned}\left(\frac{\overline{e}_{\mathcal{Y}}-t_i}{\binom{n}{2}-M}\right)^{x_i} &\geq \left(\frac{\overline{e}_{\mathcal{Y}}}{\binom{n}{2}-M}\right)^{x_i}\cdot\left(1-\frac{t_i}{\overline{e}_{\mathcal{Y}}}\right)^{x_i}\\ &\overset{(\mathcal{E}_{edges})}{\geq} \left(\frac{(1-\tilde{C}s^{-1/2})(1-p)|\Pi_{\mathcal{Y}}|}{(1-p)\binom{n}{2}}\right)^{x_i}\cdot e^{-\frac{2t_ix_i}{\overline{e}_{\mathcal{Y}}}}\\ &\overset{(4.15)}{\geq} \left(\frac{|\Pi_{\mathcal{Y}}|}{\binom{n}{2}}\right)^{x_i}\cdot\left(1-\tilde{C}s^{-1/2}\right)^{x_i}\cdot\frac{1}{2} \qquad (4.16)\\ &\overset{(4.14),(4.15)}{\geq} \left(\frac{s(n-s)}{2\binom{n}{2}}\right)^{x_i}(1+o(r^{-2}))^{x_i}\cdot e^{-\frac{4\tilde{C}}{n^{1/4}}}\cdot\frac{1}{2}\\ &\geq \frac{1}{4}\left(\frac{s(n-s)}{2\binom{n}{2}}\right)^{x_i}.\end{aligned}$$

Precisely the same calculation yields that the second term of the right-hand side of (4.13) is for sufficiently large n at least

$$\left(\frac{\overline{e}_{\mathcal{X}}-t_i}{\binom{n}{2}-M}\right)^{x_i+r} \geq \frac{1}{4}\left(\frac{s(n-s)}{2\binom{n}{2}}\right)^{x_i+r}. \qquad (4.17)$$

In order to simplify the last term in (4.13), first recall that $|\Pi_{\mathcal{X}}|+|\Pi_{\mathcal{Y}}|=s(n-s)$. Due to the event $\mathcal{E}_{edges}$ it holds that $\overline{e}_{\mathcal{X}}+\overline{e}_{\mathcal{Y}}\leq(1+\tilde{C}s^{-1/2})(1-p)s(n-s)$. Abbreviate $m:=\binom{n}{2}-M=(1-p)\binom{n}{2}$, and observe that we have $m\geq\frac{1}{5}n^2$. This yields

$$\left(1-\frac{\overline{e}_{\mathcal{X}}+\overline{e}_{\mathcal{Y}}}{m-t_i}\right)^{t_i-2x_i-r} \geq \left(1-\frac{(1+\tilde{C}s^{-1/2})(1-p)s(n-s)}{(1-\frac{5t_i}{n^2})(1-p)\binom{n}{2}}\right)^{t_i-2x_i-r}.$$

Note that due to $r\ll(pn)^{1/8}\leq n^{1/8}$ we have $\frac{5t_i}{n^2}\ll\frac{1}{n}$, which implies $s^{-1/2}(1-\frac{5t_i}{n^2})^{-1}\leq 2s^{-1/2}$, whenever n is sufficiently large. Thus, using that for sufficiently small y the estimate $1-y\geq e^{-2y}$ is true, we obtain for large n that

$$\begin{aligned}\left(1-\frac{\overline{e}_{\mathcal{X}}+\overline{e}_{\mathcal{Y}}}{m-t_i}\right)^{t_i-2x_i-r} &\geq \left(1-\frac{s(n-s)}{\binom{n}{2}}\right)^{t_i-2x_i-r}\left(1-5\tilde{C}s^{-1/2}\frac{s(n-s)}{\binom{n}{2}}\right)^{t_i}\\ &\geq \left(1-\frac{s(n-s)}{\binom{n}{2}}\right)^{t_i-2x_i-r}e^{-20\tilde{C}s^{-1/2}x_i} \qquad (4.18)\\ &\overset{(4.15)}{\geq} \frac{1}{2}\left(1-\frac{s(n-s)}{\binom{n}{2}}\right)^{t_i-2x_i-r}.\end{aligned}$$

By combining (4.16), (4.17), and (4.18) we obtain from (4.13) that

$$q \;\geq\; \frac{1}{32}\binom{t_i}{x_i,x_i+r}\left(\frac{s(n-s)}{2\binom{n}{2}}\right)^{2x_i+r}\left(1-\frac{s(n-s)}{\binom{n}{2}}\right)^{t_i-2x_i-r}.$$

Now it can be easily checked that we can apply Proposition 4.2 to estimate the above expression, where we set $N := t_i$, $\alpha := \frac{s(n-s)}{n(n-1)}$, $d := r$. Indeed, α reaches its maximum value when $s = \frac{n}{2}$, and we obtain for $n \geq 4$ that $\alpha \leq \frac{1}{3} < \frac{1}{2}$; furthermore, $t_i = r^2 \frac{n(n-1)}{s_i(n-s_i)}$ implies $d \leq \sqrt{\min\{\alpha, 1-2\alpha\}N}$. Hence, if we denote by C'' the constant defined by Proposition 4.2, we obtain due to $s_i \leq s \leq 2s_i$

$$q \;\geq\; \frac{1}{64} \cdot \frac{C''}{x_i} \;\geq\; \frac{C''}{64} \cdot \frac{1}{r^2} \frac{s_i(n-s_i)}{s(n-s)} \;\geq\; c_1 r^{-2},$$

which is precisely (4.12), if we choose c_1 appropriately. This completes the proof. ▪

5 Proof of Theorem 1.3

Let $n^{-1/250} \leq p \leq \frac{1}{2}$. Furthermore, define the functions

$$r_0 = r_0(p,n) := p^{-12} \log^2 n \quad \text{and} \quad s(r) := n^{2/3} \cdot (r+1)^4,$$

and abbreviate $s_0 := s(2r_0)$. Before we continue, we need to introduce some notation. Let $G \in \mathcal{G}_n$ be a graph with vertex set $[n] := \{1, \dots, n\}$. A bipartition of $[n]$ is said to be *balanced*, if it is contained in the set

$$\mathcal{B}al_n := \left\{ (A,B) \mid \left| |A| - \frac{n}{2} \right| \leq \frac{n}{100} \text{ and } \left| |B| - \frac{n}{2} \right| \leq \frac{n}{100} \right\}.$$

Recall that $\mathcal{T}(G)$ denotes the set of maximum cardinality triangle-free subgraphs of a graph G. Define the two "bad" events

$$\begin{aligned} \mathcal{B}_1 &:= \Big\{ G \in \mathcal{G}_n \mid \exists T \in \mathcal{T}(G) : \text{there is no balanced bipartition } \Pi \\ &\qquad\qquad \text{such that at most } r_0 \text{ edges of } T \text{ are not across } \Pi \Big\}, \qquad (5.1) \\ \mathcal{B}_2 &:= \Big\{ G \in \mathcal{G}_n \mid \exists \Pi : \operatorname{gap}(G;\, \Pi) \leq 2r_0 \text{ and } \operatorname{dist}(G;\, \Pi) \geq \frac{s\big(\operatorname{gap}(G;\, \Pi)\big)}{3} \Big\}. \end{aligned}$$

Moreover, let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and define the "good" event $\overline{\mathcal{B}} = \mathcal{G}_n \setminus \mathcal{B}$.

In the sequel we estimate the probability that there is a $T \in \mathcal{T}(G_{n,p})$ that is *not* bipartite. Observe that this implies the existence of a bipartition $\Pi_T = \Pi = (A,B)$ of the vertex set with the property that we can obtain T if we consider the subgraph $E(G_{n,p};\, \Pi)$ of the random graph, remove $t > 0$ edges from it and add *at least* $t + \operatorname{gap}(G_{n,p};\, \Pi)$ edges from $E(G_{n,p}) \setminus E(G_{n,p};\, \Pi)$ to it. Accordingly, for a fixed Π and any set S of edges not across Π, let

$$\begin{aligned} \mathcal{E}(\Pi, S) := \big\{ G \in \mathcal{G}_n \mid \exists X \subseteq E(G;\, \Pi) : (E(G;\, \Pi) \setminus X) \cup S \text{ is triangle-free} \\ \text{and } |S| - |X| \geq \operatorname{gap}(G;\, \Pi) \big\}. \end{aligned} \qquad (5.2)$$

Now let us assume that $G_{n,p} \in \overline{\mathcal{B}}$. Then the event that there is a $T \in \mathcal{T}(G_{n,p})$ which is not bipartite, implies the existence of a balanced partition $\Pi = (A,B)$, which can be "enhanced" by at least one and at most r_0 edges, with both endpoints in A or B, and possibly by removing any number $\leq r_0$ of edges with one endpoint in A and one in B, such that we obtain a triangle-free graph. The above definition of the event $\mathcal{E}(\Pi, S)$ thus implies that we have

$$\begin{aligned}
&\Pr[\exists T \in \mathcal{T}(G_{n,p}) : T \text{ is not bipartite}] \\
&\qquad \leq \Pr\left[G_{n,p} \in \mathcal{B}\right] + \sum_{\substack{\Pi=(A,B)\in \mathcal{B}al_n \\ S\in\binom{A}{2}\cup\binom{B}{2}, 1\leq |S|\leq r_0}} \Pr\left[G_{n,p} \in (\mathcal{E}(\Pi, S) \cap \overline{\mathcal{B}})\right]. \qquad (5.3)
\end{aligned}$$

Let C be the constant guaranteed to exist by Theorem 1.4. By applying Lemma 3.11 and Theorem 1.4, for all $r \in [1, 2r_0 + 1]$ and $\omega := \frac{3}{C} n^{1/6-1/250}$, the first term on the right hand side of (5.3) can be bounded as follows for sufficiently large n:

$$\begin{aligned}
\Pr\left[G_{n,p} \in \mathcal{B}\right] \;&\leq\; \Pr\left[G_{n,p} \in \mathcal{B}_1\right] + \Pr\left[G_{n,p} \in \mathcal{B}_2\right] \\
&\leq\; o(1) + \sum_{r=1}^{2r_0+1} \frac{1}{\omega} \;=\; o(1) + \frac{2n^{12/250}\log^2 n + 1}{\omega} \;=\; o(1). \qquad (5.4)
\end{aligned}$$

In the remainder of the proof we will bound the sum on the right hand side of (5.3). In particular, we show that the probability of the joint event $(G_{n,p} \in \mathcal{E}(\Pi, S)) \wedge (G_{n,p} \in \overline{\mathcal{B}})$ can be estimated by the probability that two appropriately defined events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur. This will allow us to use the FKG inequality in order to get a sufficient upper bound for the sum in (5.3). For a fixed $\Pi = (A, B) \in \mathcal{B}al_n$ and a non-empty set $S \subseteq \binom{A}{2} \cup \binom{B}{2}$ of edges, let

$$\begin{aligned}
\mathcal{E}_1(\Pi) &= \Big\{G \in \mathcal{G}_n \mid \operatorname{gap}(G;\, \Pi) \leq r_0 \text{ and } \forall \Pi' : \big(\operatorname{gap}(G;\, \Pi, \Pi') \leq r_0 \Rightarrow \operatorname{dist}(\Pi, \Pi') \leq s_0\big)\Big\}, \\
\mathcal{E}_2(\Pi, S) &= \Big\{G \in \mathcal{G}_n \mid \exists X \subset E(G; \Pi) : (E(G; \Pi) \setminus X) \cup S \text{ is } \triangle\text{-free, and } |X| \leq |S|\Big\}. \qquad (5.5)
\end{aligned}$$

Note that the definition of $\mathcal{E}_1$ seems overly complicated, and one could think that an event of the type "there is no partition Π with $\operatorname{gap}(G;\, \Pi) \leq 2r_0$ and $\operatorname{dist}(G;\, \Pi) \geq s_0$", which would follow directly from the definition (5.1) of $\mathcal{B}_2$, could be sufficient as well. It turns out (see Proposition 5.2), that this is in fact the most delicate point of our proof: we *need* to relax the event, so that it becomes an increasing function in an appropriately defined partial ordering of all graphs with M edges. As a consequence, we may use the FKG inequality to get a sufficient estimate.

First we show that in fact we can bound the probability from (5.3) with the joint probability of the events $\mathcal{E}_1$ and $\mathcal{E}_2$.

Proposition 5.1. *For all* $\Pi = (A, B) \in \mathcal{B}al_n$ *and* $S \subseteq \binom{A}{2} \cup \binom{B}{2}$ *with* $1 \leq |S| \leq r_0$

$$\Pr\left[(G_{n,p} \in \mathcal{E}(\Pi, S)) \wedge (G_{n,p} \in \overline{\mathcal{B}})\right] \leq \Pr\left[(G_{n,p} \in \mathcal{E}_1(\Pi)) \wedge (G_{n,p} \in \mathcal{E}_2(\Pi, S))\right]. \qquad (5.6)$$

Proof. Suppose that the event $(G_{n,p} \in \mathcal{E}(\Pi, S)) \wedge (G_{n,p} \in \overline{\mathcal{B}})$ occurs. From (5.2) we deduce that $\operatorname{gap}(G_{n,p};\, \Pi) \leq |S| \leq r_0$; hence, $G_{n,p} \notin \mathcal{B}_2$ implies that $\operatorname{dist}(G_{n,p};\, \Pi) \leq \frac{s(r_0)}{3}$. Now consider any partition Π' different from Π, which fulfills $\operatorname{gap}(G_{n,p};\, \Pi, \Pi') \leq r_0$. Clearly, we have that $\operatorname{gap}(G_{n,p};\, \Pi') \leq 2r_0$, and consequently $G_{n,p} \notin \mathcal{B}_2$ implies $\operatorname{dist}(G_{n,p};\, \Pi') \leq \frac{s(2r_0)}{3}$. Furthermore, due to the event $G_{n,p} \in \overline{\mathcal{B}}$ any two *maximum* bipartitions of $G_{n,p}$ have distance at most $\frac{s(0)}{3}$. As the distance of Π and Π' is at most the sum of their distances from any maximum bipartition *plus* the largest distance of any two maximum bipartitions, we obtain

$$\operatorname{dist}(\Pi, \Pi') \leq \operatorname{dist}(G_{n,p};\, \Pi) + \operatorname{dist}(G_{n,p};\, \Pi') + \frac{s(0)}{3} \leq s(2r_0) = s_0.$$

That is, we have $G_{n,p} \in \mathcal{E}_1(\Pi)$. The event $\mathcal{E}_2(\Pi, S)$ is easily seen to hold simultaneously, as it is a relaxation of the condition in (5.2). ▪

The next proposition states that we can estimate the probability from (5.6) with the *product* of the probabilities of the events $\mathcal{E}_1$ and $\mathcal{E}_2$. Its proof consists of the definition of an appropriate distributive lattice on graphs with respect of bipartions of the vertex set, and a subsequent application of the FKG inequality.

Proposition 5.2. *For all* $\Pi = (A, B) \in \mathcal{B}al_n$ *and* $S \subset \binom{A}{2} \cup \binom{B}{2}$ *with* $1 \leq |S| \leq r_0$

$$\Pr\left[(G_{n,p} \in \mathcal{E}_1(\Pi)) \wedge (G_{n,p} \in \mathcal{E}_2(\Pi, S))\right] \leq \Pr\left[G_{n,p} \in \mathcal{E}_1(\Pi)\right] \cdot \Pr\left[G_{n,p} \in \mathcal{E}_2(\Pi, S)\right].$$

Proof. For the proof we need a variant of the FKG inequality which we are going to state now; a far more general treatment of the topic can be found in [AS00]. A *lattice* is a partially ordered set $(S, \leq)$ (with ground set S and a partial order $\leq$ on S) in which every two elements x and y have a unique minimal upper bound and and a unique maximal lower bound, which we denote by $x \vee y$ and $x \wedge y$ respectively. $\mathcal{L}$ is called *distributive*, if for all $x, y, z \in \mathcal{L}$ we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{5.7}$$

Next, we need the concepts of *log-supermodular*, *increasing*, and *decreasing* functions. A function $f : S \to \mathbb{R}^+$ is called *log-supermodular*, if for all $x, y \in S$ it holds

$$f(x)f(y) \leq f(x \vee y)f(x \wedge y). \tag{5.8}$$

A function f is called *increasing* if $x \leq y$ implies $f(x) \leq f(y)$. Similarly, f is called decreasing if $x \leq y$ implies $f(x) \geq f(y)$. With this definitions we can state a probabilistic version of the well-known FKG inequality: if $\mathcal{L} = (\Omega, \leq)$ is a finite distributive lattice, $f : \Omega \to \mathbb{R}^+$ an increasing function, $g : \Omega \to \mathbb{R}^+$ a decreasing function, and $\mu : \Omega \to \mathbb{R}^+$ a log-supermodular probability measure on Ω, then we have

$$\mathbb{E}\left[f \cdot g\right] \leq \mathbb{E}\left[f\right] \cdot \mathbb{E}\left[g\right]. \tag{5.9}$$

We now prove Proposition 5.2. In order to do so fix some Π and S. As we want to apply inequality (5.9) we first define an appropriate partial ordering on graphs as follows. For two graphs G and H let

$$G \leq_\Pi H \;:\Leftrightarrow\; E(G; \Pi) \subseteq E(H; \Pi) \;\text{ and }\; (E(G) \setminus E(G; \Pi)) \supseteq (E(H) \setminus E(H; \Pi)). \tag{5.10}$$

Intuitively, a graph G is “smaller” than a graph H with respect to $\leq_\Pi$, if it has fewer edges across Π *and* simultaneously more edges in the parts of Π. One easily checks that for any pair of graphs G and H the unique minimal upper bound of G and H is given by

$$G \vee H = \big(E(G; A) \cap E(H; A)\big) \;\cup\; \big(E(G; B) \cap E(H; B)\big) \;\cup\; \big(E(G; \Pi) \cup E(H; \Pi)\big),$$

while the unique maximal lower bound is given by

$$G \wedge H := \big(E(G; A) \cup E(H; A)\big) \;\cup\; \big(E(G; B) \cup E(H; B)\big) \;\cup\; \big(E(G; \Pi) \cap E(H; \Pi)\big).$$

As probability space we use the $G_{n,p}$, i.e., for any $G \in \mathcal{G}_n$ $\mu(G) := \Pr[G_{n,p} = G] = p^{e(G)}(1 - p)^{\binom{n}{2}-e(G)}$. An easy calculation yields that

$$\begin{aligned}\mu(G)\mu(H) &= p^{e(G)+e(H)}(1-p)^{2\binom{n}{2}-e(G)-e(H)} \\ &= p^{e(G\vee H)+e(G\wedge H)}(1-p)^{2\binom{n}{2}-e(G\vee H)-e(G\wedge H)} = \mu(G\vee H)\mu(G\wedge H),\end{aligned}$$

i.e. μ is log-supermodular. Furthermore, it can easily be verified that the above defined operators are distributive, i.e., they fulfill (5.7). We leave the details to the reader.

Let $\mathcal{E}_1 := \mathcal{E}_1(\Pi)$ and $\mathcal{E}_2 := \mathcal{E}_2(\Pi, S)$. For $i = 1, 2$ we denote for a graph G by

$$f_i(G) := \begin{cases} 1, & \text{if } G \in \mathcal{E}_i \\ 0, & \text{otherwise} \end{cases}$$

the indicator function for the event $\mathcal{E}_i$. In the sequel we shall show that f_2 is decreasing with respect to $\leq_\Pi$, and that f_1 is increasing. This will conclude the proof, as the above discussion yields that the conditions for (5.9) are fulfilled.

First we prove that f_2 is decreasing. For this it obviously suffices to show that if $G \leq_\Pi H$, then $H \in \mathcal{E}_2$ implies that $G \in \mathcal{E}_2$. To see this observe that $H \in \mathcal{E}_2$ implies that there is a set $X \subset E(H;\, \Pi)$ such that

$$(E(H;\, \Pi) \setminus X) \cup S \text{ is triangle-free, and } |X| \leq |S|.$$

Due to $G \leq_\Pi H$ we have $E(G;\, \Pi) \subseteq E(H;\, \Pi)$, and let $X' := X \cap E(G;\, \Pi)$. Clearly, $|X'| \leq |S|$, and as $(E(H;\, \Pi) \setminus X) \cup S$ is triangle free, so is $(E(G;\, \Pi) \setminus X') \cup S$. But this means $G \in \mathcal{E}_2$, as desired.

Finally, we prove that f_1 is increasing. For this, we show that if $G \leq_\Pi H$, then $G \in \mathcal{E}_1$ implies $H \in \mathcal{E}_1$. Observe that by transitivity it is sufficient to consider the case that G and H differ in exactly one edge e. The event $G \in \mathcal{E}_1$ implies

$$\operatorname{gap}(G;\, \Pi) \leq r_0 \;\text{ and }\; \forall \Pi' \text{ such that } \operatorname{gap}(G;\, \Pi, \Pi') \leq r_0 : \operatorname{dist}(\Pi, \Pi') \leq s_0. \tag{5.11}$$

Now we make a case distinction. First assume that e joins two vertices in A or two vertices in B. Then, due to (5.10), we have $H = G \setminus \{e\}$. Note that this implies that the size of a maximum bipartition satisfies $b(G) - 1 \leq b(H) \leq b(G)$. As furthermore $E(G; \Pi) = E(H; \Pi)$ we thus have $\operatorname{gap}(H;\, \Pi) \leq \operatorname{gap}(G;\, \Pi) \leq r_0$. Now let Π' be a bipartition which has the property $\operatorname{gap}(H;\, \Pi, \Pi') \leq r_0$. Observe that $e(H;\, \Pi') \leq e(G;\, \Pi')$. We easily deduce

$$\operatorname{gap}(G;\, \Pi, \Pi') \;=\; e(G;\, \Pi) - e(G;\, \Pi') \;\leq\; e(H;\, \Pi) - e(H;\, \Pi') \;=\; \operatorname{gap}(H;\, \Pi, \Pi') \;\leq\; r_0,$$

which implies with (5.11) that $\operatorname{dist}(\Pi, \Pi') \leq s_0$.

Now assume that e joins a vertex in A with a vertex in B. Then $H = G \cup \{e\}$. In this case we have $b(G) \leq b(H) \leq b(G) + 1$ and $E(H;\, \Pi) = E(G;\, \Pi) \cup \{e\}$. This immediately implies

$$\operatorname{gap}(H;\, \Pi) = b(H) - e(H;\, \Pi) \leq b(G) + 1 - (e(G;\, \Pi) + 1) = \operatorname{gap}(G;\, \Pi) \leq r_0.$$

Now let again Π' be a bipartition with $\operatorname{gap}(H;\, \Pi, \Pi') \leq r_0$. Note that $e(G;\, \Pi') \geq e(H;\, \Pi') - 1$, as the edge e does not neccesarily join two vertices in different parts of Π'. This implies

$$\operatorname{gap}(G;\, \Pi, \Pi') = e(G;\, \Pi) - e(G;\, \Pi') \leq (e(H;\, \Pi) - 1) - (e(H;\, \Pi') - 1) \leq \operatorname{gap}(H;\, \Pi, \Pi') \leq r_0.$$

Hence, (5.11) implies $\operatorname{dist}(\Pi, \Pi') \leq s_0$, as desired. This completes the proof. ▪

As a final ingredient in our proof we need estimates for the probabilities $\Pr[G_{n,p} \in \mathcal{E}_1(\Pi)]$ and $\Pr[G_{n,p} \in \mathcal{E}_2(\Pi, S)]$. These are given by the next proposition.

Proposition 5.3. *For all* $\Pi = (A, B) \in \mathcal{B}al_n$ *and* $S \subseteq \binom{A}{2} \cup \binom{B}{2}$ *with* $1 \leq |S| \leq r_0$ *it holds for sufficiently large* n

$$\Pr[G_{n,p} \in \mathcal{E}_2(\Pi, S)] \leq e^{-\frac{p^2 n}{12}} \tag{5.12}$$

and

$$\sum_{\Pi \in \mathcal{B}al_n} \Pr[G_{n,p} \in \mathcal{E}_1(\Pi)] \leq n\binom{n}{s_0}. \tag{5.13}$$

Proof. We first bound the probability for the event $\mathcal{E}_2$. Fix an edge $e \in S$ and note that as Π is balanced there exist at least

$$\min\{|A|, |B|\} \geq \frac{n}{2} - \frac{n}{100} = \frac{49}{100}n$$

pairwise vertex-disjoint possible triangles across Π which contain the edge e.

Denote by Y the random variable which counts the number of those triangles in $G_{n,p}$. With the definition of $\mathcal{E}_2$ in (5.5) we deduce

$$\Pr[G_{n,p} \in \mathcal{E}_2(\Pi, S)] \leq \Pr[Y \leq |S|].$$

The probability that a triangle (with e) is contained in $G_{n,p}$ is p^2; hence, $\mathbb{E}[Y] \geq \frac{49}{100}p^2 n$. On the other hand, observe that $|S| \ll \mathbb{E}[Y]$ with our assumptions on p. A simple application of Lemma 2.1 yields for sufficiently large n (5.12).

Next we show (5.13). Trivially, we have

$$\sum_{\Pi \in \mathcal{B}al_n} \Pr[G_{n,p} \in \mathcal{E}_1(\Pi)] = \sum_{\Pi \in \mathcal{B}al_n} \sum_{G \in \mathcal{E}_1(\Pi)} \Pr[G_{n,p} = G]$$

Now we want to interchange the order of summation, such that the first sum goes over (a carefully chosen subset of) all graphs in $\mathcal{G}_n$. To achieve this, observe that the number of times the probability of a graph G is counted above is equal to the number of balanced partitions Π with the properties

$$\text{gap}(G;\ \Pi) \leq r_0,\ \text{ and } \forall \Pi' \text{ such that } \text{gap}(G;\ \Pi, \Pi') \leq r_0 : \text{dist}(\Pi, \Pi') \leq s_0. \tag{5.14}$$

In the following we argue that we can construct all such partitions Π by choosing an arbitrary partition Π^* such that $\text{gap}(G;\ \Pi^*) \leq r_0$, and modifying the parts of Π^* in at most s_0 vertices. To see this, assume that there is a partition Π fulfilling (5.14) with $\text{dist}(\Pi, \Pi^*) > s_0$. But then we have $\text{gap}(\Pi, \Pi^*) \leq r_0$, as the partitions satisfy $\text{gap}(G; \Pi) \leq r_0$ and $\text{gap}(G; \Pi^*) \leq r_0$, which implies $\text{dist}(\Pi, \Pi^*) \leq s_0$ – a contradiction.

Hence, as there are precisely $\sum_{t \leq s_0} \binom{n}{t}$ ways to choose at most s_0 vertices, which change the class they belong to, and for sufficiently large n the inequality $s_0 \leq \frac{n}{2}$ holds, we can conclude

$$\sum_{\Pi \in \mathcal{B}al_n} \Pr[G_{n,p} \in \mathcal{E}_1(\Pi)] \leq \sum_{G \in \mathcal{G}_n} \Bigg(\Pr[G_{n,p} = G] \cdot \sum_{\substack{\Pi \in \mathcal{B}al_n \\ \Pi \text{ fulfills (5.14)}}} 1\Bigg) \leq n\binom{n}{s_0}.$$

∎

Proof of Theorem 1.3. Recall that $n^{-1/250} \leq p \leq \frac{1}{2}$, which implies for sufficiently large n the bounds $r_0 \leq n^{13/250}$ and $s_0 \leq n^{9/10}$. As there are at most $\binom{n^2}{|S|}$ ways to choose a set S of edges out of all possible edges, the proof of the theorem can be completed with inequality (5.3) and Propositions 5.1, 5.2, and 5.3 as follows:

$$\begin{aligned}\Pr\left[\exists T \in \mathcal{T}(G_{n,p}) : T \text{ not bip.}\right] &\leq \Pr\left[\mathcal{B}\right] + \sum_{\substack{\Pi \in \mathcal{B}al_n \\ S:|S| \leq r_0}} \Pr\left[G_{n,p} \in \mathcal{E}_1(\Pi)\right] \cdot \Pr\left[G_{n,p} \in \mathcal{E}_2(\Pi, S)\right] \\ &\leq o(1) + e^{-\frac{p^2 n}{12}} \cdot \binom{n^2}{r_0} \cdot \sum_{\Pi \in \mathcal{B}al_n} \Pr\left[G_{n,p} \in \mathcal{E}_1(\Pi)\right] \\ &\leq o(1) + \exp\left\{ -\frac{p^2 n}{12} + 2r_0 \log n + (s_0+1)\log n \right\} = o(1),\end{aligned}$$

whenever n is chosen sufficiently large. ▪

6 Generalizations & Open problems

Let $\ell \geq 2$. Here we show how the proofs of the previous sections can be adapted in order to prove Theorem 1.5. As a tool, we will use a "higher-dimensional" variant of Theorem 1.4, see Theorem 6.1 below. Before we state it, we need to define the notion of distance for two ℓ-partitions, which is a straightforward generalization of the case $\ell = 2$:

$$\operatorname{dist}(\Pi, \Pi') := \min_{\substack{\pi:[\ell]\to[\ell] \\ \pi \text{ bijection}}} \sum_{\substack{1 \leq i \leq \ell \\ j:\pi(j) \neq i}} \left| V_i \cap V'_{\pi(j)} \right| .$$

The notion of the gap of two ℓ-partitions is defined in the obvious way. The following theorem is a statement about the structure of the set of (near-)optimal ℓ-partitions of the uniform random graph.

Theorem 6.1. *Let $\ell \geq 2$. There are constants $C = C(\ell) > 0$ and $\varepsilon_0 = \varepsilon_0(\ell) > 0$ such that the following holds for sufficiently large n. Let $0 \leq \varepsilon \leq \varepsilon_0$, $n^{-1/\ell} \ll p \leq \frac{1}{2}$ and $M = p\binom{n}{2}$. Furthermore, let $r \geq 1$ satisfy $r \ll (pn)^{1/8}$ and $\omega \gg 1$, and define*

$$s_0 := C \cdot \omega \cdot r^4 \cdot \sqrt{np^{-1}}.$$

Then

$$\Pr\left[\exists \Pi : \operatorname{gap}(G_{n,M};\, \Pi) = r-1 \ \text{ and } \ \operatorname{dist}(G_{n,M};\, \Pi) \geq s_0\right] \ \leq \ \omega^{-1}.$$

The proof of Theorem 1.4 can easily be adapted to show the above theorem. The sole difference is that instead of considering bipartitions we have to consider ℓ-partitions. In this case it is readily seen that (4.5) is still valid, i.e., the expected increase of the size of a maximum ℓ-partition can be estimated from below by

$$\mathbb{E}\left[b(G_{n,M+t_i}) - b(G_{n,M})\right] \geq \mathbb{E}\left[\mathcal{I}_{t_i}\right] + \mathbb{E}\left[\mathbb{1}_{\mathcal{O}}\right],$$

where b denotes the size of maximum ℓ-partition, $\mathcal{I}_{t_i}$ the increase of the size of a fixed maximum ℓ-partition Π^* of $G_{n,M}$, and $\mathbb{1}_{\mathcal{O}}$ the indicator variable for the event that a partition Π with gap $r-1$ becomes an optimal bipartition after adding t_i random edges to $G_{n,M}$. $\mathbb{E}\left[\mathcal{I}_{t_i}\right]$

can be routinely estimated from below. Moreover, to estimate $\mathbb{E}[\mathbb{1}_{\mathcal{O}}]$ we proceed exactly as in the proof of Theorem 1.4: we estimate the probability that the number of edges added that increased the size of Π is at least r plus the number of edges which increased the size of Π^*. We leave the solely technical but straightforward details to the reader.

With the above observations, it is not very surprising that the proof of Theorem 1.3 can be adapted in order to prove Theorem 1.5 – in fact, it turns out that it does not make a difference for our proofs if we consider ℓ-partitions instead of bipartitions of the $G_{n,p}$. However, some details are significantly more tedious than it is above the case, and we shall elaborate more on this issue.

We proceed in three steps, mimicking the proof Theorem 1.3. Let $\mathcal{F}(G_{n,p})$ denote the set of the maximum $\mathcal{K}_\ell$-free subgraphs of $G_{n,p}$, and let $F \in \mathcal{F}(G_{n,p})$. First, we prove a statement similar to that of Lemma 3.8: for every $\varepsilon > 0$ we can find a.a.s. a partition $\Pi = (V_1, \ldots, V_{\ell-1})$ such that all but $\varepsilon p n^2$ edges of F go across Π, and all parts of Π have approximately the same size. The proof is, as in the case of maximum triangle-free graphs, an application of the sparse version of Szemeredi's regularity lemma (Theorem 3.4) and the probabilistic embedding lemma (Theorem 3.6), followed by an application of the stability lemma which results in the desired $(\ell-1)$-partition. The calculations differ only in minor technical details, which are again left to the reader.

Second, we show that in fact we can find a better $(\ell-1)$-partition, i.e., we prove a general version of Lemma 3.11.

Lemma 6.2. *For $\ell \geq 4$ and $p \geq n^{-\frac{1}{100\ell^3}}$ the following holds a.a.s. For every $F \in \mathcal{F}(G_{n,p})$ there is a partition $\Pi_F = \Pi = (V_1, \ldots, V_{\ell-1})$ of the vertex set such that all but $2p^{-5\ell^2}\log^2 n$ edges of F go across Π. Furthermore, $|V_i| = \frac{n}{\ell-1} + o(n)$ for $1 \leq i \leq \ell-1$.*

Before we give the proof details let us state an auxiliary result that we will use several times. It is a statement about the number of pairwise edge-disjoint copies of complete graphs in subgraphs of the $G_{n,p}$.

Proposition 6.3. *$G_{n,p}$ has for every $k \geq 3$ a.a.s. the following property. There are constants $c = c(k)$, $c' = c'(k)$ such that for* every *k disjoint subsets of its vertices $V_1, \ldots, V_k$, where $|V_i| =: s \geq cp^{-k^2}\log n$, the number of pairwise edge-disjoint $\mathcal{K}_k$'s with one vertex in each V_i is $\geq c'ps^2$ and $\leq 2ps^2$.*

Proof. The upper bound is easy to obtain, as the number of edges between the sets V_1 and V_2 is a.a.s. at most $2ps^2$, and each copy of $\mathcal{K}_k$ contains one of those edges. For the lower bound, let us fix $V_1, \ldots, V_k$. We apply Lemma 3.10 $\binom{k}{2}$ times, with $c = 1/2$, and where we assign to k from that lemma the values $1, \ldots, k$, and with $U = V_1, \ldots, V_k$. This defines $\binom{k}{2}$ exceptional classes $X_{i,j}$, where $i \neq j$, such that all j-tuples of vertices out of $[n] \setminus X_{i,j}$ have at most $\frac{3}{2}p^j s$ and at least $\frac{1}{2}p^j s$ common neighbors in V_i, and $|X_{i,j}| = \mathcal{O}\left(p^{-j}\log n\right)$. Let $V_i' := V_i \setminus (\cup_{j=1}^{k} X_{i,j})$, and note that for sufficiently large n we have $|V_i'| \geq \frac{1}{2}|V_i|$.

The proof of the lower bound proceeds in two steps. First, we show that the number of K_k's with one vertex in each V_i' is at least $\frac{1}{2}(\frac{s}{2})^k p^{\binom{k}{2}}$, with probability $\geq 1 - n^{-ks}$. Then we show that for every edge joining (without loss of generality) V_1' to V_2', $G_{n,p}$ is a.a.s. such that the number of K_k's containing that edge (and having one vertex in each of the sets $V_3', \ldots, V_k'$) is $\leq C_k s^{k-2} p^{\binom{k}{2}-1}$. By combining the above statements the lower bound is proved.

To see the first claim, let $V_1, \ldots, V_k$ be disjoint subsets of the vertices of size s, and let for $1 \leq i \leq k$ $W_i \subseteq V_i$ be subsets of them such that $s/2 \leq |W_i| \leq s$. Denote by X the number of $\mathcal{K}_k$'s with one vertex in each of the W_i's. Clearly, $\mathbb{E}[X] \geq (\frac{s}{2})^k \cdot p^{\binom{k}{2}}$ and $\mathbb{E}[X] \leq s^k \cdot p^{\binom{k}{2}}$. We show that X is sharply concentrated by using a variant of Azuma's inequality, see [JŁR00, Corollary 2.27]. Let E be the union of all possible edges between any two sets W_i and W_j, and denote by $\mathcal{G}_E$ the set of all graphs with subsets of E as edges. The inequality says that $\Pr[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-t^2/2S}$, where $S = \sum_{e \in E} c_e^2$, and $c_e \geq \max_{G \in \mathcal{G}_E} |X(G) - X(G \setminus \{e\})|$. Clearly, for every edge $e \in E$ we have $c_e \leq s^{k-2}$, which yields $2S \leq 2|E|s^{2(k-2)} \leq Cs^{2k-2}$, for a suitably chosen constant $C = C(k)$. With $c := 15k2^{2k+2}C$ we obtain for sufficiently large n

$$\Pr[|X - \mathbb{E}[X]| \geq \mathbb{E}[X]/2] \leq 2e^{-\frac{s^{2k}2^{-2k}p^{2\binom{k}{2}}}{4Cs^{2k-2}}} \leq 2e^{-\frac{2^{-2k+2}}{C}s^2p^{k^2}} \leq e^{-3ks\log n}. \tag{6.1}$$

The number of ways to choose the sets $W_1, \ldots, W_k$ is at most $n^{ks} \cdot 2^{ks} \leq e^{2ks\log n}$; comparing this with the above bound yields that the $G_{n,p}$ has the property that for every k disjoint sets of vertices $V_1, \ldots, V_k$ of size s, the number of $\mathcal{K}_k$'s, which have one endpoint in each of the subsets $W_1 \subseteq V_1, \ldots, W_k \subseteq V_k$, such that $|W_i| \geq \frac{s}{2}$, is at least $\frac{1}{2}(\frac{s}{2})^k p^{\binom{k}{2}}$, with probability $\geq 1 - n^{-ks}$.

Next we show the second claim. Let e be an edge that joins a vertex in V_1' to one in V_2'. The number τ_3 of triangles with e and a vertex $v_3 \in V_3'$ is at most $\frac{3}{2}p^2 s$, as the common neighborhood of the endpoints of e in V_3 has at most this size. Similarly, the number τ_i of $\mathcal{K}_i$'s with e, and vertices $v_3 \in V_3'$, …, $v_i \in V_i'$ is at most τ_{i-1} times the maximum size of the common neighborhood of $i-1$ non-exceptional vertices in V_i'. It follows

$$\tau_i \leq \tau_{i-1} \cdot \frac{3}{2}p^{i-1}s \leq \cdots \leq \left(\frac{3}{2}\right)^{i-3} \cdot \tau_3 \cdot p^{\binom{i}{2}-3}s^{i-3} \leq \left(\frac{3}{2}\right)^{i-2} \cdot p^{\binom{i}{2}-1}s^{i-2}.$$

Hence, $\tau_k \leq C_k s^{k-2}p^{\binom{k}{2}-1}$, and the proof is completed. ■

The next result is a generalized version of the above statement. Its proof follows the same lines as the proof of Proposition 6.3, but the details are slightly more tedious.

Proposition 6.4. *$G_{n,p}$ has for every $k \geq 3$ a.a.s. the following property. There are constants $c = c(k)$, $c' = c'(k)$ such that for every k disjoint subsets of its vertices $V_1, \ldots, V_k$, with $|V_1| := s \geq cp^{-k^2}\log n$ and for $i \geq 2$ $|V_i| =: s' \geq |V_1|$, the number of pairwise edge-disjoint K_k's with one vertex in each V_i is $\geq c'pss'$ and $\leq k^2pss'$.*

Proof of Lemma 6.2. The main strategy is very similar to the strategy used in the proof of Lemma 3.11, but we have to make some important modifications. We will sketch only the relevant steps, and the missing details can easily be filled in by considering the respective steps in the original proof.

First of all, note that due to the discussion before Lemma 6.2 there is a.a.s. a partition $(V_1^F, \ldots, V_{\ell-1}^F)$ of F such that

$$\sum_{i=1}^{\ell-1} e(F;\, V_i^F) \leq \varepsilon^5 pn^2, \text{ and } |V_i^F| = \frac{n}{\ell-1} \pm \varepsilon^5 n,$$

for any sufficiently small positive ε. Moreover, with similar arguments as in the proof of Lemma 3.11 it can be shown that every $(\ell-1)$-partition $(P_1, \ldots, P_{\ell-1})$ of $G_{n,p}$ with gap at most $\varepsilon^5 pn^2$ satisfies a.a.s. $(1-\varepsilon)\frac{n}{\ell-1} \le |P_i| \le (1+\varepsilon)\frac{n}{\ell-1}$. We will assume that $G_{n,p}$ has all those properties simultaneously, and make also all other additional assumptions on $G_{n,p}$ made in Lemma 3.11.

In accordance with Lemma 3.11, we call a partition Π *optimal* with respect to F, if $e(F;\,\Pi)$ attains its maximum over all possible partitions. By exploiting our assumptions we see that all optimal $(\ell-1)$-partitions $\Pi = (V_1, \dots, V_{\ell-1})$ of F satisfy $(1-\varepsilon)\frac{n}{\ell-1} \le |V_i| \le (1+\varepsilon)\frac{n}{\ell-1}$ and $\sum_{i=1}^{\ell-1} e(F;\, V_i) \le \varepsilon^5 pn^2$.

Let $\Pi = (V_1, \dots, V_{\ell-1})$ be any fixed optimal $(\ell-1)$-partition of F. A *horizontal* edge is an edge in F that joins two vertices in the same V_i, and a *missing* edge joins in $G_{n,p}$ two vertices in different V_i's, but is not contained in F. As in Lemma 3.11 we can then define the horizontal degree and missing degree of any vertex v with respect to F.

Next, for all $2 \le i \le \ell-1$ we denote by $\mathcal{B}_i(U)$ the (minimal) set of vertices, such that for every $v_1, \dots, v_i \notin \mathcal{B}_i(U)$ we have $\big|\,|\bigcap_{j=1}^{i} \Gamma(G_{n,p};\, v_j) \cap U| - p^i|U|\big| \le \frac{1}{4}p^i|U|$. Let

$$X_1^{V_i} = \left(\bigcup_{j=1}^{\ell-1} \mathcal{B}_1(V_j) \cup \dots \cup \mathcal{B}_{\ell-1}(V_j)\right) \cap\ V_i, \qquad \text{and } X_2^{V_i}, X_3^{V_i} \text{ remain as in Lemma 3.11.}$$

Moreover, define $V_i^0 := V_i$, and $V_i^j := V_i^{j-1} \setminus X_j^{V_i}$, and $X_i := \cup_{k=1}^{\ell-1} X_k^{V_i}$. Note that due to Lemma 3.10 we have $|X_1| = \mathcal{O}(p^{-\ell+1})$, and that the number of missing edges in F incident to at least one vertex in $|X_3|$ is at least $\frac{5}{2}|X_3|\varepsilon pn$. From this we readily obtain that $|X_3| \le \varepsilon n$, as the number of missing edges is at most the number of horizontal edges: otherwise F would not be a maximum $\mathcal{K}_\ell$-free graph.

We now proceed with the following steps, mimicking the proof of Lemma 3.11.

(i) We first show that X_2 is small, i.e., $|X_2| \le \varepsilon p^{-2}$.
(ii) Set $H_3 := \cup_{i=1}^{\ell-1} E(F;\, V_i^3)$. We show that $|H_3| \le p^{-\ell^2} n \log n$.
(iii) We show $|X_3| \le p^{-\ell^2-2} \log n$.
(iv) We use (ii) to show that for all $v \in \cup_{i=1}^{\ell-1} V_i^3$ we have $d_H(v) \le p^{-2\ell^2} \log n$.
(v) Then we show that in fact $|H_3| \le p^{-5\ell^2} \log^2 n$.
(vi) Finally, we show $\sum_{i=1}^{\ell-1} e(F;\, V_i) \le 2p^{-5\ell^2} \log^2 n$.

We now show how the original proof has to be modified to prove this more general statements. To see (i), first note that $d_H(v) \le d(F; v, V_i)$ for all $v \in X_2^{V_1}$ and all $2 \le i \le \ell-1$, i.e., the neighborhoods of v in all V_i have size at least εpn. Moreover, suppose that $|X_2^{V_1}| \ge t_0 = c_\ell(\frac{\varepsilon}{p})^2$, where c_ℓ will be specified later. The number of missing edges can be bounded from below by the maximal number of pairwise edge-disjoint K_ℓ's, which have one endpoint in each of $V_2 \cap \Gamma(G_{n,p};\, v), \dots, V_{\ell-1} \cap \Gamma(G_{n,p};\, v)$, and two endpoints in V_1, namely v, and one of its neighbors in V_1. Let

$$\begin{aligned} \mathcal{F}(v) := {} & \text{maximum set of pairwise edge-disjoint } \mathcal{K}_\ell\text{'s in } G_{n,p}, \text{ which} \\ & \text{contain } v \text{ and one of its neighbors in each } (V_j)_{j=1\dots\ell-1}, \end{aligned}$$

and set $m_0 = |\bigcup_{v \in X_2^{V_1}} \mathcal{F}(v)|$. Note that $|\mathcal{F}(v)|$ equals the number of pairwise edge-disjoint $\mathcal{K}_{\ell-1}$'s between the neighborhoods of v, which are all of size at least εpn. By applying Proposition 6.3 we see that a.a.s. $f_i \ge C_\ell p \cdot (\varepsilon pn)^2$, for some $C_\ell > 0$. Then we perform a similar calculation as in (3.5), and it remains to estimate $|\mathcal{F}(v) \cap \mathcal{F}(w)|$. Observe that $|\mathcal{F}(v) \cap \mathcal{F}(w)|$

Figure 2: A chord in the case $\ell = 4$. The black edges are in $G_{n,p}$, and the red edge is missing.

can be crudely bounded from above by the number of edges between the common neighborhoods of v and w in $V_2, \ldots, V_{\ell+1}$. But this neighborhoods have size at most $2p^2n$ (as a.a.s. no two vertices have a larger common neighborhood in $G_{n,p}$), and the number of edges between any two sets of vertices of at most this size in $G_{n,p}$ is a.a.s. $\le 2p(2p^2n)^2$, which implies $|\mathcal{F}(v) \cap \mathcal{F}(w)| \le 4\ell^2 p(p^2n)^2$. By choosing $c_\ell = \frac{C_\ell}{8\ell^2}$ we see that $m_0 \ge \frac{1}{16\ell^2}\varepsilon^4 pn^2$, which contradicts the fact $m_0 \le \varepsilon^5 pn^2$, whenever ε is sufficiently small.

Next we show how (ii) can be proved. Assume that the number of horizontal edges is maximized in V_1^3. We adapt the definition of a "chord": here, a chord consists of ℓ vertices $\mathcal{V} = \{x, y, v_2, \ldots, v_{\ell-1}\}$, such that $\{x, y\}$ is a horizontal edge in V_1^3 , and $v_i \in V_i^3$. Additionally, one of the edges joining vertices in $\mathcal{V}$ (except for $\{x, y\}$) is a missing edge, i.e., it is contained in $G_{n,p}$ but not in F, and all other edges are in $G_{n,p}$. See Figure 6 for an illustration.

Our objective is to derive an upper and a lower bound for the number N of chords, which will immediately imply the bound on $|H_3|$ claimed in (ii). We begin with the lower bound. Note that the number of triangles in $G_{n,p}$ connecting x, y, and a vertex v_2 in $V_2 \setminus X$ is due to our assumptions for sufficiently large n at least $\frac{3p^2(|V_2|-|X|)}{4} \ge \frac{3p^2n}{4\ell}$, except for at most $\mathcal{O}(\binom{p^{-2}}{2} + p^{-2}n)$ pairs x, y (as due to Lemma 3.9 there is a set Z of $\mathcal{O}(p^{-2})$ vertices such that every other pair in V_1^3 has at least the claimed number of neighbors). Hence, the number of triangles connecting any edge in V_1^3 with a vertex $v_2 \in V_2^3$ is at least

$$\left(\left(\frac{|H_3|}{\ell} - \mathcal{O}(p^{-2}n)\right)\right)\frac{3p^2n}{4\ell} \ge |H_3| \cdot \frac{p^2n}{2\ell^2}, \tag{6.2}$$

as we may assume that $|H_3| > p^{-\ell^2} n \log n$. Similarly, the number of $\mathcal{K}_4$'s connecting x, y, v_2 with a $v_3 \in V_3^3$ is at least $\frac{3p^3n}{4\ell}$, except for at most $\mathcal{O}(\binom{p^{-3}}{2} + p^{-3}n^2)$ triples x, y, v_2. Exploiting (6.2) we obtain that the number of $\mathcal{K}_4$'s with an edge in V_1^3, a vertex in V_2^3, and a vertex in V_3^3 is at least

$$\left(\left(|H_3| \cdot \frac{p^2n}{2\ell^2} - \mathcal{O}(p^{-3}n^2)\right)\right)\frac{3p^2n}{4\ell} \ge |H_3| \cdot \frac{p^5n}{2^2\ell^3}.$$

This process can be continued to count $\mathcal{K}_5$'s, $\mathcal{K}_6$'s and so on. We obtain with room to spare that the number of $\mathcal{K}_\ell$'s is at least

$$|H_3| \cdot \frac{n^{\ell-2} p^{\binom{\ell}{2}-1}}{(2\ell)^\ell}, \tag{6.3}$$

which provides us with the desired lower bound for the number of chords, as in every counted copy of the $\mathcal{K}_\ell$ there is a missing edge in F.

To obtain an upper bound for N we partition the set of chords into three classes and derive upper bounds for each one separately.

(A) Chords where $d_{H_3}(x) \le p^{-2}\log n$, and there is a $2 \le i \le \ell - 1$ such that $\{x, v_i\}$ is missing. We count those as follows. Let $x \in V_1^3$ and $y \in \Gamma(F; x) \cap V_1^3$. Then an upper bound for the number of chords with x and y is given by ℓ^2 multiplied with the number of $\mathcal{K}_\ell$'s in $G_{n,p}$, that have one vertex in each V_i $(2 \le i \le V_{\ell-1})$ and x, y. To estimate this number we use a similar argument as above: the number of $\mathcal{K}_3$'s with x, y and a vertex $v_2 \in V_2$ is at most $2p^2n$, as no two vertices have a larger neighborhood in $G_{n,p}$. Similarly, the number of $\mathcal{K}_4$'s with x, y, v_2 and $v_3 \in V_3$ is at most $2p^3n$, and so on. Putting everything together yields that the number of (A)-chords is at most

$$\sum_{\substack{x \in V_1^3 \\ d_{H_3}(x) \leq p^{-2}\log n}} \sum_{y \in \Gamma(F;\, x) \cap V_1^3} (2p^2n) \cdot (2p^3n) \cdots (2p^{\ell-1}n)$$

$$\leq |V_1^3| \cdot p^{-2}\log n \cdot (2n)^{\ell-2} p^{\binom{\ell}{2}-1} \;\leq\; (2n)^{\ell-1} p^{\binom{\ell}{2}-3} \log n.$$

(B) Chords where $d_{H_3}(x) \geq p^{-2}\log n$, and there is a $2 \leq i \leq \ell - 1$ such that $\{x, v_i\}$ is missing. Let $S = \Gamma(G_{n,p};\, x) \setminus (V_1^3 \cup \Gamma(F;\, x))$ and note that $d_M(x) = |S|$. Due to our assumptions $G_{n,p}$ is such that except of at most $Cp^{-1}\log n$ vertices, every vertex in S has at most $\frac{5}{4}pd_{H_3}(x)$ neighbors in $\Gamma(F;\, x) \cap V_1^3$. Hence, the number of $\mathcal{K}_3$'s with x, a neighbor of x in V_1^3 and a vertex adjacent to a missing edge at x is at most

$$d_M(x) \cdot \frac{5}{4}pd_{H_3}(x) + Cp^{-1}\log n \cdot d_{H_3}(x) \overset{(d_M(x) \leq 6\varepsilon pn)}{\leq} 8\varepsilon p^2 n \cdot d_{H_3}(x).$$

With a similar argument as in (A) we easily see that the number of $\mathcal{K}_4$'s with x, a neighbor of x in V_1^3, a vertex adjacent to a missing edge at x, and a vertex in any of V_i^3's (different from the ones where a vertex is already taken from) is at most $2p^3n$, as this is the maximum number of common neighbors of any three vertices in $G_{n,p}$. In the same way we can count $\mathcal{K}_5$'s, $\mathcal{K}_6$'s, etc. To conclude, the number of (B)-chords is at most

$$\sum_{\substack{x \in V_1^3 \\ d_{H_3}(x) > p^{-2}\log n}} 8\varepsilon p^2 n \cdot d_{H_3}(x) \cdot (2p^3n) \cdots (2p^{\ell-1}n) \;\leq\; 8\varepsilon (2n)^{\ell-2} p^{\binom{\ell}{2}-1} \cdot |H_3|.$$

(C) Chords with the property that there are indexes $2 \leq i, j \leq \ell - 1$ such that $\{v_i, v_j\}$ is missing. In the sequel we assume that $G_{n,p}$ is such that for every subset U of the vertex set of size at least $p^{-2}\log^2 n$ there is a set Z_U of at most $p^{-3}\log n$ vertices such that every vertex pair in $[n] \setminus (U \cup Z_U)$ has more than $\frac{1}{2}p^2|U|$ and less than $\frac{3}{2}p^2|U|$ common neighbors in U (this statement is very similar to Lemma 3.10, and the proof is omitted).

We count (C)-chords as follows. Let $e = \{x, y\} \in H_3$ and $v \in V^3$ be one of the common neighbors of x, y that is one of the endpoints of the missing edge in the chord. Note that there are at most $2p^2n$ candidates for v. Moreover, let M_v be the set of non-exceptional missing neighbors of v, i.e., the set of vertices incident to missing edges $e = (v, w)$, where $w \in V^3$. Now let $v' \in M_v \cap \Gamma(G_{n,p};\, x) \cap \Gamma(G_{n,p};\, y)$. Observe that due to our assumptions the following is true.

- If $|M_v| \geq p^{-2}\log^2 n$, then there are at most $\frac{3}{2}p^2|M_v|$ ways to choose v', except for at most $\binom{p^{-3}\log n}{2} + n \cdot p^{-3}\log n \leq 2np^{-3}\log n$ pairs x, y.
- Otherwise, there are at most $p^{-2}\log^2 n$ ways to choose v'.

Without loss of generality let $x, y \in V_1^3$, $v \in V_2^3$ and $v' \in V_3^3$. Having chosen x, y, v, v', the number of $\mathcal{K}_\ell$'s containing those vertices, and one vertex in each of $V_4, \ldots, V_{\ell-1}$ is at most $(2n)^{\ell-4}p^{\binom{\ell}{2}-6}$ (this is seen by exactly the same counting argument that we have already used in (A) and (B)). Putting everything together yields that the number of (C)-chords is at most

$$\begin{aligned}
&\sum_{e\in H_3}\sum_{v}\sum_{v'}(2n)^{\ell-4}p^{\binom{\ell}{2}-6}\\
&\le \sum_{e\in H_3}\sum_{v}\left(\frac{3}{2}p^2|M_v|+p^{-2}\log^2 n\right)\cdot(2n)^{\ell-4}p^{\binom{\ell}{2}-6}\\
&\qquad\qquad\qquad + \sum_{v}\sum_{v'}2np^{-3}\log n\cdot(2n)^{\ell-4}p^{\binom{\ell}{2}-6}\\
&\overset{(d_M(v)\le 6\varepsilon pn)}{\le} \sum_{e\in H_3}2p^2n\cdot\left(\frac{3}{2}p^2\cdot 6\varepsilon pn+p^{-2}\log^2 n\right)\cdot(2n)^{\ell-4}p^{\binom{\ell}{2}-6}\\
&\qquad\qquad\qquad + n\cdot 2pn\cdot 2np^{-3}\log n\cdot(2n)^{\ell-4}p^{\binom{\ell}{2}-6}\\
&\le |H_3|\cdot 5\varepsilon(2n)^{\ell-2}p^{\binom{\ell}{2}-1}+(2n)^{\ell-1}p^{\binom{\ell}{2}-9}\log n.
\end{aligned}$$

By combining the results from (A), (B) and (C) with (6.3) we see that $|H_3|$ satisfies

$$|H_3|\cdot\frac{n^{\ell-2}p^{\binom{\ell}{2}-1}}{(2\ell)^{\ell}}\le |H_3|\cdot 13\varepsilon(2n)^{\ell-2}p^{\binom{\ell}{2}-1}+2(2n)^{\ell-1}p^{\binom{\ell}{2}-9}\log n,$$

from which (ii) follows readily for large n.

The proof of (iii) is identical to the proof of the analogous statement in Lemma 3.11, and we omit a detailed exposition. To see (iv), observe that the total number of horizontal edges in F is due to (i)-(iii) at most

$$|H|\le|H_3|+|X_1\cup X_2\cup X_3|n<2p^{-\ell^2-2}n\log n.$$

Suppose that there is a vertex v in V_1^3 with $d_{H_3}(v)\ge p^{-2\ell^2}\log n$ – we handle vertices in the other sets V_i^3 analogously. We will show that this implies that the number of missing edges is at least $2p^{-\ell^2-2}n\log n$, which contradicts the bound on $|H|$ derived above.

In order to give a lower bound for the number of missing edges, we estimate from below the maximum number of edge-disjoint $\mathcal{K}_\ell$'s in $G_{n,p}$, which contain v, one of the vertices counted in $d_{H_3}(v)$, and $\ell-1$ vertices in $\Gamma(F;\, v, V\setminus V_1)$, such that there is precisely one vertex in each $V_2,\dots,V_{\ell-1}$. For this we count the maximum number of edge-disjoint $\mathcal{K}_{\ell-1}$'s between the sets of vertices $\Gamma(F;\, v,V_2),\dots\Gamma(F;\, v,V_{\ell-1})$, and $\Gamma(F;\, v,V_1^3)$ in $G_{n,p}$.

Note that for $i\ge 2$ we have $d(F;\, v,V_i)\ge c_\ell pn$, for some $c_\ell>0$ depending only on ℓ, and due to our assumption $\Gamma(F;\, v,V_1^3)\ge p^{-2\ell^2}\log n$. We apply Proposition 6.4 with $k=\ell-1$, $V_1=\Gamma(T;v,V_1^3)$, and $V_i=\Gamma(T;v,V_i)$ (truncated to their first $c_\ell pn$ vertices), which yields that there is a constant $c>0$ such that there are at least $cp^2d_{H_3}(x)n$ pairwise edge-disjoint copies of $\mathcal{K}_{\ell-1}$ with one endpoint in $\Gamma(F;\, v,V_1^3)$ and in each $\Gamma(F;\, v,V_i)$. But then the number of missing edges is $\ge cp^{-2\ell^2+2}n\log n$, which completes the proof of (iv).

Next we prove (v). Let m be the number of missing edges in F. Our aim is to show that $m\ge|H_3|\cdot\frac{p^{3\ell^2}}{\log n}n$, and hence $|H_3|$ must satisfy

$$|H_3|\cdot\frac{p^{3\ell^2}}{\log n}n\le 2p^{-\ell^2-2}n\log n,\qquad\text{as }|H|<2p^{-\ell^2-2}n\log n.$$

This completes the proof of (v). To show the claimed bound for m, assume that the number of edges in H_3 is maximized in V_1^3, and let $R \subseteq H_3$ be a *matching* of maximum cardinality that joins vertices from V_1^3. Note that by using (iv) we obtain $|R| \geq \frac{1}{\ell}|H_3|p^{2\ell^2}(\log n)^{-1}$. We now proceed in two steps. First, we bound from below the number of $\mathcal{K}_\ell$'s in $G_{n,p}$, which contain an edge in R, and a vertex in each of the sets $V_2^3, \dots, V_{\ell-1}^3$. In the second step, we estimate from above the maximum number of $\mathcal{K}_\ell$'s, that contain any edge in R, and an additional (fixed) edge e' connecting any two vertices in the sets V_i^3 and V_j^3, where $1 \leq i < j \leq \ell - 1$. By dividing these two numbers (and by dividing the result by $\binom{\ell}{2}$) we readily obtain a lower bound for the number of missing edges.

To obtain the first goal note that the number of τ_3 of triangles with $e \in R$ and a vertex $v_2 \in V_2^3$ is at least $\frac{3}{4}p^2|V_2| - |X_1 \cup X_2 \cup X_3| \geq c_3p^2n$, for some $c_3 > 0$. Similarly, the number τ_i of $\mathcal{K}_i$'s with e and $i-2$ vertices $v_2 \in V_2^3, \dots, v_{i-1} \in V_{i-1}^3$ is at least

$$\tau_i \geq \tau_{i-1} \cdot \left(\frac{3}{4}p^{i-1} \cdot |V_{i-1}| - |X_1 \cup X_2 \cup X_3|\right) \geq \cdots \geq c_i \cdot n^{i-2}p^{\binom{i}{2}-1},$$

where c_i depends only on c_3 and on i. Setting $i = \ell$ yields that the number of $\mathcal{K}_\ell$'s in $G_{n,p}$, which contain an edge in R, and a vertex in each of the sets $V_2^3, \dots, V_{\ell-1}^3$, is at least $|R|c_\ell p^{\binom{\ell}{2}-1}n^{\ell-2} \geq |H_3| \cdot p^{3\ell^2-1} \cdot (\log n)^{-1} \cdot n^{\ell-2}$.

To obtain the second goal we distinguish two cases: either e' has a common endpoint with one edge in R, or $i \geq 2$, i.e., e' joins vertices in V_i^3 and V_j^3, where $2 \leq i < j \leq \ell - 1$. In the former case, let us denote by e the edge of R, to which e' is adjacent to, and observe that e is unique. This means that 3 vertices of the $\mathcal{K}_\ell$'s that we want to count are specified (the one endpoint of e, the intersection of e and e', and the other end of e'); hence, the number of $\mathcal{K}_\ell$'s with e and e' is at most $n^{\ell-3}$. In the latter case we want to count $\mathcal{K}_\ell$'s in $G_{n,p}$, which have a vertex in V_i^3 and V_j^3, where $2 \leq i < j \leq \ell - 1$. These $\mathcal{K}_\ell$'s have exactly one vertex in each V_x^3, such that $x \notin \{1, i, j\}$, and two vertices in V_1^3, which are endpoints of an edge in R. As the number of admissible indexes x is $\ell - 4$, the number of $\mathcal{K}_\ell$'s with e' is at most $|R| \cdot n^{\ell-4} \leq n^{\ell-3}$ (observe that trivially $|R| \leq n$, as R is a matching). Putting all together, the number of $\mathcal{K}_\ell$'s, that contain exactly one edge in R, and an additional edge e' connecting any two vertices in the sets V_i^3 and V_j^3, where $1 \leq i < j \leq \ell - 1$, is at most $2n^{\ell-3}$.

By combining the last two results we conclude that the number of $\mathcal{K}_\ell$'s, which contain exactly one edge from R and are otherwise edge-disjoint, is at least

$$\frac{|H_3| \cdot p^{3\ell^2-1} \cdot n^{\ell-2}}{\log n} \cdot \frac{1}{2n^{\ell-3}} \cdot \frac{1}{\binom{\ell}{2}} = |H_3| \cdot \frac{p^{3\ell^2}}{\log n} n,$$

which is the desired lower bound for the number of missing edges.

To complete the proof we show (vi). Let d be the maximum degree of a vertex $v \in X$, and suppose that $d \geq p^{-2\ell^2} \log n$. Without loss of generality we may assume that $v \in V_1$. Then the degree of v in every V_i is also at least d, as otherwise the chosen partition would have not been maximal. By applying Proposition 6.3 we readily obtain that the number of edge-disjoint $\mathcal{K}_{\ell-1}$'s joining the neighborhoods of v is at least cpd^2, for a $c > 0$, which implies that there are at least that many missing edges.

On the other hand, the total number of horizontal edges is at most $|X|d + |H_3|$. By exploiting (i)-(v) this is at most $2p^{-\ell^2-2} \log nd + p^{-5\ell^2} \log^2 n$. Hence, d satisfies

$$cpd^2 \leq 2p^{-\ell^2-2} \log nd + p^{-5\ell^2} \log^2 n,$$

from which we deduce $d \leq p^{-3\ell^2} \log n$. This completes the proof. ▪

Proof of Theorem 1.5 Theorem 1.5 can be proved in a completely analogous way as Theorem 1.3 (see Section 5). The definitions of several events, as well as the partial ordering of graphs with respect to partitions of the vertex set all generalize in an obvious and natural way from bipartitions to $(\ell-1)$-partitions. We leave the straightforward details to the reader.

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