Physarum Computations

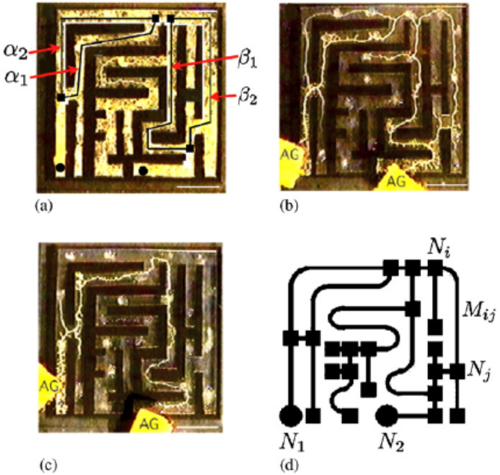
joint work with Luca Becchetti, Ruben Becker, Vincenzo Bonifaci, Michael Dirnberger, Andreas Karrenbauer, Pavel Kolev, Tim Mehlhorn, and Girish Varma

SODA 2012, ICALP 2013, J. Theoretical Biology 2012, Journal of Physics D: Applied Physics 2017, ArXiv 2017

max planck institut
informatik

SIC Saarland Informatics Campus

The Wetlab Experiment: Physarum Finds Near-Shortest Paths



Physarum, a slime mold,

single cell, several nuclei

builds evolving networks

Nakagaki, Yamada, Tóth, Nature 2000

show video

The Video of the Wetlab Experiment

2008 Ig Nobel Prize

For achievements that first make people LAUGH
then make them THINK

COGNITIVE SCIENCE PRIZE: Toshiyuki Nakagaki, Ryo Kobayashi, Atsushi Tero, Ágotá Tóth
for discovering that slime molds can solve puzzles.

REFERENCE: "Intelligence: Maze-Solving by an Amoeboid Organism," Toshiyuki Nakagaki, Hiroyasu Yamada, and Ágota Tóth, Nature, vol. 407, September 2000, p. 470.

Outline of Talk

- The maze experiment (Nakagaki, Yamada, Tóth). ✓
- A mathematical model for the dynamics of Physarum (Tero et al.).
- Convergence against the shortest path.
- Positive undirected linear programs.
- Approach:
  - Analytical investigation of simple systems.
  - A simulator.
  - Formulizing conjectures and killing them.
  - Proving the surviving conjecture.
  - Generalizing to positive undirected linear programs.
- Network formation.

Mathematical Model (Tero et al.)

- Physarum is a network of tubes (pipes);
- Flow (of liquids and nutrients) through a tube is determined by concentration differences at endpoints of a tube, length of tube, and diameter of tube;
- Tubes adapt to the flow through them: if flow through a tube is high (low) relative to diameter of the tube, the tube grows (shrinks) in diameter.
- Mathematics is the same as for flows in an electrical network with time-dependent resistors.
- Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph, nodes s_0 and s_1.
- Each edge e has a positive length c_e (fixed) and a positive diameter $x_e(t)$ (dynamic).
- Initial state $x(0) > 0$.
- Send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$r_e(t) = c_e / x_e(t).$$

- $q_e(t)$ is resulting flow across e at time t.
- Dynamics:

$$\dot{x}_e(t) = \frac{d}{dt} x_e(t) = |q_e(t)| - x_e(t).$$

We will write x_e and q_e instead of $x_e(t)$ and $q_e(t)$ from now on.

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The Questions

Does the system convergence for all (!!!) initial conditions?

If so, what does it converge to? Fixpoints?

How fast does it converge?

Beyond shortest paths?

Inspiration for distributed algorithms?

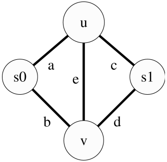
Convergence against Shortest Path

Theorem (Convergence (SODA 12, J. Theoretical Biology), Bonifaci/M/Varma)

Dynamics converge against shortest path, i.e.,

- *potential difference between source and sink converges to length of shortest source-sink path,*
- $x_e \to 1$ *for edges on shortest source-sink path,*
- $x_e \to 0$ *for edges not on shortest source sink path*

this assumes that shortest path is unique; otherwise . . .



Miyaji/Onishi previously proved convergence for parallel links and Wheatstone graph.

Does the Dynamics Solve a Larger Class of Problems?

What could this larger class of problems be?

How should we reinterpret q?

Undirected Shortest Paths as an LP

Shortest path in an undirected graph is a min-cost flow problem in an undirected graph with infinite edge capacities.

Recall min-cost flow in a directed graph

c_e = cost of the edge e, $c_e > 0$

f_e = flow over the edge e, $f_e \geq 0$.

for all vertices v: $\mathit{outflow}_v - \mathit{inflow}_v = \mathit{supply}_v$.

minimize $\sum_e c_e f_e$.

The LP for min-cost flow in a directed graph

minimize $\sum_e c_e f_e$

subject to $Af = b$ and $f \geq 0$, where

A = node-arc incidence matrix, i.e., for $e = (u, v)$, $A_e = e_u - e_v$

b = supply vector, i.e., $b = e_{source} - e_{sink}$

Undirected Shortest Paths as an LP

Modelling undirected edges

- Make all edges bidirected.
- Orient the edges of the graph arbitrarily and allow negative flows. A negative flow across an edge (u, v) is really a positive flow in the direction from v to u.

Undirected Shortest Paths as an LP

minimize $\sum_e c_e |f_e|$

subject to $Af = b$, where

$A =$ node-arc incidence matrix, i.e., for $e = (u, v)$, $A_e = e_u - e_v$ and

$b =$ supply vector, i.e., $b = e_{source} - e_{sink}$.

The Generalization: Positive Undirected LPs

Modelling undirected edges

- Make all edges bidirected.
- Orient the edges of the graph arbitrarily and allow negative flows. A negative flow across an edge (u, v) is really a positive flow in the direction from v to u.

Positive ($c > 0$) Undirected LPs

minimize $\sum_e c_e |f_e|$

subject to $Af = b$, where A and b are arbitrary.

Remark: Solution to $Af = b$ of minimal weighted one-norm.

What is the Proper Generalization of q?

Definition of q

q is the electrical flow that sends one unit of current from s_0 to s_1 with respect to the resistances $r_e = c_e / x_e$.

Tomson's principle

The electrical flow is a feasible flow f that minimizes the energy $\sum_e r_e f_e^2$ of the flow. It is unique.

$$
\begin{aligned}
q &= \operatorname*{argmin}_f \left\{ \sum_e r_e f_e^2 ; f \text{ is a feasible flow} \right\} \\
&= \operatorname*{argmin}_f \left\{ \sum_e r_e f_e^2 ; Af = b \right\} \\
&= R^{-1} A^T (A R^{-1} A^T)^{-1} b \quad \text{where } R^{-1} = diag(x_e / c_e).
\end{aligned}
$$

Theorem

Positive Undirected LP

Assume $c > 0$. Consider

(*) minimize $\sum_e c_e x_e$, subject to $Af = b$ and $|f| \leq x$.

Convergence of the Physarum Dynamics

The Physarum dynamics

$$\dot{x} = |q| - x$$

with a positive start vector $x(0) > 0$ converges to an optimum solution of (*). Here

$$q = \operatorname*{argmin}_f \left\{ \sum_e r_e f_e^2 ; Af = b \right\} \quad \text{and} \quad r_e = c_e / x_e .$$

Our Approach

Shortest Paths

- Analytical investigation of simple systems, in particular, parallel links, and
- experimental investigation (computer simulation) of larger systems,
  - to form intuition about the dynamics,
  - to kill conjectures,
  - to support conjectures.
- Proof attempts for conjectures surviving the computer experiments.

Positive Undirected LPs

Generalize the proofs. General structure unchanged, but details very different.

Computer Simulation (Discrete Time)

Discrete Dynamics (Euler discretization)

Compute

$$x_e(t+1) = x_e(t) + h \cdot (|q_e(t)| - x_e(t))$$

for $t = 1, 2, 3, \ldots$ and a small step-size h.

Computer Experiments

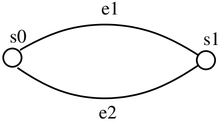
We simulated 1000 systems with up to 10000 nodes. Always observed convergence to shortest path.

Speed of convergence is determined by ratio of length of second shortest path to length of shortest path.

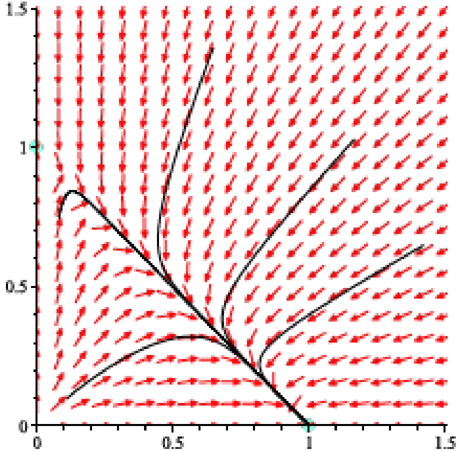
Two Parallel Links (Miyaji/Ohnishi)

A visualization of the dynamics.
Arrows show the vector $(\dot{x}_1, \dot{x}_2)$.

Trajectories in black.



e_i has length c_i, $c_1 < c_2$, and diameter x_i



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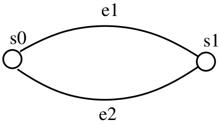


e_i has length c_i, $c_1 < c_2$, and diameter x_i

$\Delta = \Delta(t) =$ potential difference between source and sink

$q_i = \frac{x_i}{c_i} \cdot \Delta$

$\dot{x}_i = q_i - x_i = \frac{x_i}{c_i}\Delta - x_i$

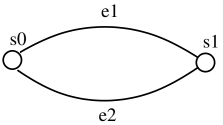
Fixpoints: $\dot{x}_1 = \dot{x}_2 = 0$:

$\dot{x}_i = 0 \quad \text{iff} \quad x_i = 0 \text{ or } \Delta = c_i.$

Thus $x_2 = 0$, $\Delta = c_1$, and $x_1 = 1$
or $\quad x_1 = 0$, $\Delta = c_2$, and $x_2 = 1$.

Fixpoints are the source-sink paths.

Two Parallel Links (Miyaji/Ohnishi)



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Convergence

Consider $V = \frac{c_1 x_1 + c_2 x_2}{x_1 + x_2}$.

Then

- $V \geq 0$,
- $\dot{V} \leq 0$, and
- $\dot{V} < 0$ if $q \neq x$.

Thus x converges to a fixpoint.

V is called a Lyapunov function.

The Structure of the Convergence Proof

Fixpoints: The points x with $\dot{x} = 0$, i.e., $|q| = x$.

The fixpoints are exactly the source-sink paths. This assumes that all paths have different length. Thus, if the system converges, it converges against some source-sink path.

Convergence

- In order to prove convergence, one needs to find a Lyapunov function, i.e., a function L mapping x to real numbers such that
  - $L(x) \geq 0$ for all x,
  - $\frac{d}{dt}L(x) \leq 0$, and
  - $\dot{L} = 0$ if and only if $\dot{x} = 0$.
- In order to prove convergence against the shortest path, one needs some additional arguments.

Lyapunov Functions?

First idea: the energy of the flow decreases over. time

Not true, even for parallel links.

Theorem

For the case of parallel links:

$$\sum_{i \geq 2} c_i \ln x_i - c_1 \ln x_1, \sum_i q_i c_i, \quad \frac{\sum_i x_i c_i}{\sum_i x_i}, \quad \textit{and}\ (p_s - p_t) \sum_i x_i c_i$$

decrease over time

computer experiment: the obvious generalizations to general graphs (replace i by e) do not work.

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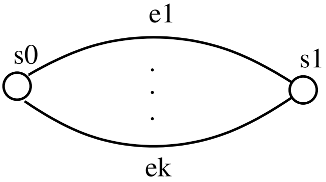
A not so Obvious Generalization



$$\frac{\sum_i x_i c_i}{\sum_i x_i} \quad \Rightarrow \quad \frac{\sum_e x_e c_e}{\text{minimum total } x\text{-value of a } s_0\text{-}s_1 \text{ cut}}$$

LEDA came handy.

Lyapunov Functions?

Computer experiment:

$$V := \frac{\sum_e x_e c_e}{\text{minimum total } x\text{-value of a } s_0\text{-}s_1 \text{ cut}} \quad \text{decreases}$$

Theorem (Lyapunov Function)

$$V + \left(\sum_{e \in \delta(\{\, s_0 \,\})} x_e - 1\right)^2 \quad \textit{decreases.}$$

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Generalization of the Scaling Factor

scaling factor = minimum total x-value of a s_0-s_1 cut

Min-Cut = Max-Flow

Interpret the x-values as edge capacities

$$\begin{aligned} \text{minimum total } x\text{-value} \ldots &= \text{minimum capacity of a } s_0\text{-}s_1 \text{ cut} \\ &= \text{maximum flow from } s_0 \text{ to } s_1 \end{aligned}$$

Scaling Factor: $sf = \max\{\,\alpha;\ Af = \alpha b;\ |f| \le x\,\}$.

LP-duality: There are vectors d_1 to d_K that only depend on A and b but are independent of x such that $sf = \min_i d_i^T x$.

Then V becomes $\dfrac{\sum_e x_e c_e}{\min_i d_i^T x}$

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Convergence against Optimum Solution

Theorem (Convergence)

Assume $c \geq 0$ and $c^T|f| > 0$ for every f with $Af = 0$.
Assume $\min c^T|f|$ subject to $Af = b$ has a unique solution.

The Physarum dynamics

$$\dot{x} = |q| - x,$$

where

$$q = \operatorname*{argmin}_f \left\{ \sum_e r_e f_e^2 ; Af = b \right\} \quad \text{and} \quad r_e = c_e / x_e,$$

converges against the optimal solution of the linear program above.

If the optimum solution is not unique, ...

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Convergence of discretization ($c = 1$), Straszak and Vishnoi.

Discretization and Speed of Convergence

$$x_e(t+1) = x_e(t) + h(|q_e(t)| - x_e(t))$$

Theorem (Epsilon-Approximation of Shortest Path), Bechetti/Bonifaci/Dirnberger/Karrenbauer/M, ICALP 13

Let *opt* be the length of the shortest source-sink path.

Let $\varepsilon > 0$ be arbitrary. Set $h = \varepsilon/(2mL)$, where L is largest edge length and m is the number of edges.

After $\widetilde{O}(nmL^2/\varepsilon^3)$ iterations, solution is $(1+\varepsilon)$ optimal, i.e., $V = \sum_e c_e x_e$ is at most $(1+\varepsilon)opt$.

Arithmetic with $O(\log(nL/\varepsilon))$ bits suffices.

Related Work: Directed Physarum

$$\dot{x}_e(t) = q_e(t) - x_e(t)$$

No biological significance is claimed.

Ito/Johansson/Nakagaki/Tero (2011)

prove convergence to shortest directed source-sink path.

Johannson/Zou (2012) and D. Straszak/N. Vishnoi (2016)

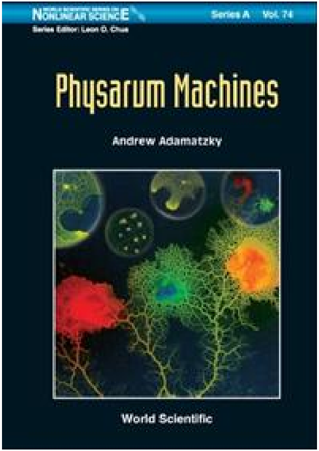
prove that directed dynamics solves any linear program with monotone objective function (all coefficients of c are positive)

$$\max c^T x \quad \text{subject to} \quad Ax = b \text{ and } x \geq 0.$$

Becker/Bonifaci/Karrenbauer/Kolev/M

established an improved convergence result.

Adamatzky's Book



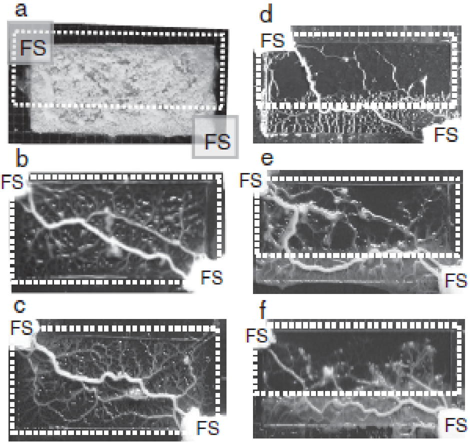
many examples of Physarum computations

- shortest paths
- network design
- Delaunay diagrams
- puzzles

also Youtube-videos: search for Physarum

Open Problems

Nonuniform Physarum



$$\dot{x}_e(t) = a_e(|q_e(t)| - x_e(t))$$

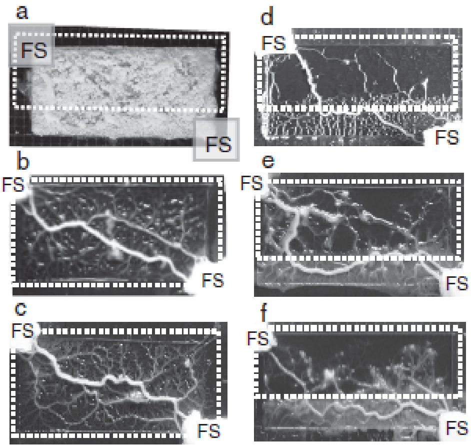
a_e reactivity of e

We have a heuristic argument for the details of the convergence process. Have verified them in computer simulations.

No convergence proof

Becchetti/Bonifaci/Karrenbauer/M (ICALP 13) incorrectly claimed a convergence proof.

Nonuniform Physarum



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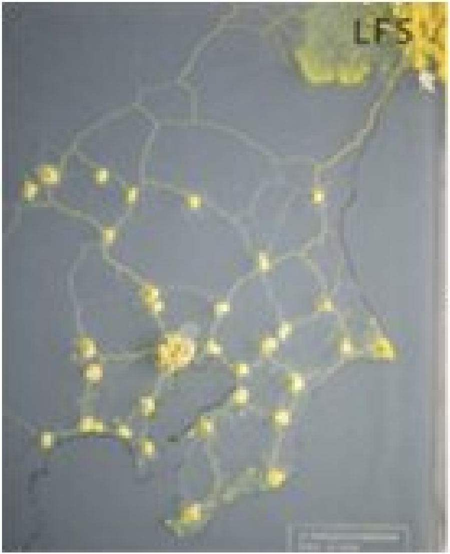
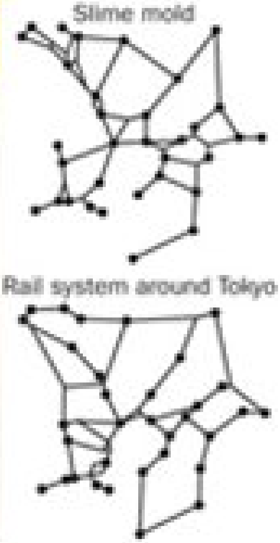
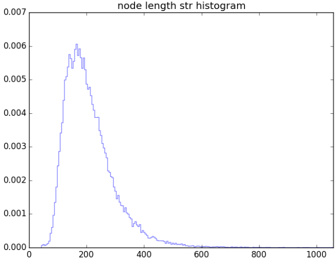
Network Design: Science 2010



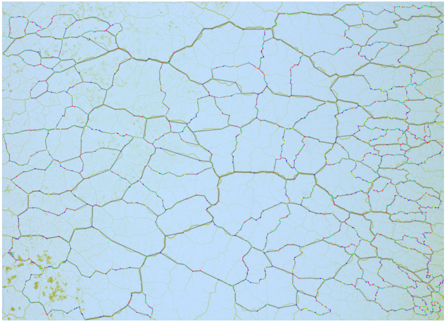


test

Observables (Wet-Lab Experiments)



Histogram for edge lengths. Abscissa shows values in pixel.



Have verified experimentally that cut capacity orthogonal to growing direction is constant.

Dirnberger/Mehlhorn/Mehlhorn, J. Phys. D, 2017, Dirnberger/Mehlhorn, J. Phys. D, 2017.

My Current Projects

Understand the principles of network formation. What does the network optimize?

Nonuniform Versions of Physarum.

Can I use Physarum as an inspiration for approximation algorithms?