



Controlled Perturbation for Delaunay Triangulations

**slides prepared by Christian Klein for SODA 2005,
extended for lectures on algorithm engineering by Kurt Mehlhorn**

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Overview



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- Idealistic Algorithms
- Controlled Perturbation
- Delaunay Triangulations
- General Randomized Construction
- Experimental Results

Idealistic Algorithms



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Simplifying assumptions when designing computational geometry algorithms:

- The Real RAM Assumption (we can compute with reals)
Only have fixed precision floating point arithmetic!
Implementing Real RAM is costly!
- The Non-Degeneracy Assumption, e.g. no 3 collinear points, no 4 cocircular points
Occur in practice!
Need High Precision to detect!
Handling all Degeneracies complicates code!

Predicates



Geometric algorithms base decisions on **predicates**.

Example: Given $(d + 1)$ points $p_0, \dots, p_d \in \mathbb{R}^d$, the **orientation test** gives the position of p_d relative to the hyperplane spanned by p_0, \dots, p_{d-1} .

$$\mathit{orient}(p_0, \dots, p_d) := \text{sign} \begin{vmatrix} p_{01} & \dots & p_{0d} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ p_{d1} & \dots & p_{dd} & 1 \end{vmatrix}.$$

Evaluates to zero iff all points lie in a common hyperplane.

\rightsquigarrow **Degeneracy**



The Controlled Perturbation Framework

Overview



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- Solve problem not for given input but for “nearby” **perturbed** input.
 - Choose perturbation carefully: such that the perturbed input
 - is in **general position**.
 - can be solved with **fixed precision arithmetic**.
- ↪ Can run **idealistic algorithm** on perturbed input.

The sales pitch for controlled perturbation: it **numerically** perturbs the inputs such that they can be processed **correctly** and **fast** by **simple** algorithms.

Previous Work



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Controlled Perturbation was proposed by Halperin et. al. and applied to

- spherical arrangements [Halperin, Shelton 1998]
- polyhedral arrangements [Halperin, Raab 1999]
- arrangements of circles [Halperin, Leiserowitz 2003].
- We describe a general framework (guard predicates) for controlled perturbation,
- apply it to Delaunay triangulations and convex hulls in any dimension,
- and analyze the use of controlled perturbation in randomized algorithms.

Guard Predicates



To guard an expression E , e.g., the expression for the orientation predicate, against round-off errors, we need a guard predicate \mathcal{G}_E :

If \mathcal{G}_E evaluates to true when evaluated with floating point arithmetic, the evaluation of E with floating point arithmetic yields the correct sign.

use error bounds [e.g. Mehlhorn, Näher] to derive \mathcal{G}_E , namely

- replace branch on sign of E by
- if $|E| > B$ (as computed in last lecture) branch on sign of E else abort

Modify idealistic algorithm A as follows:

- Guard each predicate E .
- If any guard fails abort algorithm, else report result.

↪ **Guarded Algorithm A_g .**

Fact: A_g follows execution path of A and has same running time.

Controlled Perturbation: The Basic Scheme

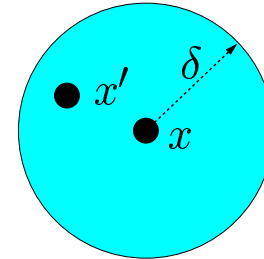


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Controlled Perturbation Scheme:

Input: $S = \{x_1, \dots, x_n\}$, Perturbation value $\delta > 0$

- * Compute random δ -perturbation S' of S , i.e., replace each point x by a random point x' in the δ -disc centered at x .
- * Run A_g on S' .
- * If A_g fails, restart.
- * Report result.



Question: What is a good value of δ ? Answer will depend on the **precision** p of floating point computation (this was called ϵ in the last lecture) and input, e.g. the **maximum coordinates** M .

Controlled Perturbation: Usage Szenarios



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- Usage 1: keep the precision p fixed
 - set δ to some initial value δ_0
 - run controlled perturbation up to c times
 - if all executions abort, double δ and repeat
- Usage 2: keep δ fixed
 - set p to some initial value p_0 , e.g., 52 (double precision floats)
 - run controlled perturbation up to c times
 - if all executions abort, double p and repeat
- what is the appropriate choice of c ?
- can we expect reasonable values of δ and p ?
- second question can be answered experimentally and analytically

On the Choice of c ?



- in the second szenario (fixed δ , variable p), the running time will depend on p , because the cost of arithmetic will depend on the mantissa length in use.
- assume running time is $p^k \cdot T$, where T depends on the input but not on p .
- if only addition and multiplications and school method is used, $k = 2$.
- let $f(p)$ be the property of abortion when precision p is used. We assume that $f(p)$ decreases and goes to zero as p increases
- let p^* be such that $f(p^*) \leq 1/2$. $p^* = 2^{i^*} p_0$
- expected running time is bounded by

$$\sum_{i \geq 0} \left(\prod_{j < i} f(2^j p_0)^c \right) \cdot c \cdot (2^i p_0)^k \cdot T = \sum_{i \leq i^*} c(2^i p_0)^k T + \sum_{i > i^*} (1/2)^{c(i-i^*)} \cdot c(2^i p_0)^k T$$

$$\leq c(2^{i^*} p_0)^k T + c(2^{i^*} p_0)^k T \sum_{i \geq i^*} \frac{2^{k(i-i^*)}}{2^{c(i-i^*)}} = O(c(p^*)^k T) \quad \text{provided that } c > k$$

Randomized Algorithms



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Let A be a randomized algorithm .

Theorem: If δ is such that A_g has constant failure probability $1/c$ on a δ -perturbed input, then the expected asymptotic running time of controlled perturbation is the running time of A .

see next slide for a proof

Application to RICs: RICs process the n input objects one by one. In order to guarantee an overall failure probability $1/c$, we need failure probability $1/cn$ for any of the n insertions.

Randomized Algorithms



Let A be a randomized algorithm,

$T(x, \pi)$ its running time on input x with random bits $\pi \in \{0, 1\}^m$,

$T(x) = 2^{-m} \sum_{\pi} T(x, \pi)$ its expected running time on input x

$U_{\delta}(x)$ the set of all δ -perturbations of x

$T_{\delta}(x) = \mathbb{E}_{x' \in U_{\delta}(x)}[T(x')] = \frac{1}{|U_{\delta}(x)|} \sum_{x' \in U_{\delta}(x)} T(x')$ the δ -smoothed running time at x .

For A_g we have $T_g(x, \pi) = \mathcal{O}(T(x, \pi))$ for all x and π .

Let $\chi(x, \pi) = 1$ if $A_g(x, \pi)$ aborts, zero otherwise. Then

$$p_{\delta}(x) = \sum_{\pi} \sum_{x' \in U_{\delta}(x)} \frac{\chi(x', \pi)}{2^m \cdot |U_{\delta}(x)|}$$

is the probability that A_g fails on a random $x' \in U_{\delta}(x)$

running time T_g of A_g on x : $T_g(x) = T_{\delta}(x) + p_{\delta}(x)T_g(x) = \frac{T_{\delta}(x)}{1-p_{\delta}(x)}$.

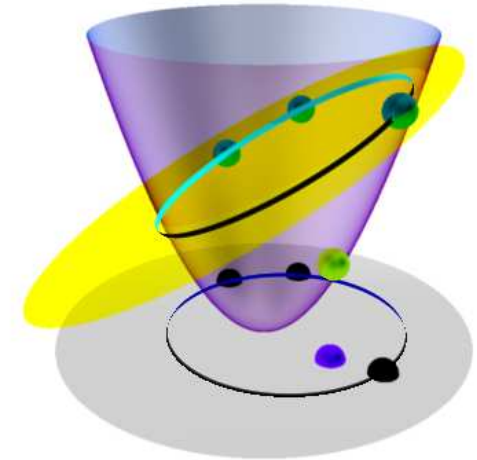
Predicates and Guards



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Delaunay RIC uses predicates
 $orient(p, q, r)$ and $incircle(p, q, r, t)$,

where $incircle(p, q, r, t)$
 $= orient(l(p), l(q), l(r), l(t))$
with $l((x, y)) = (x, y, x^2 + y^2)$



Can derive **error bounds**

$$B_{Orient} = 24 \cdot M^2 2^{-p}, \quad B_{Incircle} = 432 \cdot M^4 2^{-p}.$$

Guards ensure predicates have **absolute values** greater than error bounds.

Delaunay Triangulations I



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A triangulation is called **Delaunay triangulation** if the interior of the circumcircle of any triangle contains no point.

RIC for Delaunay triangulation in $\mathcal{O}(n \log n)$:

- Start with infinite Triangle.
- Pick a new point p at random.
- Use history to locate triangle containing p .
- Split Triangle and “legalize” triangulation.

Expected number of edges generated during the algorithm is $6n$ and at most twice as many triangles are generated.

for details of the algorithm see LEDAbook or deBerg/vanKreveld/Overmars/Schwarzkopf or ...

Legalize Triangulation



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- initialize a list L to the three new triangles with vertex p
- invariant: all triangles in L will have p as a vertex
- while L is non-empty do
 - let (p, a, b) be one of the triangles in p .
 - let (a, b, c) the other triangle sharing the edge (a, b)
 - if (p, a, b, c) form a convex quadrilateral and p lies inside the circumcircle of (a, b, c) , replace the triangles (p, a, b) and (a, b, c) by the triangles (p, a, c) and (p, b, c) . Put the new triangles on L .
- connection to 3d convex hull of lifted points

Experimental Results



Input, pts	avg. δ (lazy)	avg. δ (standard)
Flower, 400	0.05877	0.001
Flower, 2000	0.20473	0.0023
Flower, 10000	0.79845	51.76
Grid, 441	0.00308	0.001
Grid, 2601	0.00675	0.0043
Grid, 10201	0.01299	0.105
Grid, 160801	0.05181	>grid

lazy: choose δ so that alg works and perturb points only if necessary

standard: use δ as given by analysis and perturb every point



Controlled Perturbation for Delaunay Triangulations

The Analysis

Delaunay Triangulations

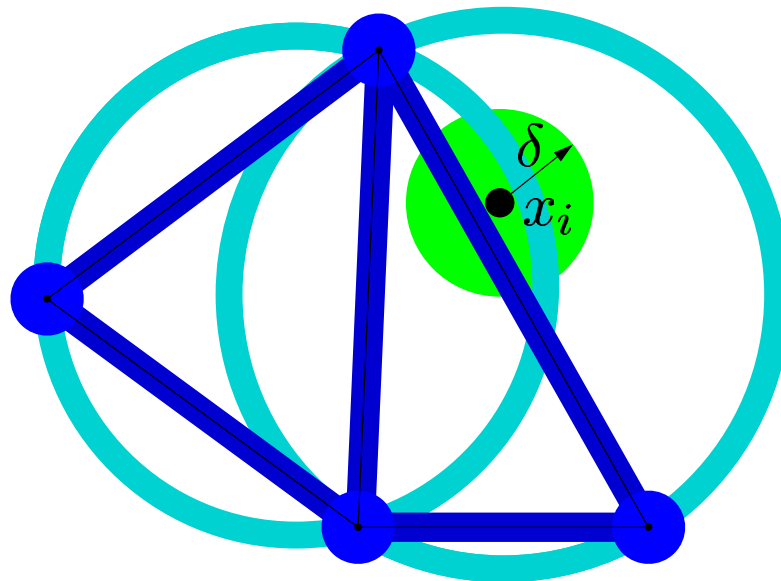


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Want δ such that guards hold with reasonable probability when inserting point x_i^{pert} .

Areas where guards fail: **Forbidden Areas**.

\rightsquigarrow only a small fraction of δ -disc should be forbidden; more precisely, a fraction $1/(cn)$. This guarantees failure probability $1/(cn)$ for single insertion and $1/c$ overall. Concrete, $c = 2$.



Delaunay Triangulations



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Geometrically, guards hold when evaluating

\rightsquigarrow $orient(p, q, x_i)$, if points form **triangle of area at least $B_{Orient}/2$** .

since $orient(p, q, x_i) =$ signed area of triangle $\Delta(p, q, x_i)$

\rightsquigarrow $incircle(p, q, r, x_i)$, if x_i lies **outside $B_{Incircle}/area(\Delta)^{3/2}$ -annulus** around circumcircle of $\Delta(p, q, r)$.

since $incircle(p, q, r, x_i) \geq area(\Delta)^{3/2} \cdot dist(x_i, Circumcircle)$

We further require (as a kind of induction hypothesis)

- $dist(p, q) \geq \xi$ for all points
- $area(\Delta(p, q, r)) \geq \xi_{\Delta} > B_{Orient}/2$ for all created triangles.

Delaunay Triangulations



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Delaunay Triangulations



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- average number of edges constructed is at most $6n$
- more than $24n$ edges with probability at most $1/4$
- we restrict to executions with at most $24n$ constructed edges
- each point generates a forbidden region of size $\pi\xi^2$
- total forbidden region due to points: $n\pi\xi^2$
- each edge e generates a forbidden region
 - length of edge is at least ξ
 - no point in strip of half-width $2\xi_\Delta/\xi$ around $\ell(e)$ guarantees that all triangles (e, x) have area at least ξ_Δ
 - strip intersects disk of radius δ in a region of area at most $2\delta 4\xi_\Delta/\xi$
 - total forbidden region due to edges: $24n8\delta\xi_\Delta/\xi$

Delaunay Triangulations

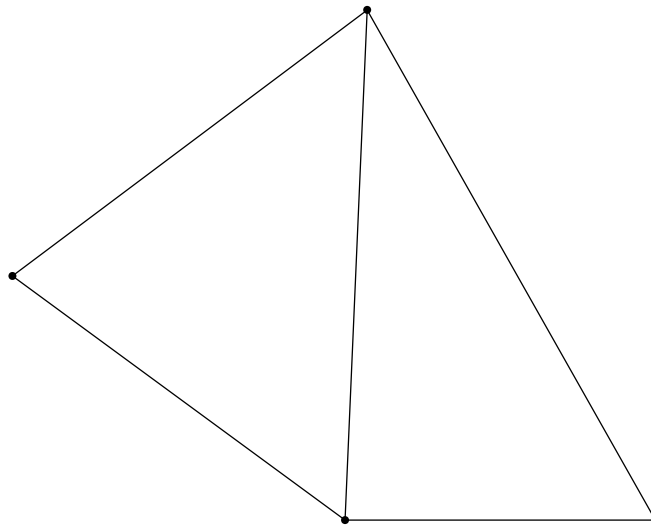


MAX-PLANCK-GESELLSCHAFT

Lemma: If

$$\pi\delta^2/(4n) \geq (\quad),$$

the guarded Delaunay algorithm will succeed with probability at least $1/2$.



Delaunay Triangulations

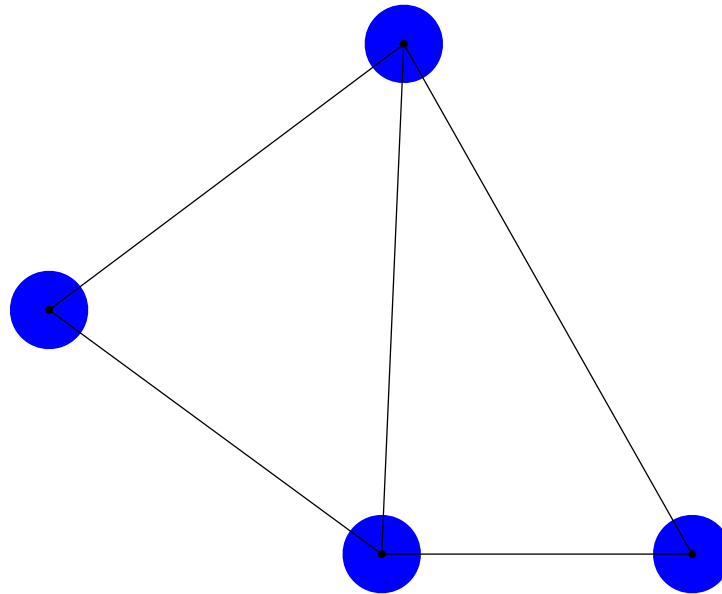


MAX-PLANCK-GESELLSCHAFT

Lemma: If

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Delaunay Triangulations

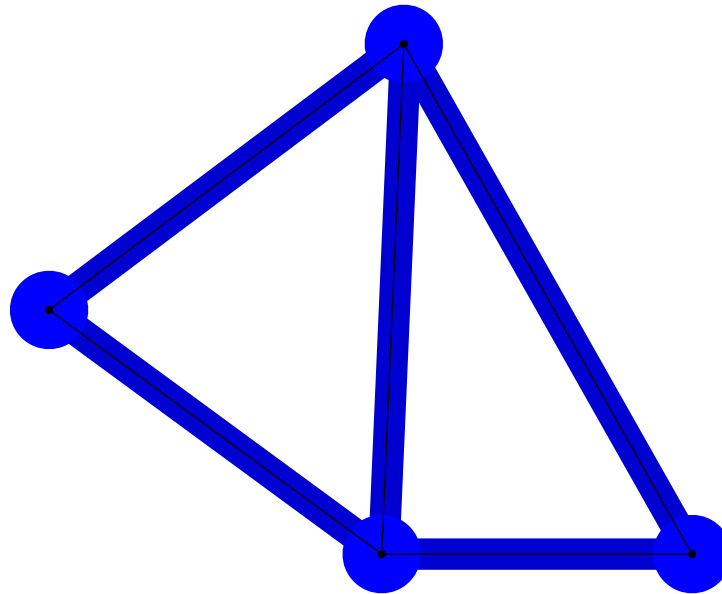


MAX-PLANCK-GESELLSCHAFT

Lemma: If

$$\pi\delta^2/(4n) \geq (n\pi\xi^2 + 24n \cdot 8\delta\xi_\Delta/\xi),$$

the guarded Delaunay algorithm will succeed with probability at least $1/2$.



Delaunay Triangulations

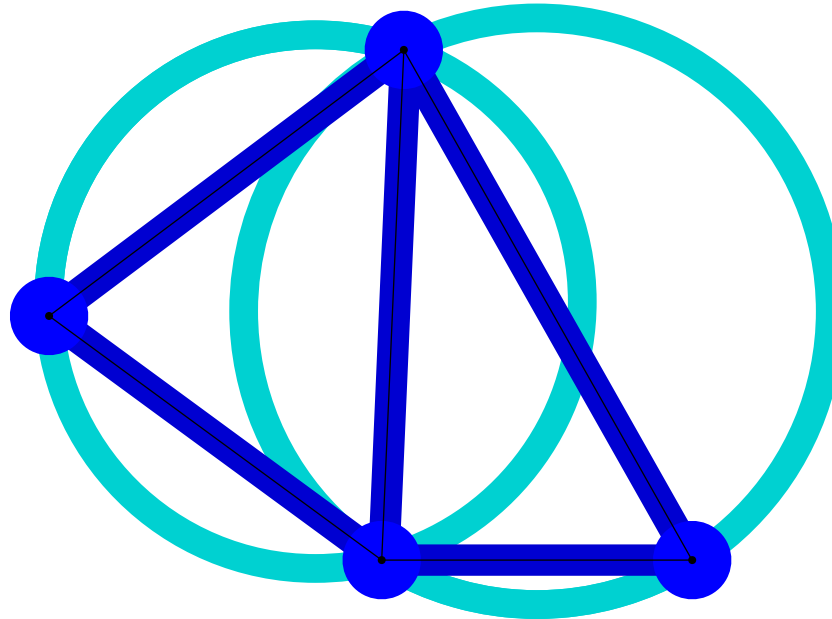


MAX-PLANCK-GESELLSCHAFT

Lemma: If

$$\pi\delta^2/(4n) \geq (\mathbf{n}\pi\xi^2 + \mathbf{24n} \cdot 8\delta\xi_{\Delta}/\xi + \mathbf{2n} \cdot 4\pi\delta B_{Incircle}/\xi_{\Delta}^{3/2}),$$

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Delaunay Triangulations



MAX-PLANCK-GESELLSCHAFT

Lemma: If

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the guarded Delaunay algorithm will succeed with probability at least $1/2$.

Theorem: If the guarded algorithm is executed with precision p , where $p \geq C(\log M - \log \delta + \log n + 1)$ for a suitable constant C , it succeeds with probability at least $1/2$.

Open Problems



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- apply approach to problems which require non-rational arithmetic
- Voronoi diagrams of line segments
- Arrangements of Conics
- Arrangements of High Degree Curves