

# Midterm Exam

## Optimization, SS 2005

### General Rules

### Problems

1. **Safe Playgrounds** (10)

The city council of Safety City decides to build a new playground in Danger Street. Three children are living in this street, namely Alice, Bob and Charlie. Since walking along the street is very dangerous the city council wants to minimize the maximum total distance a child has to walk per week along the street in order to reach the playground. The total length of the street is 400 meters. Alice lives at the beginning of the street and wants to visit the playground three times a week. Bob lives 225 meters away from Alice and will go to the playground twice a week. Charlie lives at the end of the street and his parents only allow him to visit the playground on Sundays. How far away from the beginning of the street should the playground be built?

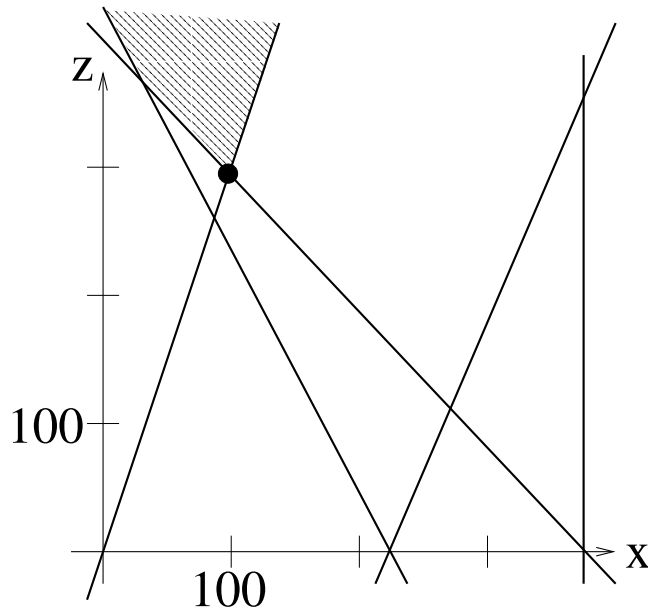
- (a) Formulate the problem as a linear programming problem.
- (b) Sketch the feasible region of your LP and solve it graphically.
- (c) Prove optimality of your solution using the dual linear program.

**Solution**

a) Let  $x$  be the position to build the playground. As the playground should be on the street, we have  $0 \leq x \leq 400$ . We have to minimize the maximum distance a child has to walk per week. Alice has to walk per week  $3x$ , Bob has to walk per week  $2|225 - x|$  and Charlie has to walk per week  $400 - x$ . Let  $z$  be a variable to describe the objective function value, i.e. the value of  $z$  should be at least the maximum of the above three values. Hence we can formulate the problem by the following LP:

$$\begin{array}{ll}
 \min & z \\
 \text{s.t.} & z \geq 3x \\
 & z \geq 2 * (225 - x) \\
 & z \geq 2 * (x - 225) \\
 & z \geq 400 - x \\
 & x \leq 400 \\
 & x \geq 0
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ll}
 \min & z \\
 \text{s.t.} & z - 3x \geq 0 \\
 & z + 2x \geq 450 \\
 & z - 2x \geq -450 \\
 & z + x \geq 400 \\
 & -x \geq -400 \\
 & x \geq 0
 \end{array}$$

b)



The optimal point is  $x = 100$  and  $z = 300$  with objective function value 300.

c) The dual of the LP above is

$$\begin{aligned}
 \max \quad & 450y_2 - 450y_3 + 400y_4 - 400y_5 \\
 \text{s.t.} \quad & y_1 + y_2 + y_3 + y_4 = 1 \\
 & -3y_1 + 2y_2 - 2y_3 + y_4 - y_5 \leq 0 \\
 & y \geq 0
 \end{aligned}$$

As the first and third constraints are active for the optimal point, we conclude that only  $y_1$  and  $y_4$  are different from 0.  $y_1 = 1/4$  and  $y_4 = 3/4$  gives a feasible solution of objective function value 300. Hence the primal and dual solutions are both optimal.

2. **Optimal Basic Feasible Solution** (10)

For which real numbers  $\alpha$  and  $\beta$  is  $x^* = (0, 4, 2, 5, 0)^T$  an optimal basic feasible solution of the following linear program?

$$\begin{array}{rcll} \max & 7x_1 & + & 6x_2 & + & 5x_3 & - & 2x_4 & + & \alpha x_5 & & \\ \text{s.t.} & x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & \beta \\ & 4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 9 \\ & 2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 15 \\ & 3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 3 \\ & & & & & & & & & & & x_i \geq 0 \end{array}$$

**Solution**

We check feasibility of the  $x^*$ .

$$\begin{aligned} (1, 3, 5, -2, 2)^T x^* &= 12 \Rightarrow \beta \geq 12 \\ (4, 2, -2, 1, 1)^T x^* &= 9 \\ (2, 4, 4, -2, 5)^T x^* &= 14 < 15 \\ (3, 1, 2, -1, -2)^T x^* &= 3 \end{aligned}$$

For  $x^*$  to be basic feasible solution 5 linearly independent constraints must be fulfilled with equality. Two nonnegativity constraints, (2) and (4) are fulfilled with equality. One constraint is missing, hence (1) must also be fulfilled with equality, so  $\beta = 12$ .

Now we use complementary slackness to find conditions for  $x^*$  to be optimal solution.

Dual of this problem is

$$\begin{aligned} (D) \min & \beta y_1 + 9y_2 + 15y_3 + 3y_4 \\ & y_1 + 4y_2 + 2y_3 + 3y_4 \geq 7 & (1) \\ & 3y_1 + 2y_2 + 4y_3 + y_4 \geq 6 & (2) \\ & 5y_1 - 2y_2 + 4y_3 + 2y_4 \geq 5 & (3) \\ & -2y_1 + y_2 - 2y_3 - y_4 \geq -2 & (4) \\ & 2y_1 + y_2 + 5y_3 - 2y_4 \geq \alpha & (5) \\ & y_i \geq 0 \end{aligned}$$

Consider primal LP and it's dual

$$\begin{array}{ll} \max c^T x & \min b^T y \\ Ax \leq b & (P) \quad y^T A \geq c^T \quad (D) \\ x \geq 0 & y \geq 0 \end{array}$$

Feasible solutions  $x^*$  and  $y^*$  for (P) and (D) resp. are optimal solutions for (P) and (D) resp. if and only if they satisfy  $((y^*)^T A - c^T)x^* = 0$  and  $(y^*)^T(Ax^* - b) = 0$ .

Complementary slackness condition implies that  $x_2^*, x_3^*, x_4^* > 0 \Rightarrow (6), (7), (8)$  must be fulfilled with equality. Moreover since (3) is not fulfilled with equality  $y_3^*$  must be equal to 0.

From the system of equations corresponding to (6), (7), (8) and the fact, that  $y_3 = 0$  we get unique solution  $y^* = (1, 1, 0, 1)$ . In order to check feasibility of the  $y^*$  we substitute it to (5) and (9).

$$\begin{aligned}(1, 4, 2, 3)^T y^* &= 8 > 7 \\ (2, 1, 5, -2)^T y^* &= 1 \geq \alpha\end{aligned}$$

Hence for  $y^*$  to be feasible solution for the dual  $\alpha$  must be not greater than 1. Moreover  $\forall y_i^* > 0$  corresponding constraints of the primal are fulfilled with equality.

Hence  $y^*$  and  $x^*$  satisfy complementary slackness condition for  $\alpha \leq 1$  and  $\beta = 12$ . It implies that  $x^*$  is optimal basic solution if and only if  $\alpha \leq 1$  and  $\beta = 12$ .

### 3. Duality (10)

For matrices  $A \in \mathbb{R}^{n \times r}$ ,  $B \in \mathbb{R}^{n \times s}$ ,  $C \in \mathbb{R}^{m \times r}$ ,  $D \in \mathbb{R}^{m \times s}$  and vectors  $c \in \mathbb{R}^r$ ,  $d \in \mathbb{R}^s$ ,  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$  consider the following linear program with variables  $x \in \mathbb{R}^r$ ,  $y \in \mathbb{R}^s$ .

$$\begin{aligned} \max \quad & c^T x + d^T y \\ \text{s.t.} \quad & Ax + By \leq a \\ & Cx + Dy = b \\ & x \geq 0 \end{aligned}$$

Form its dual and show weak duality.

#### Solution

The Primal program:

$$\begin{aligned} \max \quad & c^T x + d^T y \\ \text{s.t.} \quad & Ax + By \leq a \quad (1) \quad :u \\ & Cx + Dy = b \quad (2) \quad :v \\ & x \geq 0 \end{aligned}$$

$x \in \mathbf{R}^r, y \in \mathbf{R}^s$ .

The Dual program:

$$\begin{aligned} \min \quad & u^T a + v^T b \\ \text{s.t.} \quad & u^T A + v^T C \geq c^T \quad (3) \\ & u^T B + v^T D = d^T \quad (4) \\ & u \geq 0 \end{aligned}$$

$u \in \mathbf{R}^n, v \in \mathbf{R}^m$ .

Weak duality:

Let  $(x, y)$  and  $(u, v)$  be primal - dual feasible. Then  $c^T x + d^T y \leq u^T a + v^T b$ .

Proof:

$$\begin{aligned} c^T x + d^T y &\stackrel{(3)}{\leq} (u^T A + v^T C)x + d^T y \stackrel{(4)}{=} (u^T A + v^T C)x + (u^T B + v^T D)y = \\ &= u^T(Ax + By) + v^T(Cx + Dy) \stackrel{(2)}{=} u^T(Ax + By) + v^T b \stackrel{(1)}{\leq} u^T a + v^T b \\ &\text{(q.e.d)} \end{aligned}$$

#### 4. Vertices of Polyhedra (10)

- (a) Show that any non-empty polyhedron without vertices contains a point with at least one negative coordinate.
- (b) Let  $P, S \subseteq \mathbb{R}^n$  where  $P$  is a polyhedron and  $S$  is an arbitrary set.  
*Show:* If  $\text{conv}(P \cap S) = P$  then  $S$  contains all vertices of  $P$ .

#### Solution

a) Let  $P \subseteq \mathbb{R}^n$  be a non-empty polyhedron without vertices. Then  $P$  contains a line, that is, there are  $x, d \in \mathbb{R}^n$  such that  $d \neq 0$  and  $\{x + \lambda d \mid \lambda \in \mathbb{R}\} \subseteq P$ . Let  $i \in [n]$  such that  $d_i \neq 0$ . Let  $\lambda = -(|x_i| + 1)/d_i$  and  $y = x + \lambda d$ . Then  $y_i = x_i + \lambda d_i = x_i - |x_i| - 1 \leq -1$ . Hence  $y$  is a point of  $P$  having a negative coordinate.

*Remark.* The german version of the exam accidentally asked to show that any non-empty polyhedron without vertices contains a point with at least one **non**-negative coordinate. The proof remains same but you should choose  $\lambda = +(|x_i| + 1)/d_i$ .

b): Let  $P$  be a polyhedron in  $\mathbb{R}^n$  and  $S \subseteq \mathbb{R}^n$  such that  $\text{conv}(P \cap S) = P$ . Assume that there is a vertex  $x$  of  $P$  such that  $x \notin S$ . Since  $x \in P = \text{conv}(P \cap S)$ ,  $x$  is a convex combination of elements of  $P \cap S$ . Hence there are  $k \in \mathbb{N}$ ,  $s_1, \dots, s_k \in P \cap S$ ,  $\lambda_1, \dots, \lambda_k \in [0, 1]$  such that  $x = \sum_{i \in [k]} \lambda_i s_i$  and  $\sum_{i \in [k]} \lambda_i = 1$ . Since  $x \notin S$ , we have  $\lambda_i \neq 1$  for all  $i \in [k]$ , and thus  $k \geq 2$ . Let  $y = \frac{1}{1-\lambda_k} \sum_{i \in [k-1]} \lambda_i s_i$ . Since  $\frac{1}{1-\lambda_k} \sum_{i \in [k-1]} \lambda_i = 1$  and  $P$  is convex,  $y \in P$ . Hence  $x = (1 - \lambda_k)y + \lambda_k s_k$  is a non-trivial convex combination of two points of  $P$ . This contradicts the assumption that  $x$  is a vertex of  $P$ .

*Remark:* I proved that a non-trivial convex combination of many points can be written as convex combination of two points of the polyhedron. I assume that similar reasonings appeared in the lecture at least once. Therefore mentioning this fact would have been sufficient.

5. **Prove or Disprove** (6×2)

Prove or disprove each of the following statements. For (e) and (f) you don't have to provide concrete counterexamples or algebraic proofs but you should argue in terms of the simplex method.

- (a) All non-empty polyhedra of the form  $\{x \mid Ax \leq b, x \geq 0\}$  contain at least one vertex.

**Solution.** True.

Let  $P := \{x \mid Ax \leq b, x \geq 0\}$  be a non-empty polyhedron for some matrix  $A$  and vector  $b$ . Suppose  $P$  would not contain a vertex. Then, since it is non-empty, it contains a line, i.e.,  $x + \lambda y \in P$  for some  $y \neq 0$  and all  $\lambda \in \mathbb{R}$ . W.l.o.g.  $y \not\geq 0$ . Thus for a sufficiently large  $\hat{\lambda}$  we have  $P \ni x + \hat{\lambda}y \not\geq 0$  which is a contradiction. Note that the statement of 5.a) is implied by 4.a).

- (b) If a polyhedron contains at least two distinct points then it contains at least one non-vertex.

**Solution.** True.

Let  $x, y \in P$  be two distinct points in polyhedron  $P$ . Let  $P = \{x \mid Ax \leq b\}$  for some matrix  $A$  and vector  $b$ . Consider the point  $z := \frac{1}{2}(x + y)$ . Since

$$Az = A\frac{1}{2}(x + y) = \frac{1}{2}(Ax + Ay) \leq \frac{1}{2}(b + b) \leq b$$

we have  $z \in P$ . Since  $z$  is a proper convex combination of two points in  $P$  it is not an extreme point. Hence  $z$  is a non-vertex of  $P$ .

- (c) A non-empty polyhedron of the form  $P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$  is unbounded if and only if

$$\max_{x \in P} x_k \text{ or } \min_{x \in P} x_k$$

is unbounded for at least one  $1 \leq k \leq n$ .

**Solution.** True. We show each entailment by proving its contraposition.

The optimal objective values  $\max_{x \in P} x_k$  and  $\min_{x \in P} x_k$  are bounded for each  $1 \leq k \leq n$  if and only if there exists a  $C$  such that  $\max_{x \in P} x_k \leq C$  and  $-C \leq \min_{x \in P} x_k$  for each  $1 \leq k \leq n$ . This is equivalent to:  $|x_k| \leq C$  for each entry  $x_k$  of each point  $x \in P$ . And this is our definition of a bounded set.

- (d) If a non-empty polyhedron of the form  $P = \{x \mid Ax \leq b\} \subseteq \mathbb{R}^n$  is unbounded then

$$\max_{x \in P} \mathbb{1}^T x \text{ or } \min_{x \in P} \mathbb{1}^T x$$

is unbounded. ( $\mathbb{1} \in \mathbb{R}^n$  denotes the all-ones vector.)

**Solution.** False.

Consider  $P = \{(x \ y) \in \mathbb{R}^2 \mid x + y \leq 0, -x - y \leq 0\}$ . It contains the line  $\lambda(x + y) = 0$ , i.e. is unbounded. Moreover both LPs are bounded as  $x + y = 0$  for all feasible  $(x \ y) \in P$ .

- (e) In general, the (primal) simplex method strictly improves the objective function value in each step.

**Solution.** False.

If the step is degenerated, the basic feasible solution remains the same and hence its objective function.

- (f) The basic feasible solutions in two consecutive steps of the simplex method are either same or adjacent.

**Solution.** True.

Consider a polyhedron in standard form. Recall that two basic feasible solutions are adjacent, if they are distinct and there are  $n - 1$  linearly independent linear constraints that are satisfied with equality for both.

Let  $B_1$  and  $B_2$  be bases of two consecutive simplex iterations. Then the bases are adjacent, i.e.  $|B_1 \cup B_2| = m + 1$ . For each  $j \notin B_1 \cup B_2$ , the inequality  $x_j \geq 0$  is satisfied at equality for both basic feasible solutions. Furthermore the  $m$  equality constraints are satisfied at equality. All those constraints are linearly independent (as they are a subset of the  $n$  linearly independent constraints that are active for the basic feasible solution corresponding to  $B_1$ ). Thus, we have at least  $n - (m + 1) + m = n - 1$  linearly independent constraints that are active for both solutions. Hence, the solutions are adjacent if they are not the same.