Problem Set 4

Approximation Algorithms

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Problem 1 (Local search). **1 point**

Consider the scheduling problem of minimizing the makespan on m identical parallel machines, $P||C_{max}$. The local search *move* neighborhood as introduced in the lecture consists of all schedules that can be obtained by selecting a job j on some machine i and a machine $k \neq i$ and moving job j from i to k.

(i) Show that a move-optimal schedule has a value $C_{\max} \leq \left(2 - \frac{2}{m+1}\right) C^*_{\max}$, where $C^*_{\max}$ is the value of an optimal solution.

(ii) Show that the bound is tight.

Problem 2 (Facility location). **1 point**

1. Consider the facility location problem where the connection costs are now c_{ij}^2. The c_{ij}'s still satisfy the triangle inequality. What is the approximation guarantee that we can achieve using the Jain-Vazirani algorithm?

2. It is reasonable to assume that as more cities connect to a facility, it's opening cost reduces per facility. Hence, assume that the opening cost of a facility is now $f_i + l \cdot s_i$, when there are l cities connected to facility i. How can we use the Jain-Vazirani algorithm to solve this variant?

Problem 3 (Lagrangean relaxation). **2 points**

The general Lagrangean relaxation technique is the following. Suppose we are given an optimization problem

$$
\begin{array}{rcl}
 & \min & f_0(x) \\
\text{s.t.} & & \\
f_i(x) & \leq & 0,\ i = 1, \cdots, k \\
h_j(x) & = & 0,\ j = 1, \cdots, l \\
x & \in & \mathbb{R}^n
\end{array}
$$

Let p^* be the optimum value of this problem. The basic idea of Lagrangean relaxation is to augment the objective function with a weighted sum of constraints to obtain an unconstrained optimization problem. We define the Lagrangean of the above problem as follows.

$$L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{k} \lambda_i \cdot g_i(x) + \sum_{j=1}^{l} \mu_j \cdot h_j(x)$$

The variables λ_i, μ_j are called the *dual variables*, or Lagrange multipliers associated with the problem. The Lagrangean dual is the function $g(\lambda, \mu)$ defined as

$$g(\lambda, \mu) = min_{x \in \mathbb{R}^n} L(x, \lambda, \mu) \, .$$

That is, for a fixed value of λ and μ, the value of g is the minimum of the function $L(x, \lambda, \mu)$.

1. Prove that $g(\lambda, \mu)$, with any $\lambda \geq 0$ is a lower bound on f, i.e., $g(\lambda, \mu) \leq p^*$, for all $\lambda \geq 0, \mu$.

2. In many cases, we can give an explicit description of the Lagrangean Dual. Suppose the optimization problem is the linear program $min\{c^T x | Ax = b\}$. What is the Lagrangean Dual. Can we write an explicit description of the dual problem?

3. Do the same for the linear programming problem $min\{c^T x | Ax \geq b, x \geq 0\}$?

Problem 4. **2 points**

Obtain a factor 6 approximation algorithm for a common generalization of the facility location and k-median problems. In this problem, we are given facilities with opening costs f_i and connection cost of city j to facility i equal to c_{ij}. The c_{ij}'s satisfy triangle inequality. We are also given a parameter k. The objective is to open *at most* k facilities and connect each city to an open facility such that the total of the connection costs and opening costs of the facilities is minimized.