Exact Algorithms for a Geometric Packing Problem (Extended Abstract)

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1 Introduction

The following packing problem arises in the connection with lettering of maps: given n points $p_1, \ldots, p_n$ in the plane determine the supremum σ_{opt} of all reals σ, for which there are n pairwise disjoint, axis-parallel squares $Q_1, \ldots, Q_n$ of side length σ, where for each i, $1 \le i \le n$, p_i is a corner of Q_i. In [**?**] Formann and Wagner showed that the related decision problem is NP-complete, that there is an $O(n \log n)$ approximation algorithm which finds a packing with side length $\sigma \ge \sigma_{opt}/2$, and that there is no better approximation algorithm unless P equals NP.

In this paper we discuss *exact* algorithms for the decision problem: Given a set X of n points $p_1, \ldots, p_n$ in the plane, decide whether there are unit-size, pairwise disjoint, axis-parallel squares $Q_1, \ldots, Q_n$ where for each i, $1 \le i \le n$, p_i is a corner of Q_i. It is easy to see that the optimization problem can be reduced to $O(n^2)$ decision problems; one only has to observe that σ_{opt} is the x-distance or y-distance of two points in X or half of such a distance.

The goal of our research is to develop algorithms which can solve packing problems with up to one hundred points exactly. Applying such an algorithm to subproblems of larger packing problems would allow to compute *upper* bounds for σ_{opt}. Combined with the *lower* bound given by the approximation algorithm of Formann and Wagner, we hope to obtain very good (within a few percent) estimates of σ_{opt}. Our experiments substantiate this hope.

Let X denote our set of points. In [**?**] it was shown that if the placement problem for X has a solution then no square with side length 3 can contain more than 25 points of X. In view of this observation we make (for this extended abstract but not for the full paper) the simplifying assumption that X is sparse, where a set X of points is called *sparse* if any two points of X have distance at least one.

A naive algorithm checks all 4^n placements of the n squares and thus has running time $O(4^n poly(n))$. It can be used to solve problems with at most 10 points in a few seconds on a SUN SPARC. In section 2 we give an asymptotically much faster algorithm.

Theorem 1 *The decision problem can be solved in time* $4^{O(\sqrt{n})}$.

The algorithm is based on a variant of the planar separator theorem [LT77] given in [LM84]. Unfortunately, this algorithm is of only theoretical interest and does not extend the range of problems which can be solved in realistic amounts of time.

In section 3 we give a simple sweep line algorithm. It sweeps the plane orthogonally to a carefully chosen direction d. Call an element of $[0, \pi)$ a *direction* and call

$$\mathcal{C}_X(d) = \max_{y \in \mathbb{R}^2} |\{x \in X \; ; \; |(x_1 - y_1)\cos d + (x_2 - y_2)\sin d| \leq \sqrt{2}\}|$$

the *complexity* of X in direction d. Note that $\mathcal{C}_X(d)$ is the largest number of elements of X contained in any stripe of width $2\sqrt{2}$ orthogonal to vector $(\cos d, \sin d)$. Let $\mathcal{C}_X$ be the minimal value of $\mathcal{C}_X(d)$ where d ranges over all directions, and let $\mathcal{C}(n)$ be the maximal value of $\mathcal{C}_X$ for any set X of n points. We show

Theorem 2 *Let X be a sparse set of cardinality n.*

1. *The sweep algorithm, when used in direction d, solves the decision problem in time* $O(4^{\mathcal{C}_X(d)} poly(n))$.
2. *A direction d minimizing* $\mathcal{C}_X(d)$ *can be found in polynomial time.*
3. $\mathcal{C}(n) \leq 16\sqrt{(n \ln n)/\pi}$.
4. $\mathcal{C}(n) = \Omega(\sqrt{n \log n})$.

The proof of part 4 was suggested by J. Matoušek.

We close this introduction with some definitions. Identify the four possible placements of a single square with the symbols in the set $A = \{\text{SW,SE,NE,NW}\}$ and identify the possible placements of the squares for a set X of points with the functions from X to A. Call a placement ρ *legal* if the squares are disjoint when placed as given by ρ. Let X' be a subset of X. Call a placement ρ' for X' *extendible* if there is a legal placement ρ for X which extends ρ', i.e., $\rho(x) = \rho'(x)$ for all $x \in X'$.

2 A Proof of Theorem 1

We sketch a proof of Theorem 1. This following Lemma is essential.

Lemma 1 *Let X be a sparse set of n points. Then X can be partitioned into sets X_1, S, and X_2 such that*

1. $|S| = O(\sqrt{n})$,
2. $|X_1|, |X_2| \leq 2n/3$,
3. *any point in X_1 has distance at least two from any point in X_2,*
4. *the partition can be determined in polynomial time.*

Proof: The proof is a modification of the proof of [LM84, Theorem I]. ∎

The algorithm is now as follows. Partition X into sets X_1, S, and X_2. Cycle through all placements π_S for S and use the algorithm recursively to check whether π_S can be extended to placements π_1 for $S \cup X_1$ and π_2 for $S \cup X_2$. If there is such a placement π_S then declare the problem solvable, if not declare it unsolvable. The correctness of this algorithm follows from the observation that the squares for the points in X_1 and X_2 can be placed independently since points in X_1 have distance at least two from points in X_2.

Let $T(n)$ be the maximal running time of the algorithm on an input of n points. Then T obeys the recurrence

$$T(n) = 4^{O(\sqrt{n})}(poly(n) + max\{T(n_1) + T(n_2) \quad ; n_i \leq 2n/3 + O(\sqrt{n}) \text{ for } i = 1, 2 \text{ and } n_1 + n_2 = n + O(\sqrt{n})\})$$

This recurrence has solution $T(n) = O(4^{O(\sqrt{n})}poly(n))$.

3 A Proof of Theorem 2

We first describe the algorithm and prove parts 1 and 2 of the theorem. We then turn to parts 3 and 4.

3.1 The Sweep Algorithm

Let d be a direction. The algorithm sweeps a line orthogonal to d across the plane. For a position p of the sweep line, let $B(p)$ be the set of points in X behind the sweep line and let $X(p)$ be the set of points in $B(p)$ which are within distance $2\sqrt{2}$ of the sweep line. The algorithm maintains all placements for $X(p)$

which can be extended to a legal placement of $B(p)$. The set $X(p)$ changes $2n$ times as each point enters and leaves $X(p)$ exactly once. When $X(p)$ changes the set of placements needs to be updated. We will show in the full paper that this can be done in time $O(4^{|X(p)|}poly(n))$. Since $|X(p)| \leq \mathcal{C}(d)$ this implies part 1.

Part 2 follows from the observation that the search for an optimal d can be restricted to a set of polynomially many candidate directions and that $\mathcal{C}(d)$ for a fixed direction d can be determined in linear time.

3.2 An Upper Bound for $\mathcal{C}(n)$

We prove part 3. Let $m = 16\sqrt{(n \ln n)/\pi}$ (= our claimed upper bound on $\mathcal{C}(n)$), let $k = 2n/m = \sqrt{\pi n/(64 \ln n)}$, and for i, $1 \leq i \leq k$, let $d_i = i\pi/k$. Let $\varepsilon = 2\sqrt{2}$ and suppose that for each $i \in [1..k]$ there is a stripe Q_i of width ε, containing at least m elements of X.

Given two stripes of width $1+\varepsilon$, the area of the intersection of the stripes is at most $(1+\varepsilon)^2 \sin^{-1}\gamma \leq 16 \sin^{-1}\gamma \leq 8\pi/\gamma$, where γ is the angle between the stripes.

Note that if a stripe of width ε around a given line contains a point x, then a stripe of width $1+\varepsilon$ contains x together with the 1/2-ball centered at x, and if x, y are two different elements of X, then the 1/2-balls around them are disjoint. Hence the intersection of any two stripes of width ε contains at most $(8\pi/\gamma)/(\pi/4) = 32/\gamma$ elements of X.

We say that a point $x \in X$ is *ambiguous*, if it is contained in at least two different stripes Q_i, $i = 1, \ldots, k$. Since for any i and any $s \leq k$ there are at most $2s$ elements $j \in [1..k]$, such that $|d_i - d_j| \leq \alpha s/k$ it follows that, for any fixed i, the total number of points in X in

$$Q_i \cap \bigcup_{j \neq i} Q_j$$

is at most

$$2\frac{32}{\pi/k} + 2\frac{32}{2\pi/k} + 2\frac{32}{3\pi/k} + \cdots + 2\frac{32}{(k/2)\pi/k} =$$

$$= \frac{64k}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k/2}\right) <$$

$$< \frac{64k \ln k}{\pi} \leq \frac{64k \ln n}{\pi} = \frac{m}{2}$$

for large n. Therefore each stripe Q_i, $i = 1, \ldots, k$, contains more than $\frac{m}{2}$ unambiguous points, which implies that the number of points of X should be more than $mk/2 = n$, which is a contradiction.

3.3 A Lower Bound for $\mathcal{C}(n)$

We show that $\mathcal{C}(n) = \Omega(\sqrt{n \ln n})$.

In [?] the following result was shown: Let $p \geq 2$ be an integer and let $m = 2^{p-2}$. Consider a two by two square which is partitioned into $4m$ triangles; each triangle has a base of length $2/m$ aligned with one of the sides of the square and has the center of the square as its third vertex. Then it is possible to rearrange the triangles (by translating them individually) such that the resulting figure, call it B, has area less than or equal to $4/p$.

Let $C = 100mp$, let B' be the figure B scaled up by factor C and let B'' be the set of grid points contained in B'.

Claim 1 *B' contains at most $5 \cdot 10^4 m^2 p$ grid points.*

Proof:

B' is the union of $4m$ triangles each with perimeter less than $400mp$. Such a triangle can contain at most $500mp$ grid points such that the unit square centered at the grid point intersects the boundary of the triangle.

For every grid point contained in B' the unit square centered at the grid point is either completely contained in B' or intersects the boundary of one of the triangles forming B'. There are at most $4C^2/p$ grid points of the former kind and at most $4m \cdot 500mp$ grid points of the latter kind. ∎

Claim 2 *$\mathcal{C}_{B''} \geq 20mp$.*

Proof: The figure B contains for each direction d a $1/4$ by $1/m$ rectangle with its longer side aligned with direction d. Thus B' contains a $25mp$ by $100p$ rectangle for each direction. Consider any such rectangle. It contains at least $2000mp^2$ grid points (since its area is $2500mp^2$ and its perimeter is less than $60mp$). Thus, if one partitions the rectangle into $100p$ rectangles with sides $25mp$ and 1 then one of the resulting rectangles contains at least $20mp$ grid points. This shows $\mathcal{C}_{B''} = 20mp$. ∎

Lemma 2 *$\mathcal{C}(n) = \Omega(\sqrt{n \log n})$.*

Proof: Choose p maximal such that $5 \cdot 10^4 \cdot m^2 p \leq n$ where $m = 2^p/4$. Then $\mathcal{C}(n) \geq 20mp$ by Claims 1 and 2. ∎

References

[LM84] T. Lengauer and K. Mehlhorn. Four results on the complexity of VLSI computation. *Advances in Computing Research*, 2:1–22, 1984.

[LT77] R. Lipton and R.E. Tarjan. A separator theorem for planar graphs. In *Conference on Theoretical Computer Science, Waterloo*, pages 1–10, 1977.