max planck institut informatik

The Physarum Computer

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joint work with Luca Becchetti, Vincenzo Bonifaci, Michael Dirnberger, Andreas Karrenbauer, Girish Varma

SODA 2012, Journal of Theoretical Biology, ICALP 2013

publications and slides are available on my homepage

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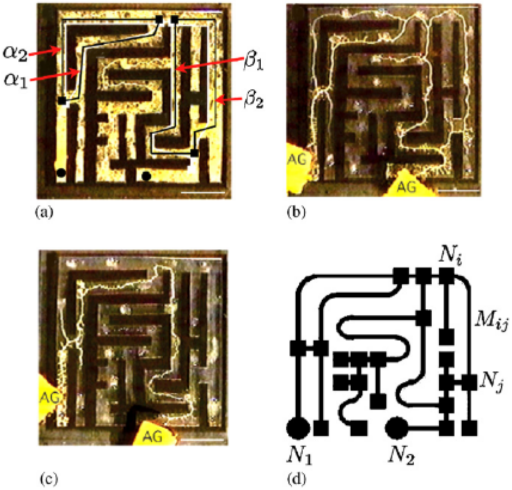
Physarum

- Physarum, a slime mold,
- single cell, several nuclei
- builds evolving networks
- lives in forests, on trees, in air-conditioners, . . . ,
- a model mechanism in biology

The Physarum Computer



Nakagaki, Yamada, Tóth, Nature 2000, use Physarum to compute shortest paths in networks

show video

made me laugh first and then made me think

Mathematical Model (Tero et al.)

- Physarum is a network of tubes (pipes);
- flow (of liquids and nutrients) through a tube is determined by concentration differences at endpoints of a tube, length of tube, and diameter of tube;
- tubes adapt to the flow through them: if flow through a tube is high (low) relative to diameter of the tube, the tube grows (shrinks) in diameter.
- mathematics is the same as for flows in an electrical network with time-dependent resistors.
- Tero et al., J. of Theoretical Biology, 553 – 564, 2007

Mathematical Model (Tero et al.)

- $G = (V, E)$ undirected graph
- each edge e has a positive length L_e (fixed) and a positive diameter $D_e(t)$ (dynamic)
- send one unit of current (flow) from s_0 to s_1 in an electrical network where resistance of e equals

$$R_e(t) = L_e / D_e(t).$$

- $Q_e(t)$ is resulting flow across e at time t
- Dynamics:

$$\dot{D}_e(t) = \frac{dD_e(t)}{dt} = |Q_e(t)| - D_e(t).$$

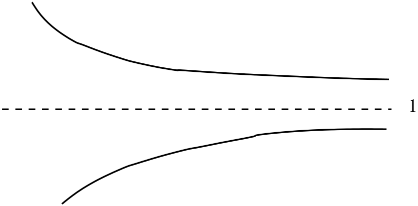
Interlude: Differential Equations

We have an equation involving a function and its derivative; want to know the function. For example,

$$\dot{D}(t) = 1 - D(t).$$

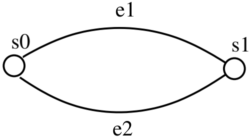
Then

$$D(t) = 1 + (D(0) - 1) \cdot e^{-t}.$$



We will write D instead of $D(t)$.

Two Parallel Links



e_i has length L_i, $L_1 < L_2$, and diameter D_i

- $Q_1 + Q_2 = 1$ always and hence

$$\frac{d(D_1 + D_2)}{dt} = (Q_1 + Q_2) - (D_1 + D_2) = 1 - (D_1 + D_2);$$

- thus $D_1 + D_2$ converges to one;
- assume $D_1 = D_2 = 1/2$;

  then resistance of first link is smaller than resistance of second link and therefore $Q_1 > Q_2$;

  thus $Q_1 > 1/2 > Q_2$ and hence D_1 grows and D_2 shrinks.

Mathematical Model II: The Node Potentials

- electrical flows are driven by electrical potentials; let p_u be the potential at node u at time t ($p_{s_1} = 0$ always)
- $Q_e = D_e(p_u - p_v)/L_e$ is flow on edge $\{ u, v \}$ from u to v
- flow conservation gives n equations, one for each vertex u

$$\sum_{e=\{ u,v \} \in E} D_e(p_u - p_v)/L_e = b_u$$

- $b_{s_0} = 1 = -b_{s_1}$ and $b_u = 0$, otherwise
- the equations above define the p_v's uniquely
- can be computed by solving a linear system
- from now on: $\Delta_e = p_u - p_v$ for $e = uv$; potential drop on e

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Computer Experiments (Discrete Time)

compute potentials with respect to initial configuration
while true **do**
update diameters: $D_e(t+1) = D_e(t) + h \cdot (|Q_e(t)| - D_e(t))$
recompute potentials
end while

In simulations, the system converges (Miyaji/Ohnishi 07/08)

- e on shortest s_0-s_1 path: D_e converges to 1
- e not on shortest path: D_e converges to 0

Miyaji/Ohnishi ran simulations only on small graphs

We ran experiments on thousands of graphs of size up to 50,000 vertices and 200,000 edges. Confirmed their findings.

The Questions

Does system convergence for all (!!!) initial conditions?

How fast does it converge?

Does discrete time simulation converge? Choice of h?

Beyond shortest paths?

Inspiration for distributed algorithms?

Convergence against Shortest Path

Theorem (Convergence (SODA 12, J. Theoretical Biology))

Dynamics converge against shortest path, i.e.,

$D_e \to 1$ *for edges on shortest source-sink path and* $D_e \to 0$ *otherwise.*

this assumes that shortest path is unique; otherwise ...

Miyaji/Onishi previously proved convergence for planar graphs with source and sink on the same face

Our Approach

- analytical investigation of simple systems, in particular, parallel links
- experimental investigation (computer simulation) of larger systems
  - to form intuition about the dynamics
  - to kill conjectures
  - to support conjectures
- proof attempts for conjectures surviving experiments

What did Evolution Optimize?

Evolution optimized dynamics so as to achieve a global objective.
Which? (Lyapunov Function)

First idea: the energy of the flow $\sum_e Q_e \Delta_e$ decreases over time

not true, even for parallel links

Theorem

For the case of parallel links:

$$\sum_i Q_i L_i, \quad \frac{\sum_i D_i L_i}{\sum_i D_i}, \quad \text{and } (p_s - p_t) \sum_i D_i L_i$$

decrease over time

computer experiment: the obvious generalizations (replace i by e) to general graphs do not work

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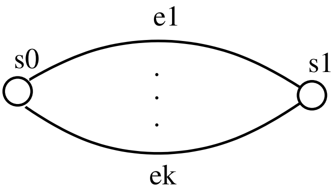
A not so Obvious Generalization



$$\frac{\sum_i D_i L_i}{\sum_i D_i} \quad \Rightarrow \quad \frac{\sum_e D_e L_e}{\text{minimum total diameter of a } s_0\text{-}s_1 \text{ cut}}$$

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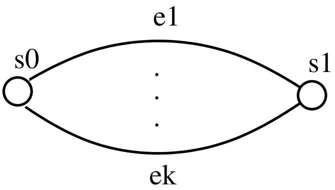

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Lemma: $V = \sum_e D_e L_e$ decreases exept if $\dot{D}_e = 0$ for all e (stationary point)

Lemma: System converges against a stationary point

Lemma: Stationary points = 1_P, where P is a source-sink path.

$$\dot{D}_e = 0 \Rightarrow D_e = Q_e = \frac{D_e}{L_e}\Delta_e \overset{D_e \neq 0}{\Rightarrow} \Delta_e = L_e.$$

Theorem: System converges to shortest path

$\Delta_{s_0,s_1} > L_{P*}$ in stationary point $\Rightarrow$ edges on P^* explode

$\dot{D}_e = Q_e - D_e = \frac{D_e}{L_e}\Delta_e - D_e = D_e(\frac{\Delta_e}{L_e} - 1) > 0$

- Assume $D_e(0) = 1$ for all e and there is a source-sink cut of capacity 1 at time zero.
- Then $D_e(t) \leq 1$ for all e and min-cut has capacity 1 at all times
- Let $V = \sum_e D_e L_e$ and $\eta = \sum_e Q_e^2 R_e$ (dissipated energy).

Lemma: $\eta \leq V$ always.

Let f be a maximum source-sink flow with $f_e \leq D_e$ for all e. Then f has value 1 and

$$\eta = \sum_e Q_e^2 R_e \leq \sum_e f_e^2 R_e \leq \sum_e D_e^2 \frac{L_e}{D_e} = \sum_e D_e L_e = V$$

The first inequality: Thompson's principle

Lemma: $\eta = \sum_e Q_e^2 R_e \le V = \sum_e L_e D_e$ always

Lemma: V is decreasing except if $D_e = Q_e$ for all e

Recall $R_e = L_e/D_e$ (or equivalently, $D_e R_e = L_e$)

$$\begin{aligned}
\dot{V} &= \sum_e L_e \dot{D}_e = \sum_e L_e(|Q_e| - D_e) = \sum_e (L_e D_e R_e)^{1/2} |Q_e| \quad - V \\
&= \sum_e (L_e D_e)^{1/2} \cdot R_e^{1/2} |Q_e| \quad - V \\
&\le \sqrt{\sum_e L_e D_e} \cdot \sqrt{\sum_e R_e Q_e^2} \quad - V \\
&= \sqrt{V} \cdot \sqrt{\eta} - V \le 0
\end{aligned}$$

Discretization and Speed of Convergence

$$D_e(t+1) = D_e(t) + h(|Q_e(t)| - D_e(t))$$

Theorem (Bechetti, Bonifaci, Dirnberger, Karrenbauer, M: ICALP 2013)

For any $\epsilon > 0$, let $h = \epsilon/(2mL)$, where L is largest edge length. Assume $L_{P^*} \geq 1$.

After $\widetilde{O}(nmL^2/\epsilon^3)$ iterations, solution is $1 + O(\epsilon)$-optimal.

Arithmetic with $O(\log(nL/\epsilon))$ bits suffices.

For $L = O(1)$, we have PTAS.

For $\epsilon = 1/(4nL)$, one can read off shortest path.

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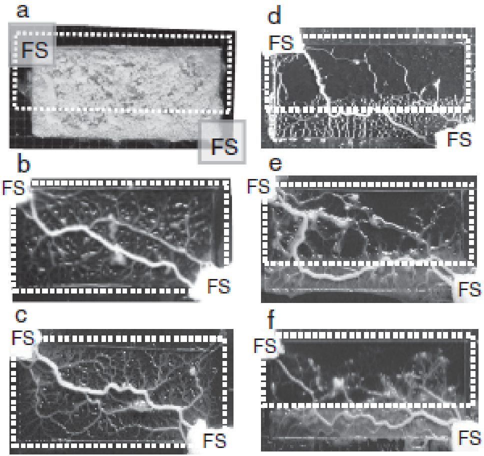
Nonuniform Physarum



$$\dot{D}_e(t) = |Q_e(t)| - a_e D_e(t)$$

a_e reactivity of e

no convergence proof is known, but simulations suggest convergence against shortest path under length function $a_e L_e$.

Nonuniform Directed Physarum

$$\dot{D}_e(t) = Q_e(t) - a_e D_e(t)$$

Theorem (BBDKM, ICALP 2013)

no biological significance claimed

convergence to shortest path according to length function $a_e L_e$

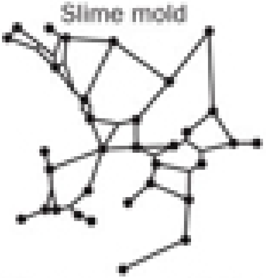
Ito/Johansson/Nakagaki/Tero (2011) prove convergence for uniform case ($a_e = 1$ for all e)

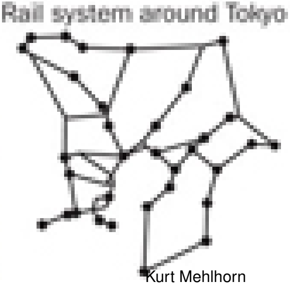
discretization converges in $\widetilde{O}(nmL^2/\epsilon^3)$ iterations to $1 + O(\epsilon)$ optimal solution (our proof requires uniformity)

Open Problems

- nonuniform Physarum, convergence, discretization, complexity
- nonuniform directed Physarum, discretization, complexity
- dependency on L or $\log L$? quasi-polynomial or polynomial
- Physarum apparently can do more, e.g., network design.
- inspiration for the design of distributed algorithms and/or approximation algs for NP-complete problems

Network Design: Science 2010





My Current Projects

Understand the principles of network formation. What does the network optimize?

Discrete Versions of Physarum.

Nonuniform Versions of Physarum

Can I use Physarum as an inspiration for approximation algorithms?

Thank you for your Attention