# Selfish Routing

## Price of Anarchy and Coordination Mechanisms

**Kurt Mehlhorn**

**joint work with Spyros Angelopoulos, Giorgos Christodoulou, and Evangelia Pyrga**

**Max Planck Institute for Informatics and Saarland University**

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# Outline

# Price of Anarchy and Coordination Mechanisms

## Global Optimum versus Selfish Behavior

consider a situation with many independent agents, e.g., traffic

Nash equilibrium = each agent optimizes its own fate

Global optimum = a solution of minimum cost

$$\text{Price of Anarchy} = \max \frac{\text{Cost of a Nash Equilibrium}}{\text{Cost of Global Optimum}}$$

Koutsoupias/Papadimitriou (99)

Coordination Mechanism = increase of costs that makes selfish agents behave differently

# Routing

## Basic Notation I

- $G = (V, E)$, a network, $s$ = source, $t$ = sink
- want to send $r$ units of flow from $s$ to $t$
- $f$ = a flow of rate $r$
- $f_e$ = flow across edge $e$

## The cost of a flow

$$C(f) = \sum_e \text{cost of } e \text{ at flow } f_e \cdot f_e$$

Observe: Cost (latency) of an edge depends on flow across it

# Routing

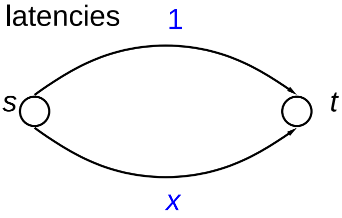
## Basic Notation II

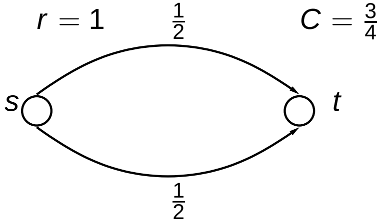
- $\ell_e(x)$ = latency (cost) of $e$ as a function of flow over $e$
- affine cost functions: $\ell_e(x) = a_e x + b_e$ with $a_e \geq 0$ and $b_e \geq 0$

## The cost of a flow

$$C(f) = \sum_e \ell_e(f_e) f_e$$

# Optimal Flow

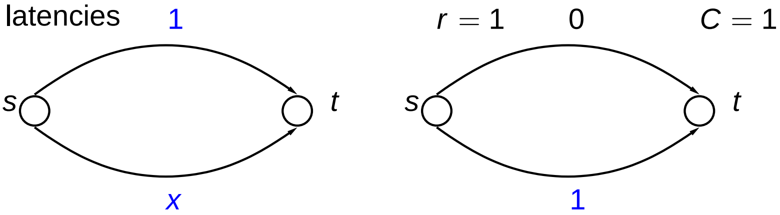



- cost of upper link$(x) = 0 \cdot x + 1$, cost of lower link$(x) = 1 \cdot x + 0$
- $f^* = f^*(r) =$ optimum flow for rate $r =$ flow of minimum cost
- here: $C(f^*) = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$
- opt-flow minimizes $f_1 \cdot f_1 + 1 \cdot f_2$ subject to $r = f_1 + f_2$, $f_i \geq 0$

  marginal costs are identical; here $\frac{d}{dx} x^2|_{x=1/2} = \frac{d}{dx} x|_{x=1/2}$
- selfish agents will deviate from optimum flow

# Nash Flow



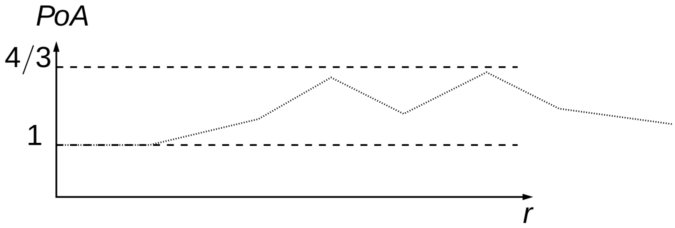

- Nash flow = no gain by deviating infinitesimally, i.e., all used edges have the same latency
- $f^N = f^N(r) =$ Nash flow for rate $r$
- here: $C(f^N) = 1 \cdot 0 + 1 \cdot 1 = 1$

## Price of Anarchy

$$PoA = \max_{r>0} \frac{C(f^N(r))}{C(f^*(r))} \geq \frac{C(f^N(1))}{C(f^*(1))} = \frac{1}{3/4} = \frac{4}{3}$$

# Remarks

- $C(f^N(r))$ and $C(f^*(r))$ are piecewise quadratic functions in $r$
- $PoA$ is quotient of piecewise quadratic functions in $r$



# Roughgarden/Tardos (02): aff. costs, $PoA \le 4/3$

proof for two links: assume Nash and Opt both use both links

let $L = \ell_1(f_1^N) = \ell_2(f_2^N)$ and assume $f_1^* \le f_1^N$

$$
\begin{aligned}
C^N - C^* &= L(f_1^N + f_2^N) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^* \\
&= L(f_1^* + f_2^*) - \ell_1(f_1^*)f_1^* - \ell_2(f_2^*)f_2^* \\
&= \left(\ell_1(f_1^N) - \ell_1(f_1^*)\right) f_1^* + \left(\ell_2(f_2^N) - \ell_1(f_2^*)\right) f_2^* \\
&\le \left(\ell_1(f_1^N) - \ell_1(f_1^*)\right) f_1^* \\
&\le \frac{\ell_1(f_1^N)f_1^N}{4} \le \frac{C^N}{4}
\end{aligned}
$$

next slide.

and hence $(1 - \frac{1}{4})C^N \le C^*$ . Thus $C^N \le \frac{4}{3}C^*$.

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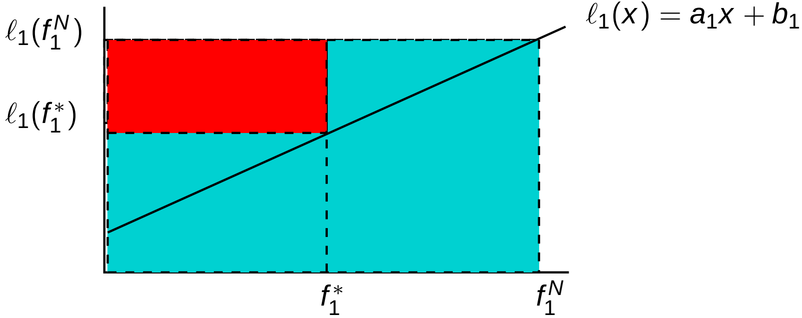
# The Key Inequality

$$\left(\ell_1(f_1^N) - \ell_1(f_1^*)\right) f_1^* \le \frac{\ell_1(f_1^N) f_1^N}{4}$$




(Correa/Schulz/Stier-Moses, 08)

# The Question

## Summary

For affine costs, the price of anarchy can be as large as $4/3$, but is never larger.

## Question

Can we reduce the price of anarchy by a coordination mechanism? In particular, by taxes or tolls? In other words

- underlying network is unchanged
- we increase the cost (latency) of some edges.
- this leads to a change of behavior of selfish agents
- such that total cost goes down
- although cost of new Nash flow is computed with respect to increased costs!!!!

# Question rephrased

Can making edges more expensive
reduce the overall cost
by leading to “better” behavior of selfish agents?

## Engineered Price of Anarchy (*ePoA*)

- replace $\ell_e$ by $\hat{\ell}_e$ with $\hat{\ell}_e(x) \geq \ell_e(x)$ for all $x$.
- $\hat{C}^N = \hat{C}^N(r) =$
  cost of Nash flow of rate $r$ for $\hat{\ell}$ computed with respect to $\hat{\ell}$
- Are there $\hat{\ell}$ such that **for all** $r$

$$ePoA(r) = \frac{\hat{C}^N(r)}{C^*(r)} < \frac{4}{3}?$$

- Observe: $\hat{C}^N$ is with respect to increased costs,
  $C^*$ is with respect to original costs.
  We want a solution that works for all $r$.

# The Answer is clearly NO

Obviously, increasing edge costs can never decrease total cost

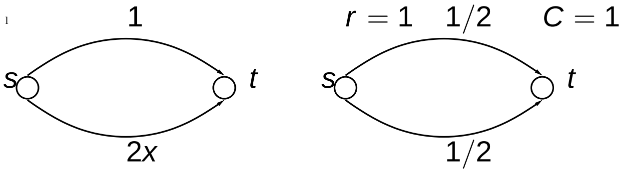
## A Negative Result

If the $\hat{\ell}$ are continuous, then $\hat{C}^N(r) \geq C^N(r)$ for all $r$

and hence $ePoA(r) \geq PoA(r)$ for all $r$

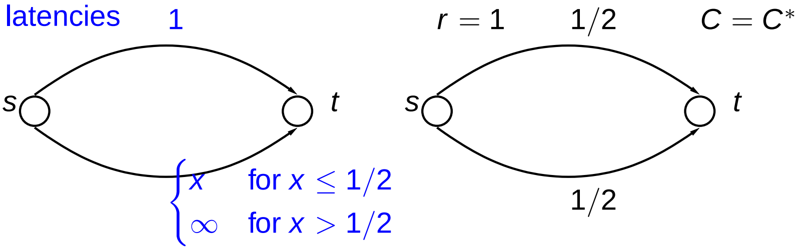
# A Non-Solution: Marginal Cost Pricing

$$\hat{\ell}(x) = \frac{d}{dx}\ell(x)x = 2a_e x + b_e$$



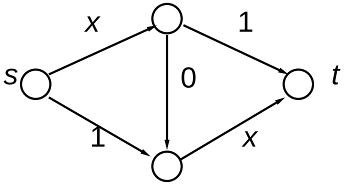

- Nash flow for marginal cost latencies = optimal flow for original latencies
- but $\hat{C}^N(1) = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$ and hence $ePoA(1) \geq 4/3$
- $\hat{C}^N(\epsilon) = 2\epsilon^2 = 2C^*(\epsilon)$ and hence $ePoA(\epsilon) = 2$.

# The Answer might be Yes




- Nash flow = Optimal flow for all $r$ and
- $\hat{C}^N = C^*$ for all $r$
- Thus $ePoA = 1$

# Braess' Paradox

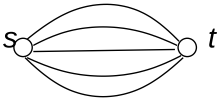



- At rate $r = 1$,
  - Opt routes 1/2 each along upper and lower path: $C^*(1) = 3/2$
  - Nash routes 1 along path $x \to 0 \to x$: $C^N(1) = 2$
  - deleting the edge of cost zero, i.e., setting its cost to $\infty$, makes the optimum flow a Nash flow, i.e., $\hat{C}^N(1) = 3/2$
  - generally, $\hat{\ell}(x) = 0$ for $x \leq 2/3$ and $\infty$ ow

In Stuttgart, after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.

# A Theorem

For any network of $k$ parallel links, there are modified latency functions $\hat{\ell}_1$ to $\hat{\ell}_k$ with $\hat{\ell}_i \geq \ell_i$ such that



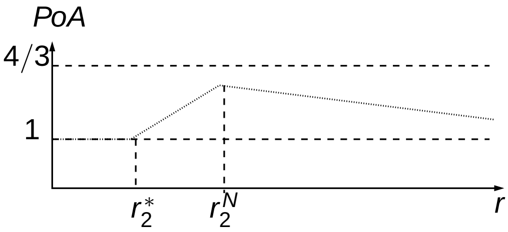

$$\frac{\hat{C}^N(r)}{C^*(r)} \leq c_k < \frac{4}{3} \quad \text{for all } r.$$

- $c_2 \leq 5/4$ by an easy argument
- $c_2 \leq 1.192$ by an involved argument
- $c_k \to 4/3$ for $k \to \infty$

# Open Problems

- improved upper bounds
  - improve upper bound for $c_2$?
  - is there a construction with $c_k \leq c < 4/3$ for all $k$
- lower bounds: we know $c_2 \geq 1.02$.
- general networks instead of parallel links
- more general cost functions, e.g., polynomial cost functions
- atomic flow, i.e., flow consists of units of fixed size instead of infinitesimal units

# Two Links, $b_1 < b_2$




$$PoA \leq \frac{4+4R}{3+4R}$$

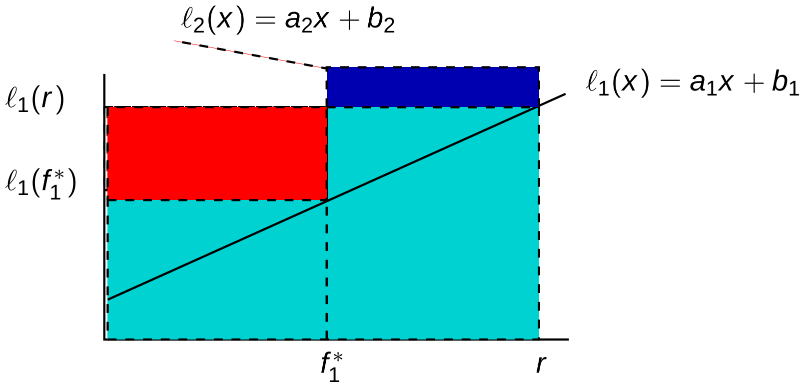
where $R = a_2/a_1$

- Nash starts to use the second link at $r = r_2^N = \frac{b_2 - b_1}{a_1}$
- worst-case PoA is at this rate, flows are:

$$\text{Nash: } (r, 0) \qquad \text{Opt: } (f_1^*, f_2^*) = (f_1^*, r - f_1^*)$$

# The Key Inequality Revised

flows are: Nash: $(r, 0)$ Opt: $(f_1^*, f_2^*) = (f_1^*, r - f_1^*)$




Opt saves the red area, but pays the blue area. $\frac{\text{red}-\text{blue}}{\text{cyan}} \leq \ldots$

# Two Links: Engineered Price of Anarchy

$$PoA \leq \frac{4+4R}{3+4R} \qquad \text{where } R = \frac{a_2}{a_1}$$

**The benign case: $R \geq 1/4$**

Then $PoA \leq \frac{5}{4}$

We do nothing, i.e. $\hat{\ell}_i = \ell_i$ for all $i = 1, 2$.

**The non-benign case: $R < 1/4$**

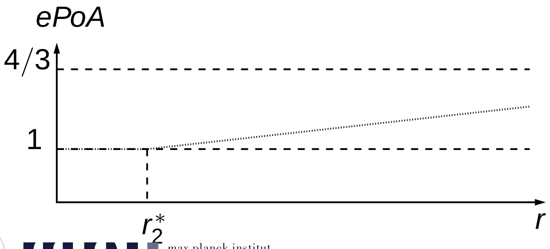
see next slide

# Non-benign Case: $R = a_2/a_1 < 1/4$

- second link is much more efficient than first
- Nash is hurt since it uses second link only at $r_2^N$.
- we modify $\ell_1$ as follows ($\ell_2$ stays unchanged)

$$\hat{\ell}_1(x) = \begin{cases} \ell_1(x) & \text{for } x \le r_2^* \\ \infty & \text{for } x > r_2^* \end{cases}$$

- this limits the flow on link 1 to $r_2^*$.




$ePoA \le 1 + R \le \frac{5}{4}$

# 2 Links: Advanced Solution

- in the non-benign case (with modified threshold)
- we modify $\ell_1$ as follows ($\ell_2$ stays unchanged)

$$\hat{\ell}_1(x) = \begin{cases} \ell_1(x) & \text{for } x \leq x_1 \text{ or } x > x_2 \\ \ell_1(x_2) & \text{for } x_1 < x \leq x_2 \end{cases}$$

- this forces Nash to use second link early, but also allows Nash to use both links at high rates

# $k$ Links

- highest link is unchanged
- consider any link which is not the highest:
- if there is no higher link that is much more efficient, we leave it unchanged
- if there is a higher link that is much more efficient, we modify the cost function such that the higher link is used earlier.

# Conclusion

- first study of coordination mechanisms for routing games
- we show that coordination mechanisms improve price of anarchy for networks of parallel links.
- many open problems
  - improved upper bounds
    - what is $c_2$?
    - is there a construction with $c_k < 4/3 - \epsilon$ for all $k$
  - lower bounds: is $ePoA > 1$ for the case of two links?
  - general networks instead of parallel links
  - more general cost functions, e.g., polynomial cost functions
  - atomic flow, i.e., flow consists of units of fixed size instead of infinitesimal units