Set-Up

set up non-certifying and certifying planarity demo. Let the non-certifying demo run during introduction

Certifying Algorithms

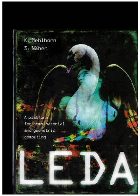
Algorithms Explaining their Work
Algorithmics meets Software Engineering

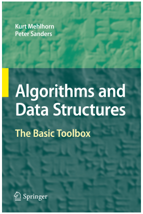
Kurt Mehlhorn

max planck institut
informatik

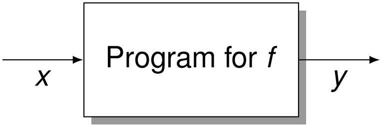
Outline of Talk

- problem definition and certifying algorithms
- examples of certifying algorithms
  - testing bipartiteness
  - matchings in graphs
  - planarity testing
  - convex hulls
  - further examples
- advantages of certifying algorithms
- universality
- formal verification and certifying algorithms
- summary





The Problem



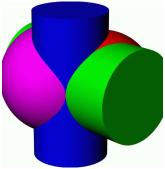
- A user feeds x to the program, the program returns y.
- How can the user be sure that, indeed,

$$y = f(x)?$$

The user has no way to know.

Warning Examples

- LEDA 2.0 planarity test was incorrect
- Rhino3d (a CAD systems) fails to compute correct intersection of two cylinders and two spheres

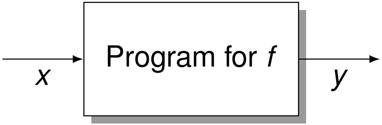


- CPLEX (a linear programming solver) fails on benchmark problem *etamacro*.
- Mathematica 4.2 (a mathematics systems) fails to solve a small integer linear program

```
In[1] := ConstrainedMin[ x , {x==1,x==2} , {x} ]
Out[1] = {2, {x->2}}

In[1] := ConstrainedMax[ x , {x==1,x==2} , {x} ]
ConstrainedMax::"lpsub": "The problem is unbounded."
Out[2] = {Infinity, {x -> Indeterminate}}
```

The Problem



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A program should justify (prove) its answers in a way that is easily checked by the user of the program.

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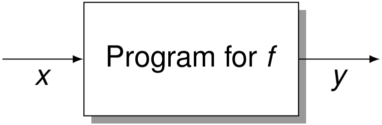

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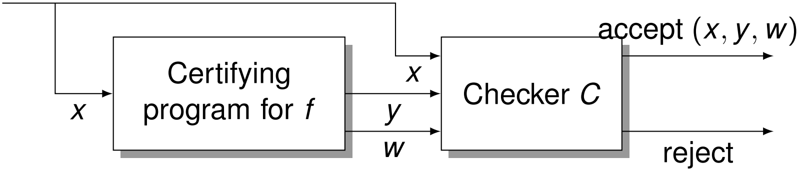
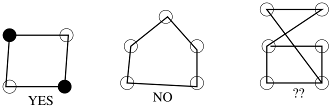
A Certifying Program for a Function f



- On input x, a certifying program returns
  the function value y and a certificate (witness) w
- w proves $y = f(x)$ even to a dummy,
- and there is a simple program C, the checker, that verifies the validity of the proof.

A First Example: Testing Bipartiteness of Graphs

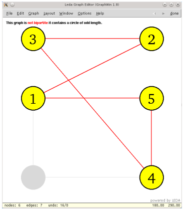
A graph is bipartite if its vertices can be colored black and white such that the endpoints of each edge have distinct colors.



Conventional algorithm outputs YES or NO

Certifying Algorithm outputs

- a two-coloring in the YES-case
- an odd cycle in the NO-case



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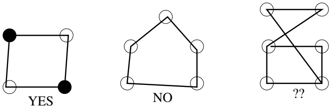

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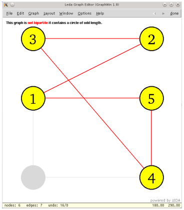

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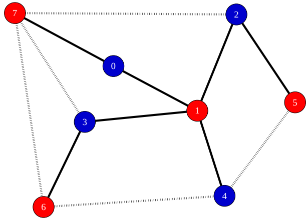
Bipartite Graphs: An Algorithm

- construct a spanning tree of G
- use it to color the vertices with colors red and blue
- check for all non-tree edges: do endpoints have distinct colors?
- if yes, the graph is bipartite and the coloring proves it.
- if no, declare the graph non-bipartite: Let $e = \{ u, v \}$ be a non-tree edge with equal colored endpoints



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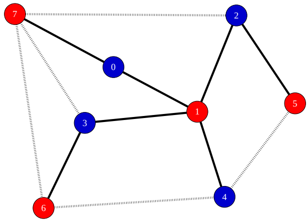

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Examples

Planarity Testing
Maximum Cardinality Matchings
Further Examples

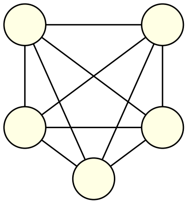
Example II: Planarity Testing

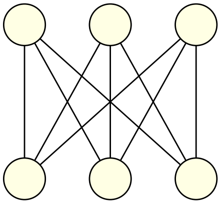
- Given a graph G, decide whether it is planar
- Tarjan (76): planarity can be tested in linear time
- A story and a demo
- Combinatorial planar embedding is a witness for planarity

Chiba et al (85): planar embedding of a planar G in linear time

- Kuratowski subgraph is a witness for non-planarity

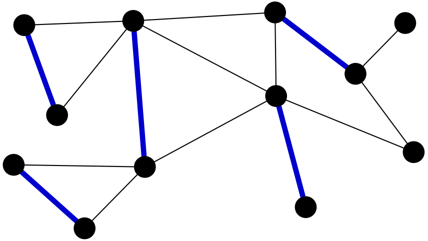
Hundack/M/Näher (97): Kuratowski subgraph of non-planar G in linear time, LEDAbook, Chapter 9





Example III: Maximum Cardinality Matchings

- A matching *M* is a set of edges no two of which share an endpoint



- The blue edges form a matching of maximum cardinality; this is non-obvious as two vertices are unmatched.
- A conventional algorithm outputs the set of blue edges.

Maximum Cardinality Matching: A Certifying Alg

Edmonds' Theorem: Let M be a matching in a graph G and let ℓ be a labelling of the vertices with non-negative integers such that for each edge $e = (u, v)$ either $\ell(u) = \ell(v) \geq 2$ or $1 \in \{ \ell(u), \ell(v) \}$. Then

$$|M| \leq n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor ,$$

where n_i is the number of vertices labelled i.

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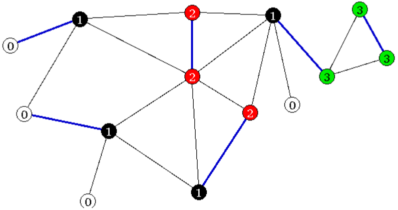

- $n_1 = 4$, $n_2 = 3$, $n_3 = 3$.
- no matching has more than $4 + \lfloor 3/2 \rfloor + \lfloor 3/2 \rfloor = 6$ edges.
- $|M| = 6$

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- Let M_1 be the edges in M having at least one endpoint labelled 1 and, for $i \geq 2$, let M_i be the edges in M having both endpoints labelled i.
- $M = M_1 \cup M_2 \cup M_3 \cup \ldots$
- $|M_1| \leq n_1$ and $|M_i| \leq n_i/2$ for $i \geq 2$.

Further Examples

- biconnectivity, strong connectivity, flows, . . . ,
- Convex Hulls
- Schmidt, Mehlhorn/Neumann/Schmidt: Three-Connectivity of Graphs
- Georgiadis/Tarjan: Dominators in Digraphs
- Wang: Arrangements of Algebraic Curves
- Mehlhorn/Sagraloff/Wang: Root Isolation for Real Polynomials
- Althaus/Dumitriu: Certifying feasibility and objective value of linear programs
- Hauenstein/Sottile: alphaCertified: certifying solutions to polynomial systems
- Cook et al: Traveling Salesman Tours
- Cheung/Gleixner/Steffy: Verifying Integer Programming Results

History

- I do not claim that I invented the concept; it is an old concept
  - al-Kwarizmi: multiplication
  - extended Euclid: gcd
  - primal-dual algorithms in combinatorial optimization
  - Blum et al.: Programs that check their work
- I do claim that Näher and I were the first (1995) to adopt the concept as the design principle for a large library project: LEDA

  (Library of Efficient Data Types and Algorithms)
- Kratsch/McConnell/M/Spinrad (SODA 2003) coin name
- McConnell/M/Näher/Schweitzer (2010): 80 page survey

How I got interested?

- till ’83: only theoretical work in algorithms and complexity
- ’83 – ’89: participation in a project on VLSI design:
  implementation work proceeds very slowly
- since ’89: LEDA, library of efficient data types and algorithms
- many implementations incorrect
- ’95: adopt exact computation paradigm (computational geometry) and certifying algorithms as design principles
- ’95 – ’99: make textbook algs certifying, reimplementation of library, LEDA book
- since ’00: additional certifying algorithms
- ’10: 80 page survey paper
- since ’12: formal verification of checkers

The Advantages of Certifying Algorithms

- Certifying algs can be tested on
  - **any** input
  - and not just on inputs for which the result is known.
- Certifying algorithms are reliable:
  - Either give the correct answer
  - or notice that they have erred $\Rightarrow$ confinement of error
- Computation as a service
  - There is no need to understand the program, understanding the witness property and the checking program suffices.
  - One may even keep the program secret and only publish the checker

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Odds and Ends

- General techniques
  - Linear programming duality
  - Characterization theorems
  - Program composition
- Probabilistic programs and checkers
- Reactive Systems (data structures)
- does apply to problems in NP (and beyond), e.g., SAT
  - output a satisfying assignment of satisfiable inputs or
  - ouput a resolution proof for unsatisfiability.

Universality

- Does every problem have a certifying algorithm? Can every program be converted into a certifying one?

- I know 100+ certifying algorithms, see survey by McConnell/M/Näher/Schweitzer (CS Review), in particular, all text-book algorithms can be made certifying
- most programs in LEDA are certifying, and
- checking a solution is never harder than finding it.

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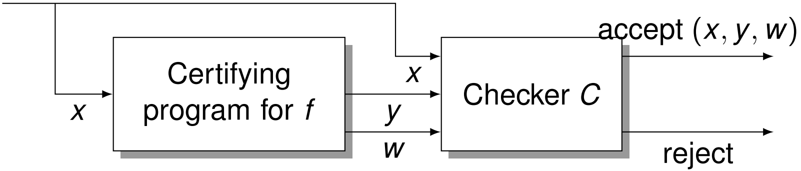

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Let us have a closer look at the checker programs.

The Maximum Cardinality Matching Checker

Edmonds' Theorem: Let M be a matching in a graph $G = (V, E)$ and let $\ell : V \to \mathbb{N}$ such that for each edge $e = (u, v)$ of G either $\ell(u) = \ell(v) \geq 2$ or $1 \in \{ \ell(u), \ell(v) \}$. Then

$$|M| \leq n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor \, ,$$

where n_i is the number of vertices labelled i.

The Checker Program has input G, M, and ℓ:

- checks that $M \subseteq E$,
- checks that M is a matching,
- checks that ℓ satisfies the hypothesis of the theorem, and
- checks that $|M| = n_1 + \sum_{i \geq 2} \lfloor n_i/2 \rfloor$

set $c[v] = 0$ for all $v \in V$;
for all $e = (u, v) \in M$: increment $c[u]$ and $c[v]$;
if some counter reaches 2, M is not a matching.

Who Checks the Checker?

How can we be sure that the checker programs are correct?

My answer up to 2011: Because they are so simple.

Because we can prove their correctness in a formal system

Isabelle/HOL

Nipkow/Paulson

- formal mathematics
- proof are machine-checked
- only kernel needs to be trusted

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definition *disjoint-edges* :: (α, β) *pre-graph* $\Rightarrow \beta \Rightarrow \beta \Rightarrow$ *bool* **where**
disjoint-edges $G\ e_1\ e_2 = ($
start $G\ e_1 \neq$ *start* $G\ e_2 \wedge$ *start* $G\ e_1 \neq$ *target* $G\ e_2 \wedge$
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definition *matching* :: (α, β) *pre-graph* $\Rightarrow \beta$ *set* $\Rightarrow$ *bool* **where**
matching $G\ M = ($
$M \subseteq$ *edges* $G \wedge$
$(\forall e_1 \in M.\ \forall e_2 \in M.\ e_1 \neq e_2 \longrightarrow$ *disjoint-edges* $G\ e_1\ e_2))$

definition *edge-as-set* :: $\beta \Rightarrow \alpha$ *set* **where**
edge-as-set e $\equiv$ {tail G e, head G e}

lemma *matching*_disjointness:
assumes *matching* G M
assumes $e_1 \in$ M **assumes** $e_2 \in$ M **assumes** $e_1 \neq e_2$
shows *edge-as-set* $e_1 \cap$ *edge-as-set* e_2 = {}
using assms
by (auto simp add: *edge-as-set*_def *disjoint-edges*_def *matching*_def)

What do we Formally Verify and How?

- Edmonds' theorem
- Checker always halts and either rejects or accepts.
- Checker accepts a triple (G, M, ℓ) iff is satisfies the assumptions of Edmonds' theorem.
- we prove Edmonds' theorem in Isabelle
- we translate checkers from C to I-Monads with AutoCorres (NICTA)
- I-Monads is a programming language defined in Isabelle
- we prove items 2 and 3 for the resulting I-Monads program in Isabelle
- since NICTA-tools are verified, this verifies the C-code of the checker
- verification revealed that one of the checkers in LEDA was incomplete

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Formal Verification: Summary

Formal Instance Correctness

If a formally verified checker accepts a triple (x, y, w),

we have a formal proof that y is the correct output for input x.

- a high level of trust (only Isabelle kernel needs to be trusted)
- a way to build large libraries of trusted algorithms

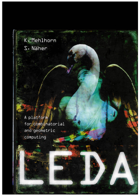
Alkassar/Böhme/M/Rizkallah: Verification of Certifying Computations, JAR 2014

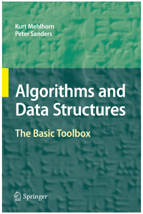
Noshinski/Rizkallah/M: Verification of Certifying Computations through AutoCorres and Simpl, NASA Formal Methods Symposium 2014

Summary

- **Only certifying algs are good algs**
- Certifying algs have many advantages over standard algs:
  - every run is a test
  - notice when they erred
  - can be relied on without knowing code
  - are a way to computation as a service
- Formal verification of checkers and formal proof of witness property are feasible
- Most programs in the LEDA system are certifying.

**When you design your next algorithm,
make it certifying.**





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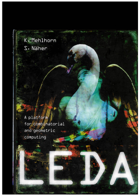

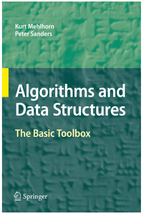

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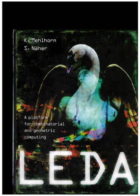

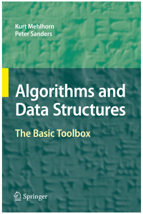

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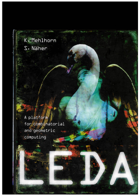

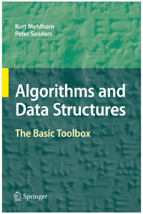

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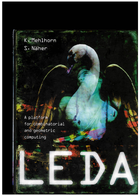
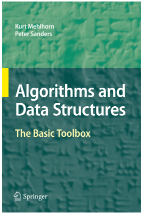
- Formal verification of checkers and formal proof of witness property are feasible
- Most programs in the LEDA system are certifying.





When you design your next algorithm,
make it certifying.