Probabilistic Analysis for the Knapsack Problem

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The Knapsack Problem

Given n items with

- weights $w_1, \ldots, w_n$
- profits $p_1, \ldots, p_n$

and a knapsack of capacity b.

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Find a subset $K \subseteq \{1, 2, \ldots, n\}$ such that

- the capacity is not exceeded, i.e., satisfy $\sum_{i \in K} w_i \leq b$,
- and the profit is maximized, i.e., maximize $\sum_{i \in K} p_i$.

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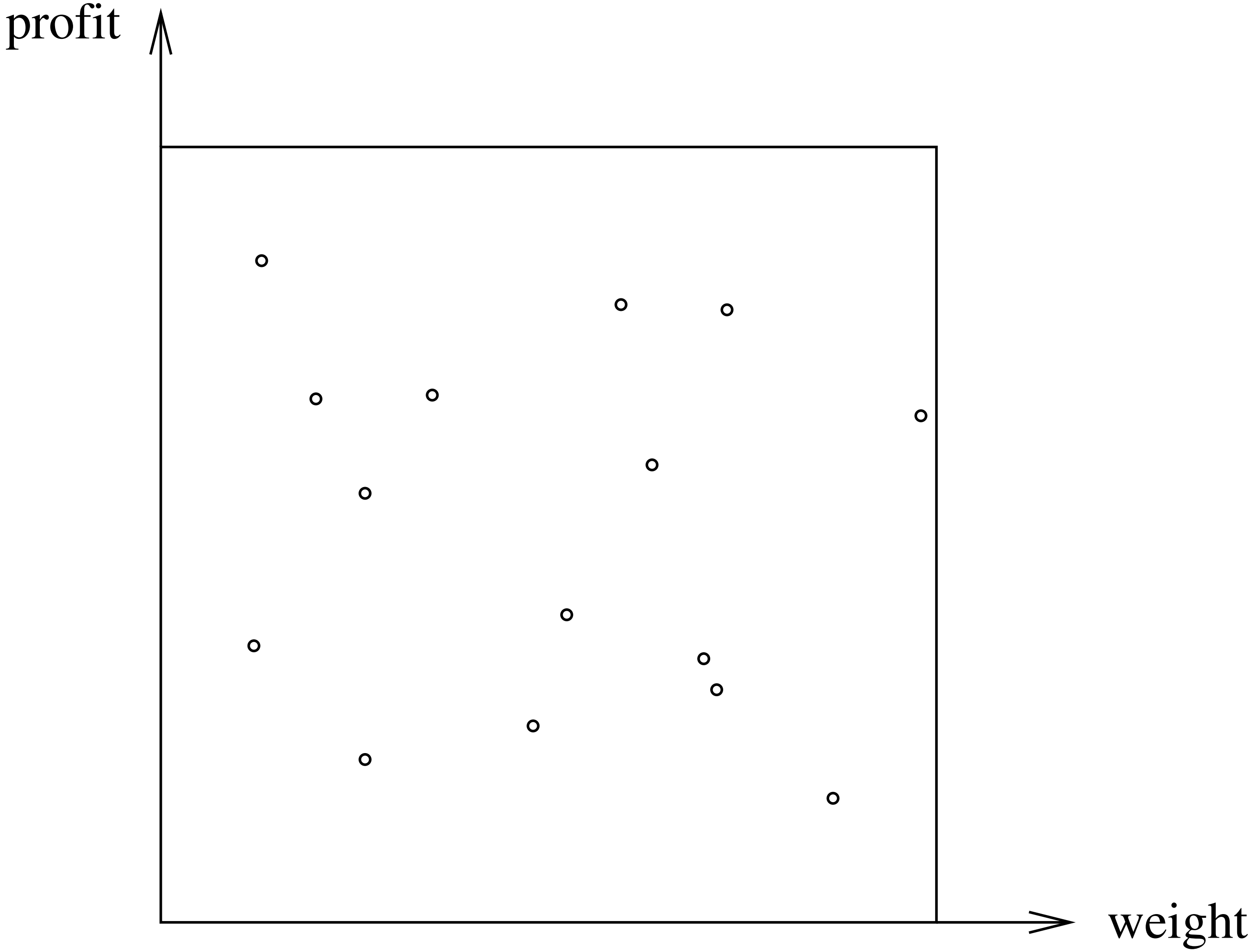
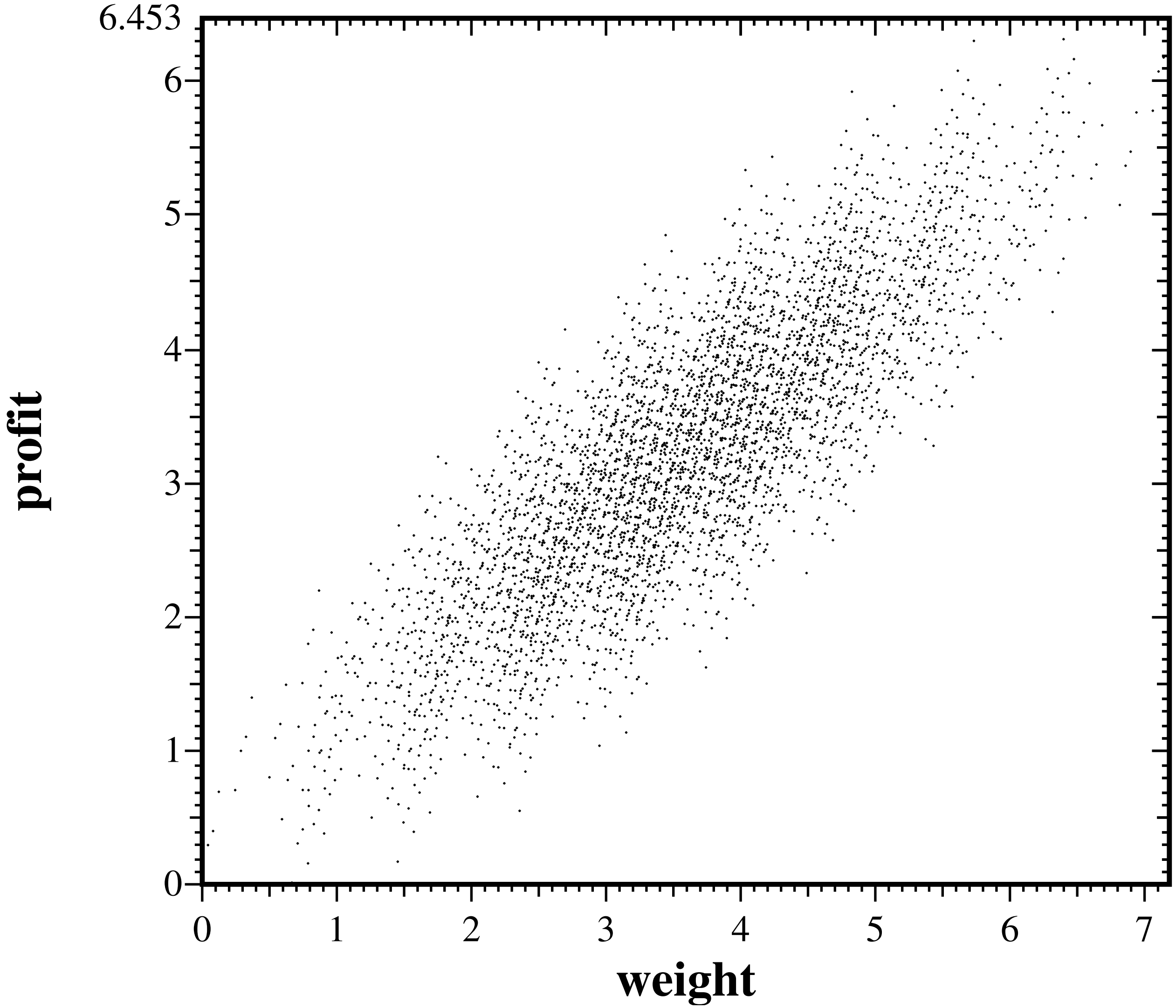
NP-hard but random instances can be solved very fast

uniform random instance



. . . find the best knapsack filling



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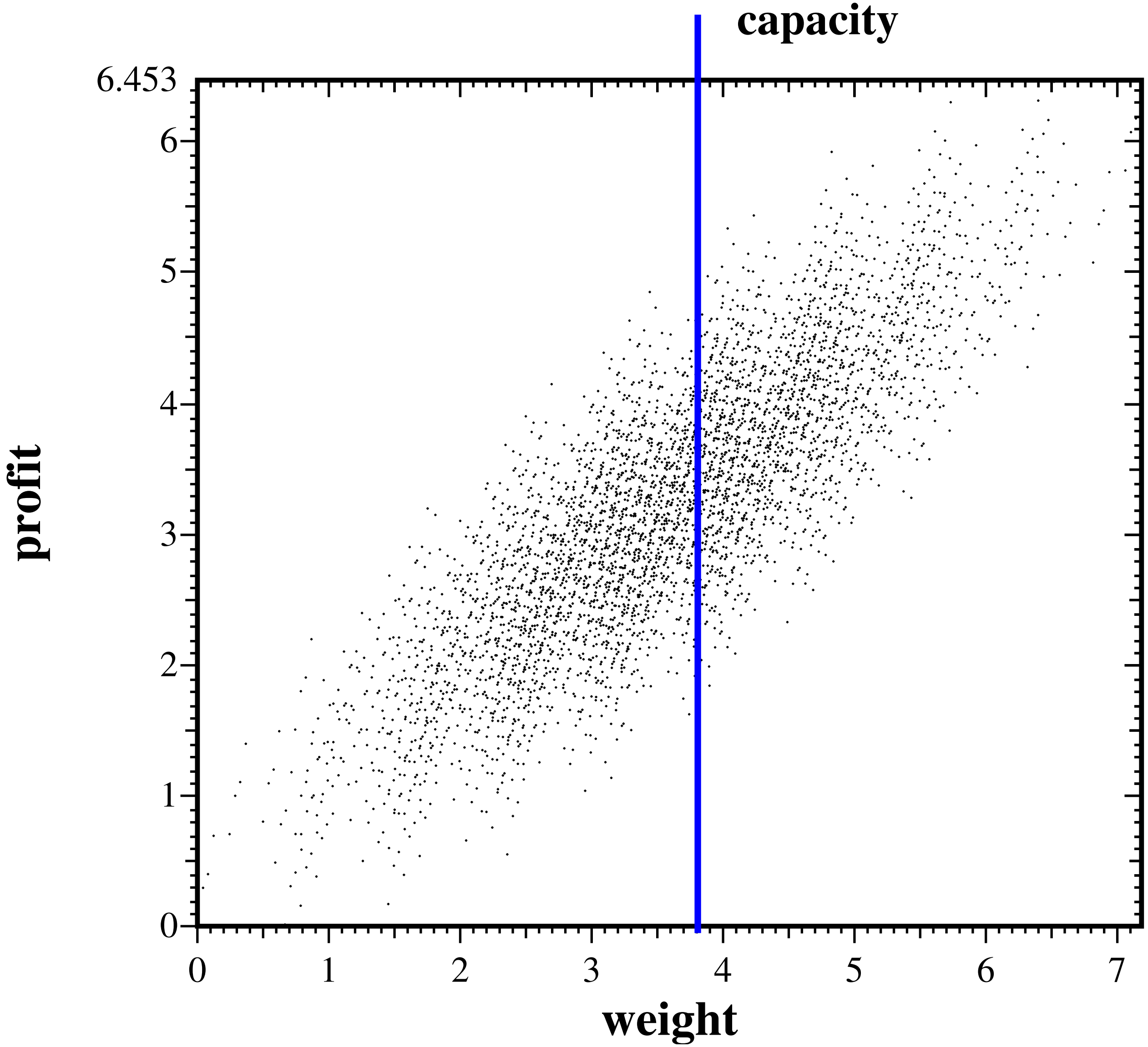

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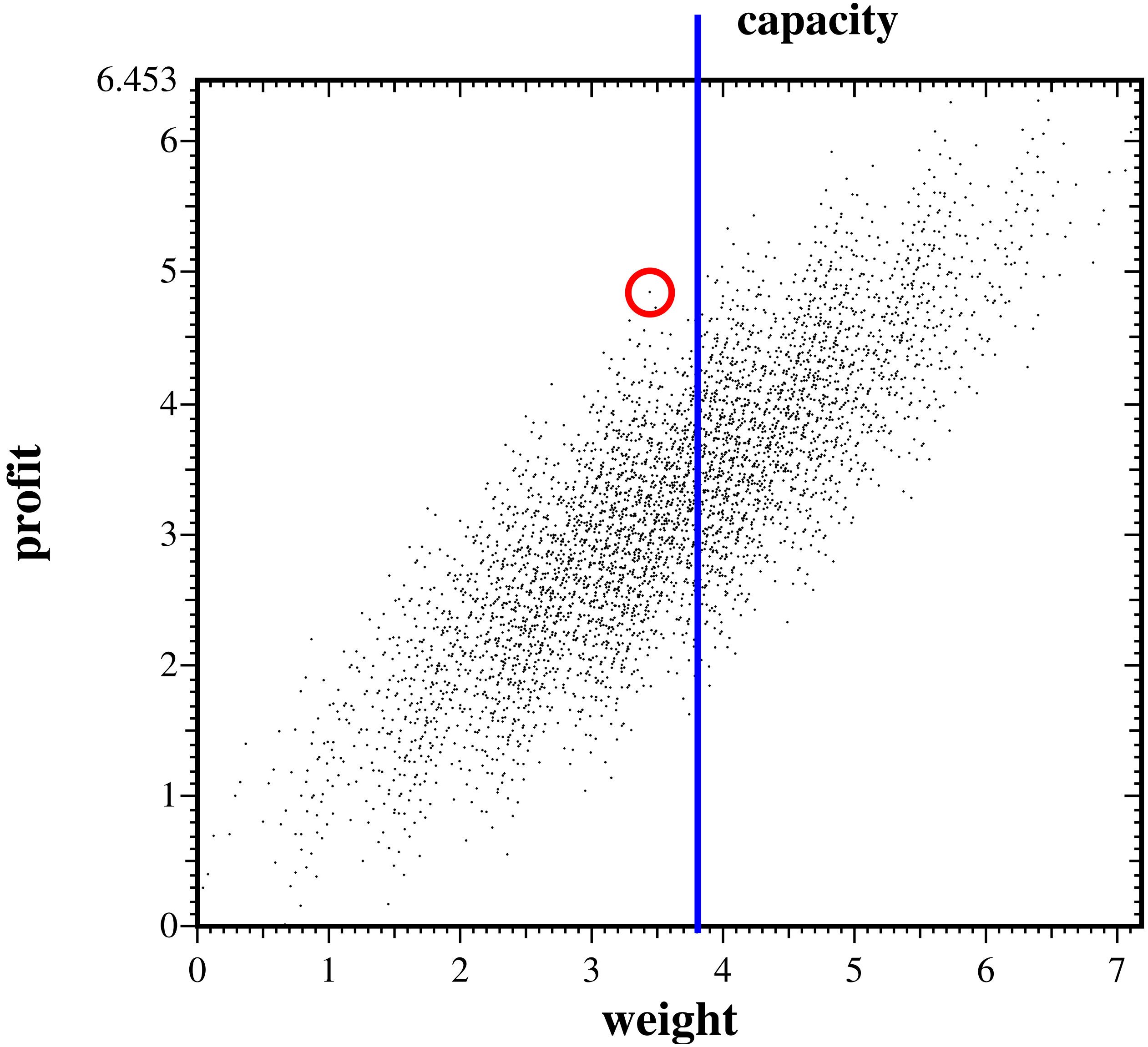

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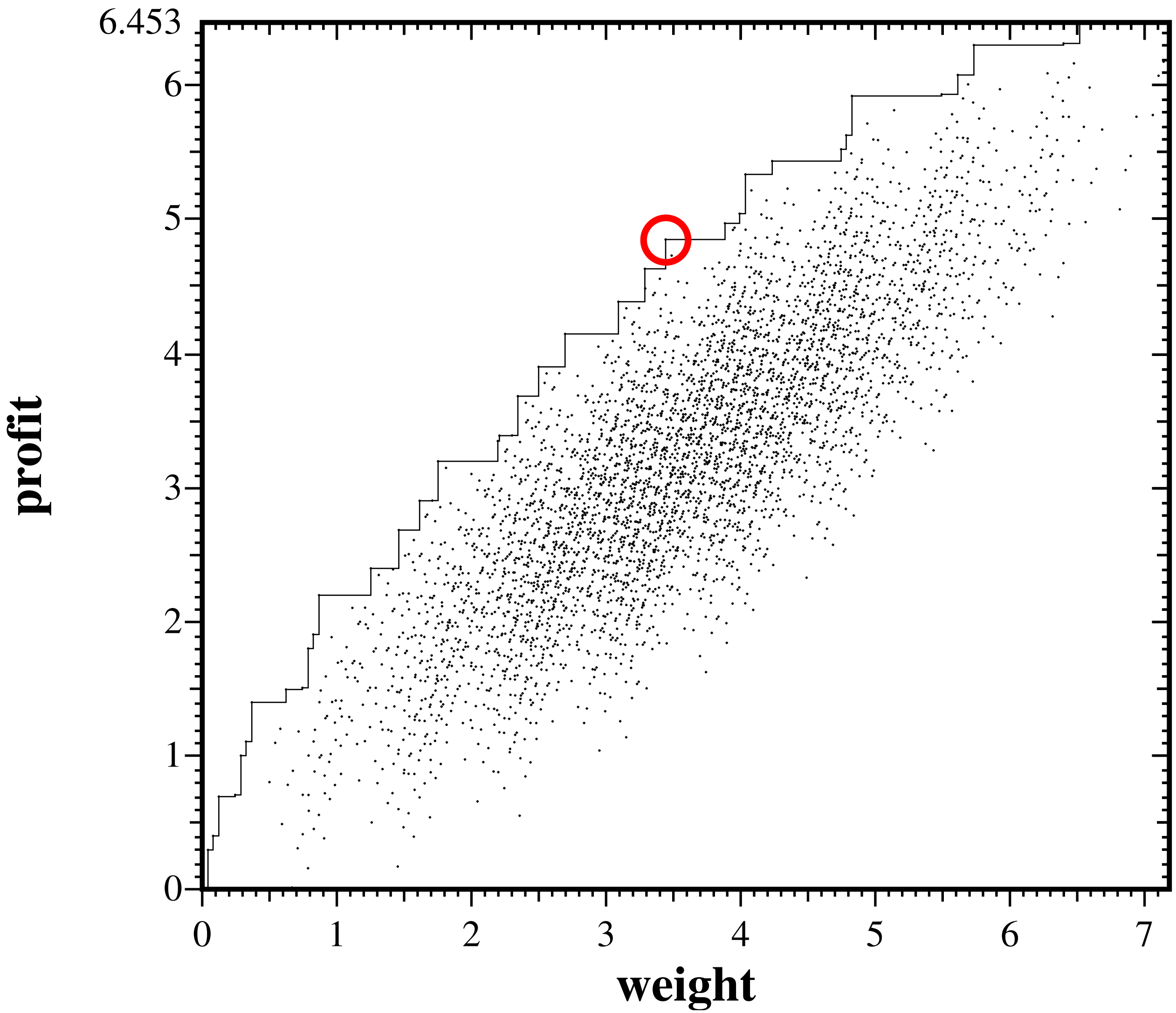

Pareto optimal solutions

A subset $K \subseteq [n]$ is called Pareto optimal if

$$p(K) > p(J), \text{ for all } J \subseteq [n] \text{ with } w(J) \leq w(K).$$

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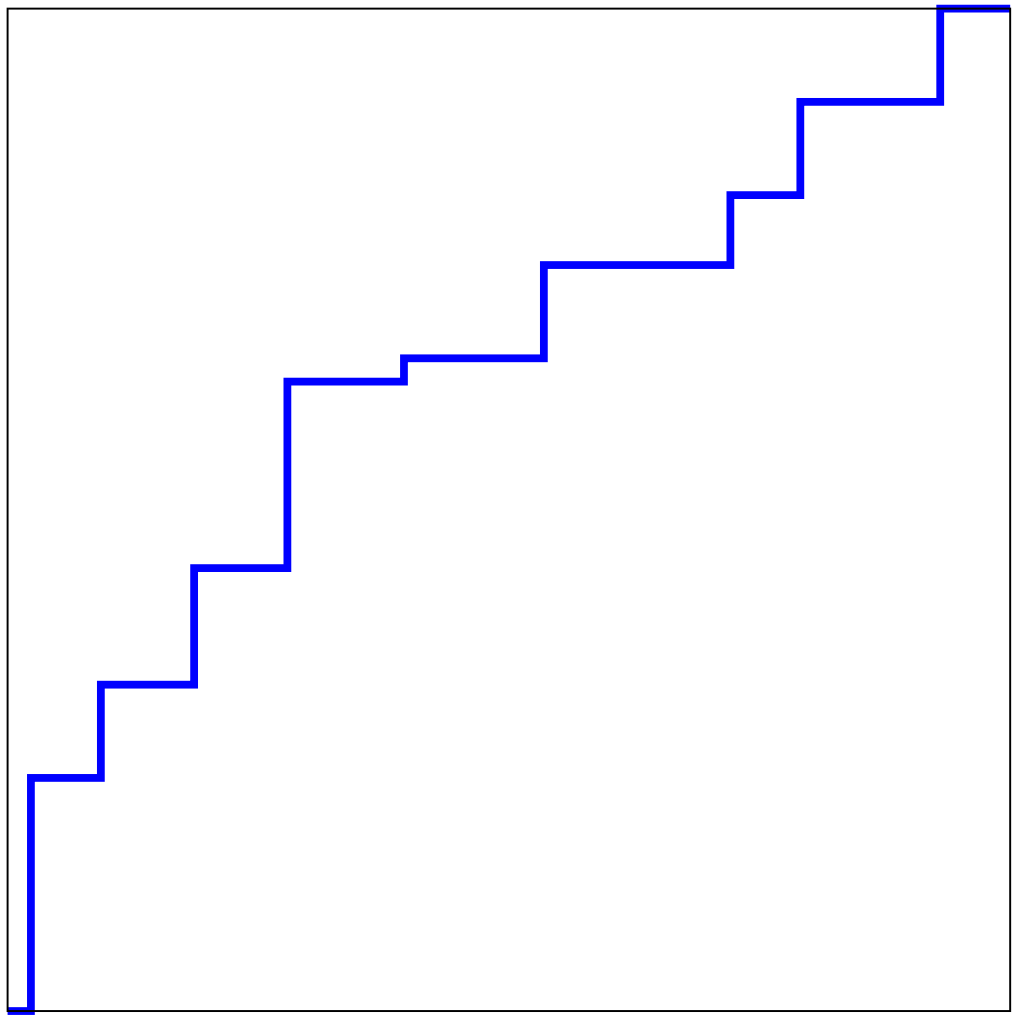
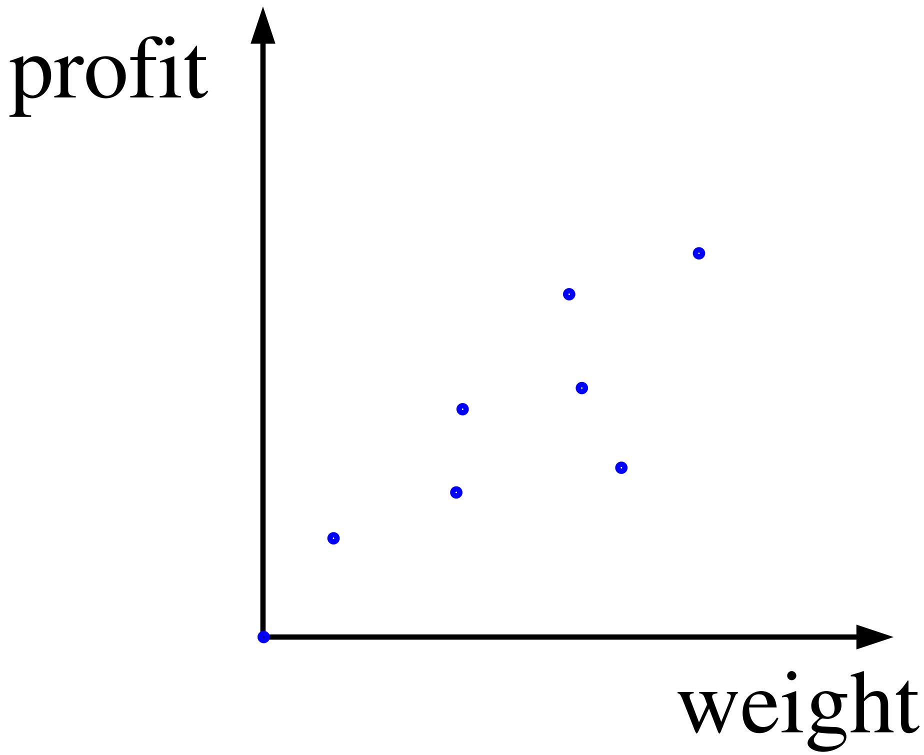
A list with all Pareto optimal solutions can be calculated efficiently ...

The Nemhauser/Ullmann algorithm (1969)

after adding elements 1 to i−1



The Nemhauser/Ullmann algorithm (1969)



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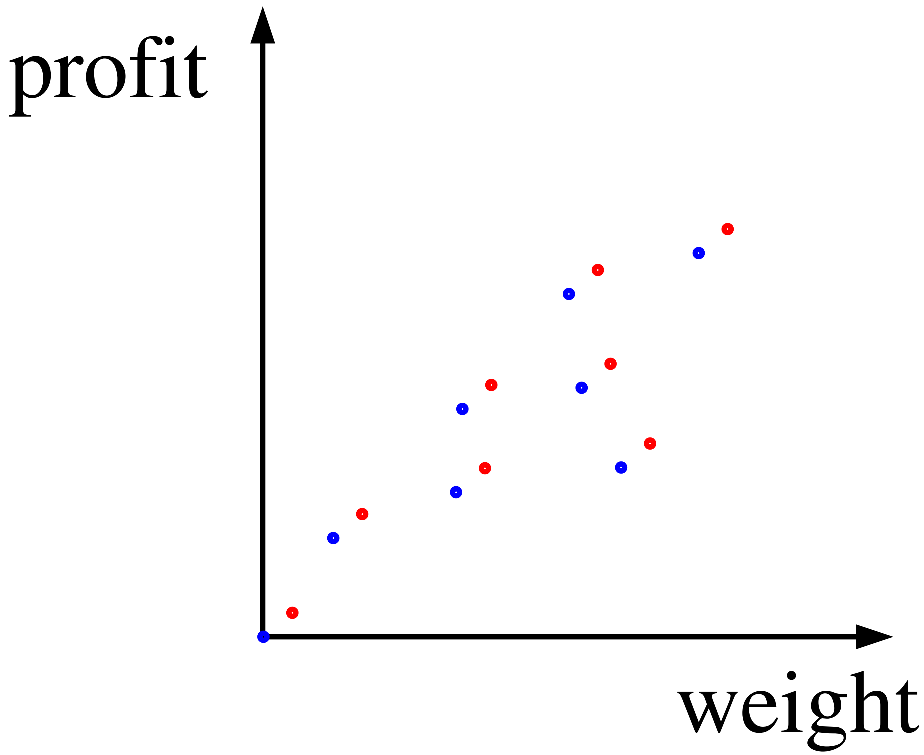

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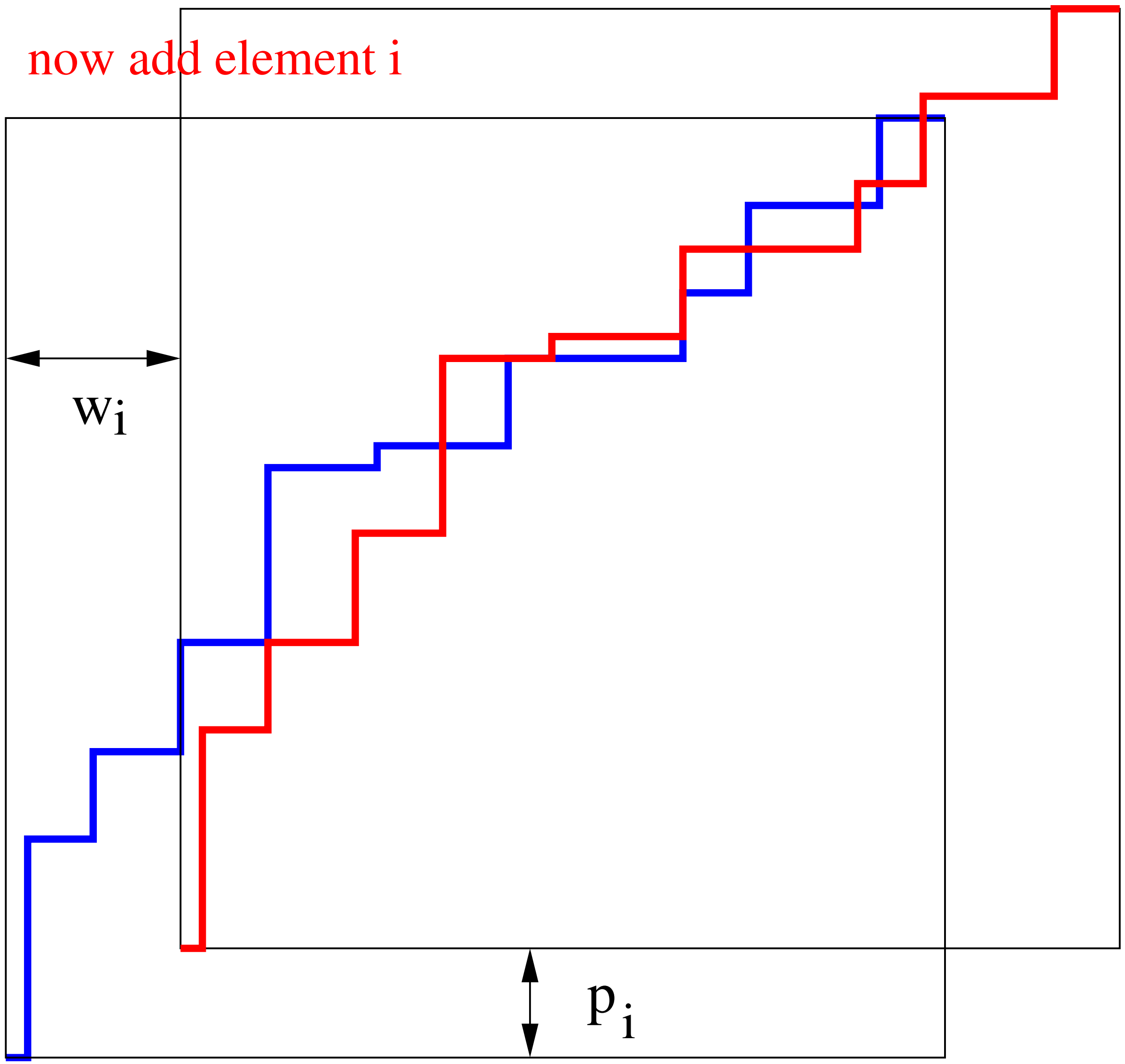

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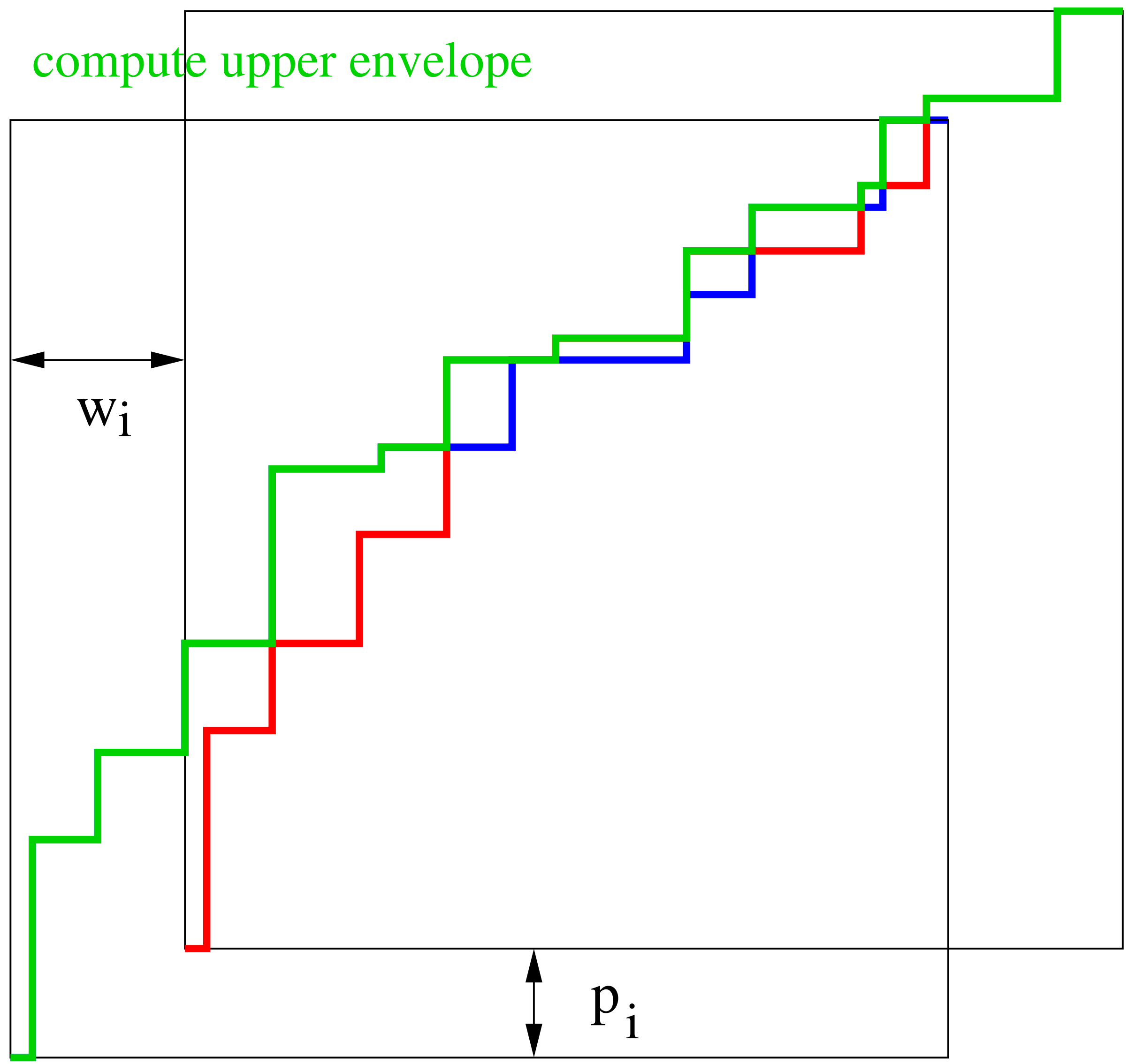

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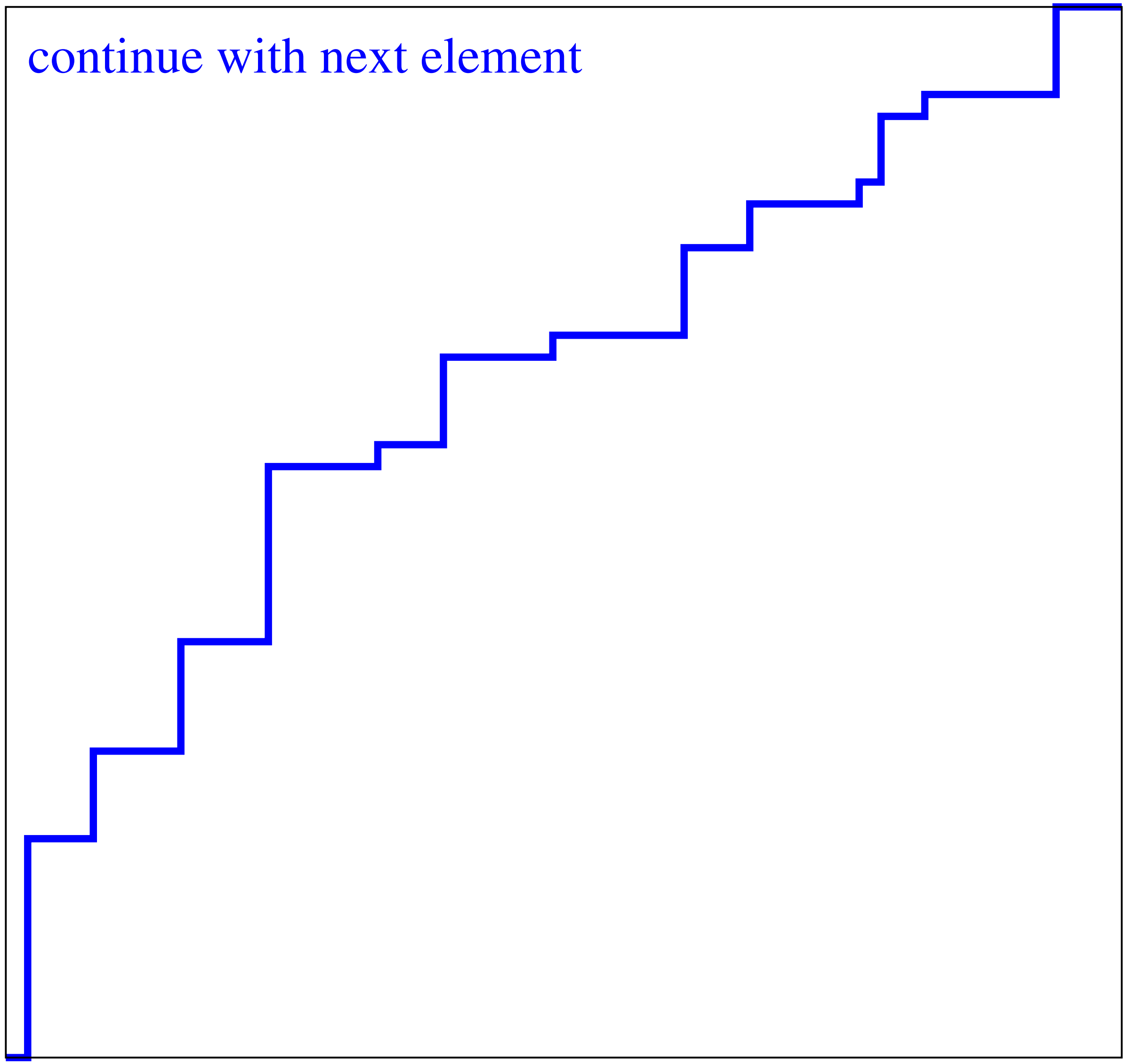

running time analysis

- $q_i \ldots$ # Pareto-optimal knapsack filling for items $\{1, \ldots, i\}$.
- running time $T = O\left(\sum_{i=1}^{n} q_{i-1}\right) \quad \Rightarrow \quad \mathbf{E}[T] = O\left(\sum_{i=1}^{n} \mathbf{E}[q_{i-1}]\right)$

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$$\forall i \geq 1, \mathbf{E}[q_i] \leq i^2 + 1$$

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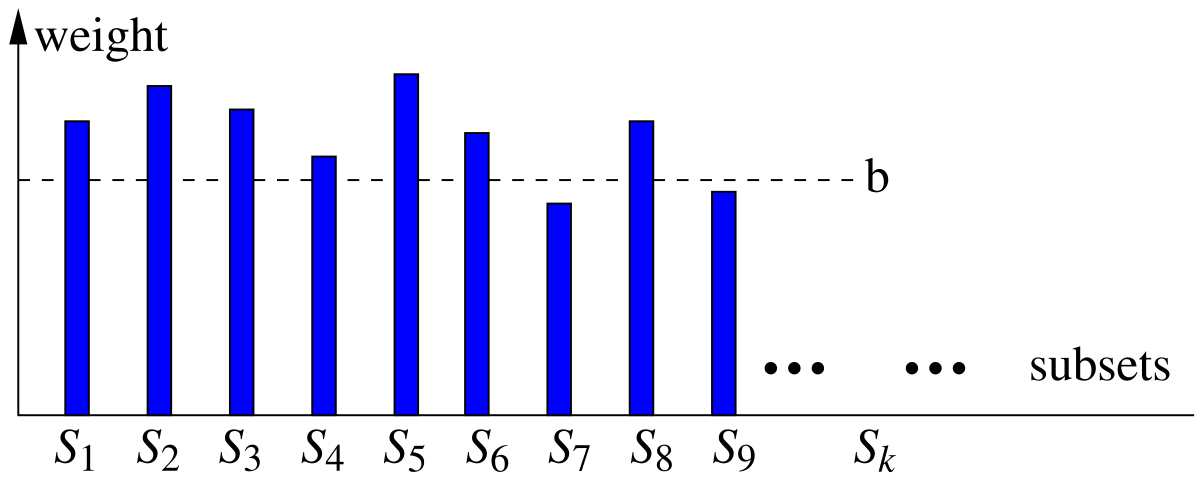
$$\forall i \geq 1, \mathbf{E}[q_i] \leq i^2 + 1$$

- thus $\mathbf{E}[T] = O\left(\sum_{i=1}^{n} i^2\right) = O(n^3)$

Pareto-Optimality

Assume profits $p_1, \ldots, p_n$ are arbitrary positive numbers.
Orderer subsets according to decreasing profit ($S_i \neq \emptyset$).



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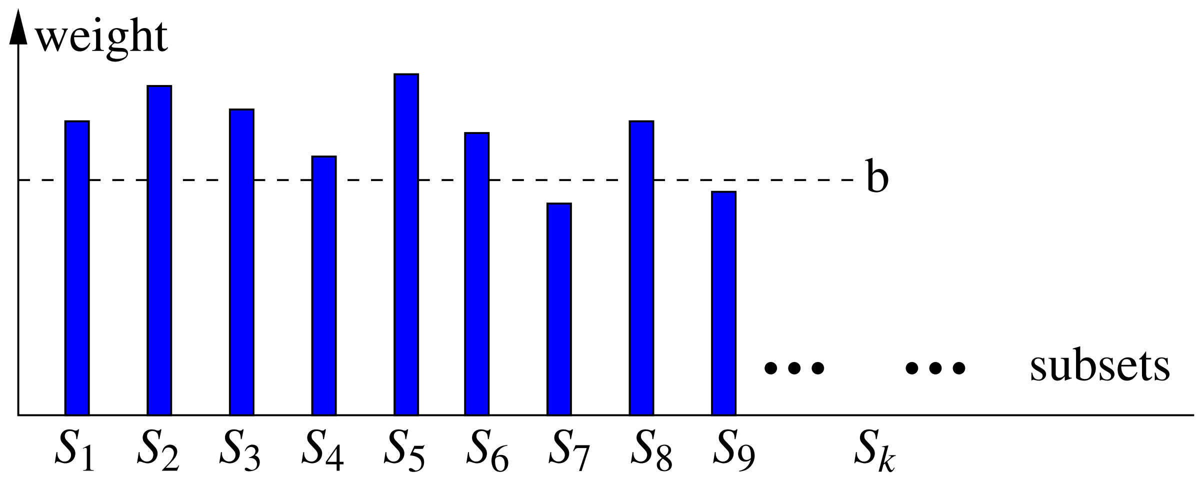

Pareto-optimal solution $\leftrightarrow$ Left-Right-Minima

$\mathcal{P}$: set of Pareto optimal solutions

Pareto-Optimality

Lemma: Let $S_1, \ldots S_k$ be an arbitrary sequence of subsets of $[n]$ with the following monotonitity property:

$$S_i \subset S_j \Rightarrow i > j$$

Let $w_1, \ldots, w_n$ be independent random variables with $\mathbf{Pr}\left[w_i = t\right] \leq \phi_i$ for all $i \in [n]$ and arbitrary t. Then

$$\mathbf{Pr}\left[\exists x \in \mathcal{P} : w^T x = t\right] \leq \sum_{i \in [n]} \phi_i.$$

Analysis for Uniform Distribution

Assume that weights w_i are drawn uniformly at random from $\{1, \ldots, M\}$.

$$\mathbf{Pr}\left[w_i = t\right] \leq \phi_i = \frac{1}{M} \quad \forall i \in [n], \text{ arbitrary } t$$

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$$P_t = \begin{cases} 1 & \exists x \in \mathcal{P} : w^T x = t \\ 0 & \text{otherwise.} \end{cases}$$

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$$q = \sum_{t=1}^{nM} P_t$$

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$$\begin{aligned} \mathbf{E}[q] = \mathbf{E}\left[\sum_{t=1}^{nM} P_t\right] = \sum_{t=1}^{nM} \mathbf{E}[P_t] &= \sum_{t=1}^{nM} \mathbf{Pr}\left[\exists x \in \mathcal{P} : w^T x = t\right] \\ &\le \sum_{t=1}^{nM} \sum_{i=1}^{n} \phi_i \le \sum_{t=1}^{nM} \frac{n}{M} = n^2. \end{aligned}$$

Analysis

Show: $\mathbf{Pr}\left[\exists x \in \mathcal{P} : w^T x = t\right] \leq \sum_{i \in [n]} \phi_i$.

- $\mathbf{stopper}(i)$ = highest ranked solution S with $i \notin S$ and $\mathbf{weight}(S) < t$
- $E_i := [\min\{\mathbf{weight}(S) \mid i \in S, S$ has higher rank than $\mathbf{stopper}(i)\} = t]$
- Show: $E := (\exists x \in \mathcal{P} : w^T x = t) \Rightarrow \cup_i E_i$

Assume there is a Pareto optimal solution S^* with weight t. Let T be the highest ranked solution with $\mathbf{weight}(T) < t$.

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Assume there is a Pareto optimal solution S^* with weight t. Let T be the highest ranked solution with $\mathbf{weight}(T) < t$.

- S^* has higher rank than T.
- monotonicity: $\exists i : i \in S^*, i \notin T \rightarrow T$ is $\mathbf{stopper}(i)$
- $\min\{\mathbf{weight}(S') \mid i \in S', S'$ has higher rank than $\mathbf{stopper}(i)\}$ is attained by S^*
- $E \Rightarrow E_i$ for at least one i.

Analysis

Show: $\mathbf{Pr}\left[\exists x \in \mathcal{P} : w^T x = t\right] \leq \sum_{i \in [n]} \phi_i$.

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Assume there is a Pareto optimal solution S^* with weight t. Let T be the highest ranked solution with $\mathbf{weight}(T) < t$. If T does not exist:

- There is no stopper (not for a single i)
- $S^* \neq \emptyset$, so let i be some element in S^*.
- $\min\{\mathbf{weight}(S') \mid i \in S', S'$ has higher rank than $\mathbf{stopper}(i)\}$ is attained by S^*
- $E \Rightarrow E_i$ for at least one i.

Analysis (cont.)

We have shown $E \Rightarrow (\cup_i E_i)$, thus

$$\mathbf{Pr}[E] \leq \sum_{i=1}^{n} \mathbf{Pr}[E_i]$$

Next we will prove $\mathbf{Pr}[E_i] \leq \phi_i$. Hence,

$$\mathbf{Pr}[E] \leq \sum_{i=1}^{n} \mathbf{Pr}[E_i] \leq \sum_{i=1}^{n} \phi_i.$$

Analysis (cont.)

Show: $\mathbf{Pr}[E_i] \leq \phi_i$ for all $i \in [n]$.

Recall definitions:

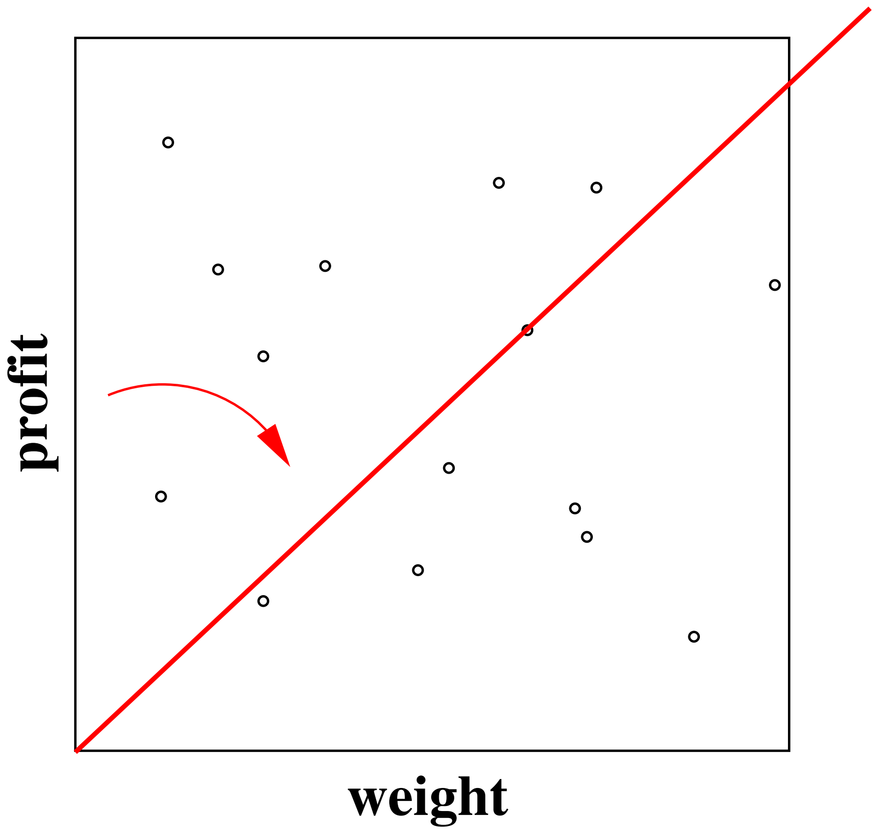
$\mathbf{stopper}(i)$ = highest ranked solution S with $i \notin S$ and $\mathbf{weight}(S) < t$

$E_i := [\min\{\mathbf{weight}(S) \mid i \in S, S$ has higher rank than $\mathbf{stopper}(i)\} = t]$

- Show claim for arbitrary i
- Fix all w_j, $j \neq i$.
- Fixes $\mathbf{stopper}(i)$ and thus the solution that attains the minimum in E_i.
- This solution has weight $c + w_i$, where c is some fixed quantity depending on w_j, $j \neq i$
- $\mathbf{Pr}[w_i = t - c] \leq \phi_i$, hence $\mathbf{Pr}[E_i] \leq \phi_i$

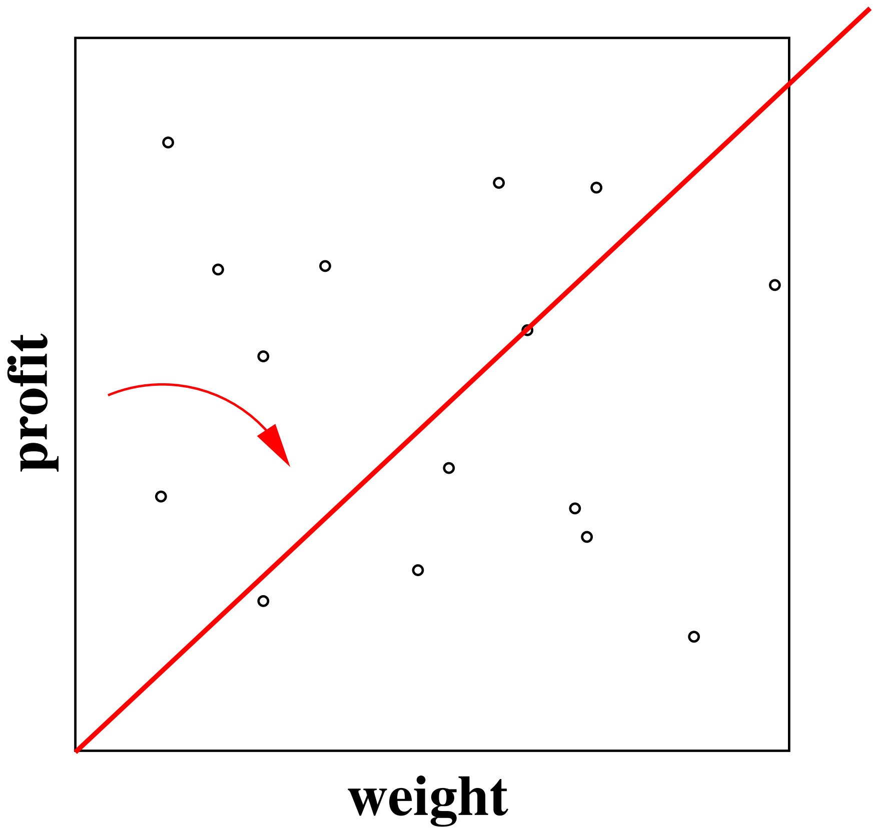
The Core Concept

optimal fractional solution: Dantzig's relaxation (1957)

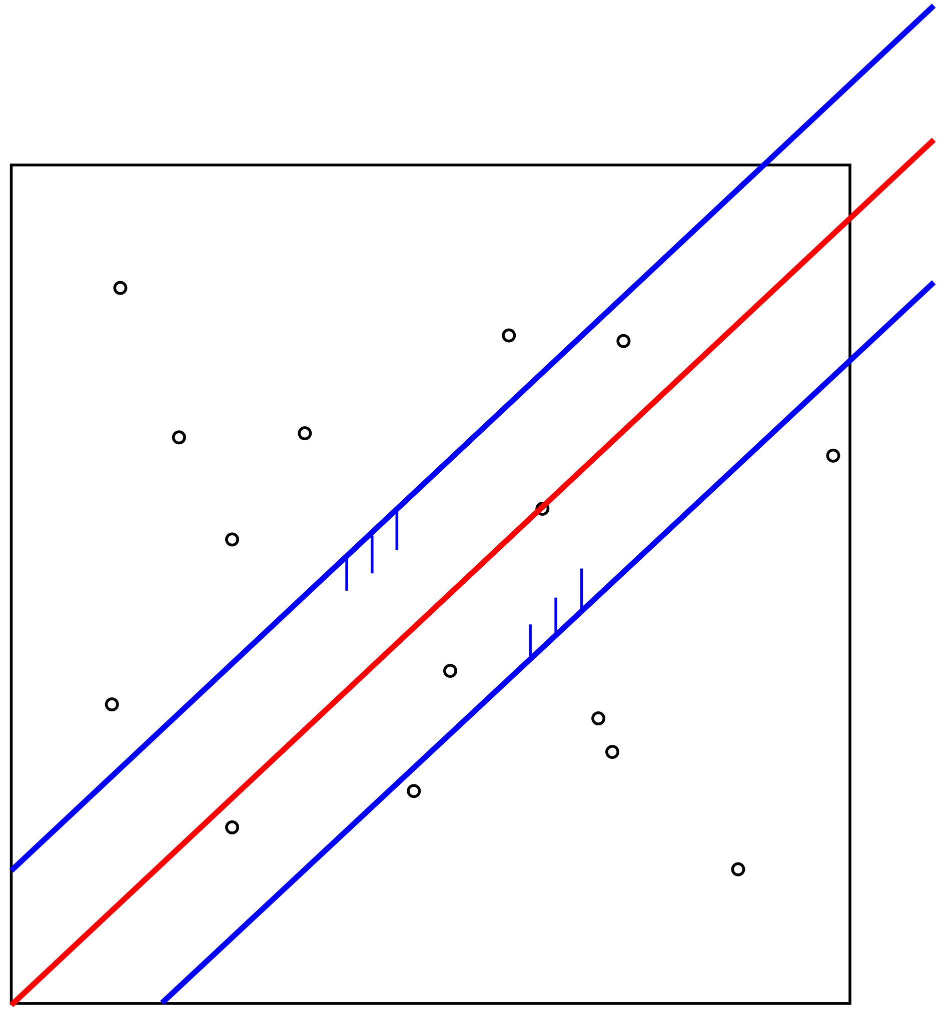


optimal fractional solution: Dantzig's relaxation (1957)

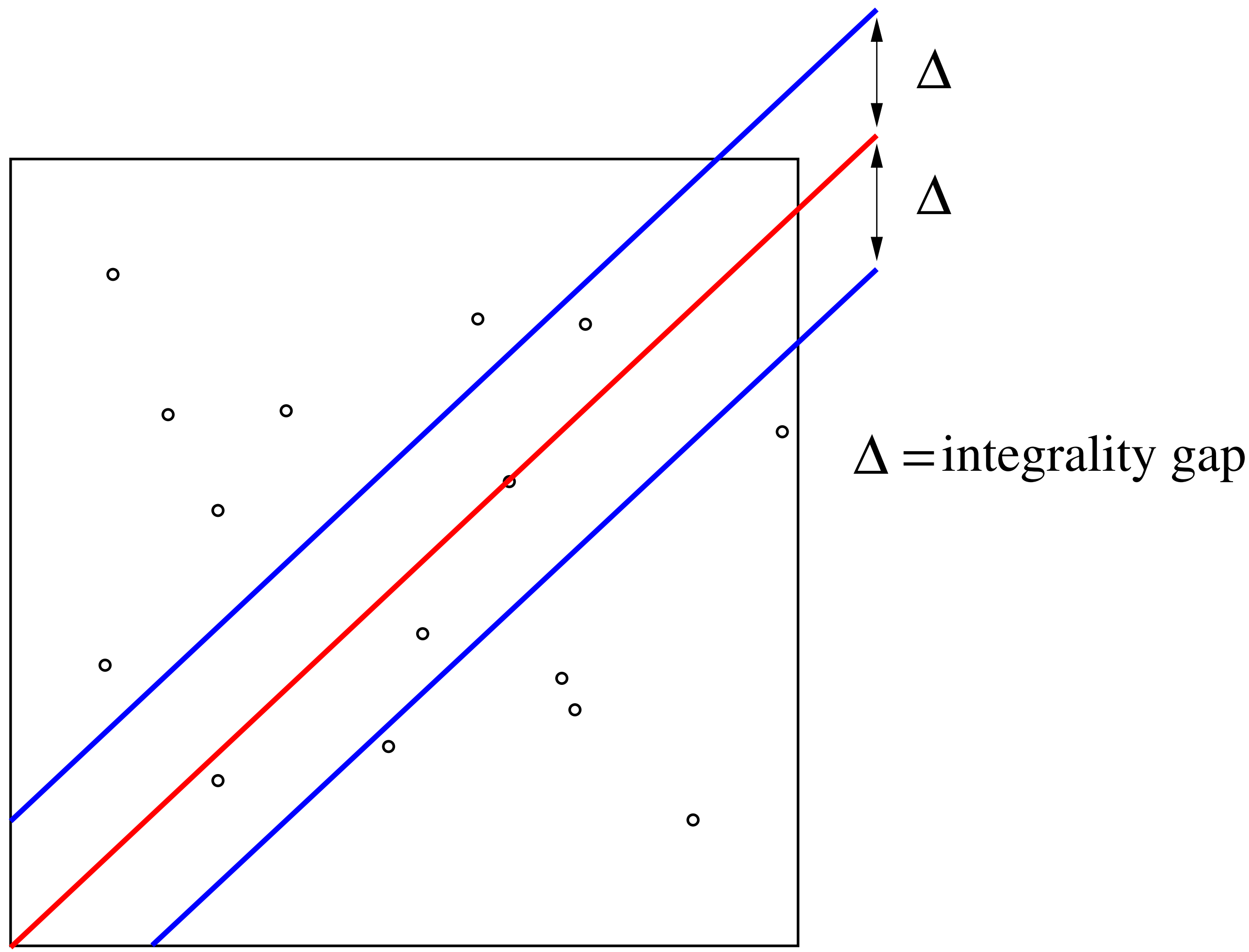
- Linear time (Balas and Zemel, 1980)



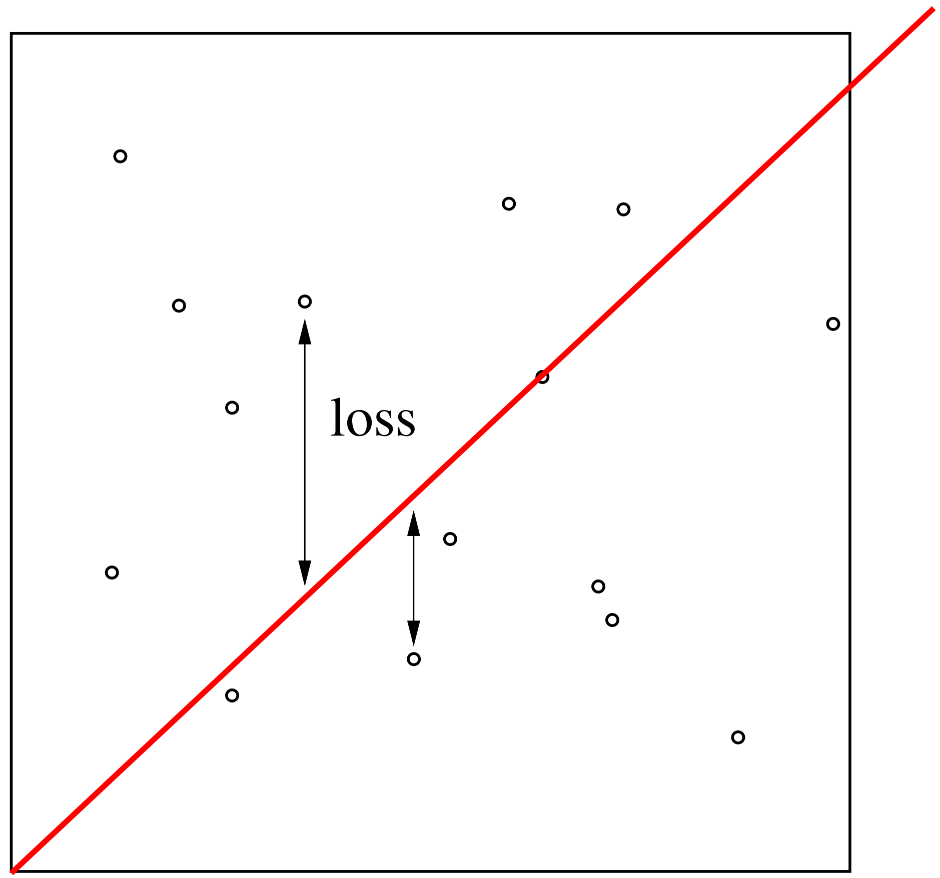
core algorithms (Balas and Zemel, 1980)



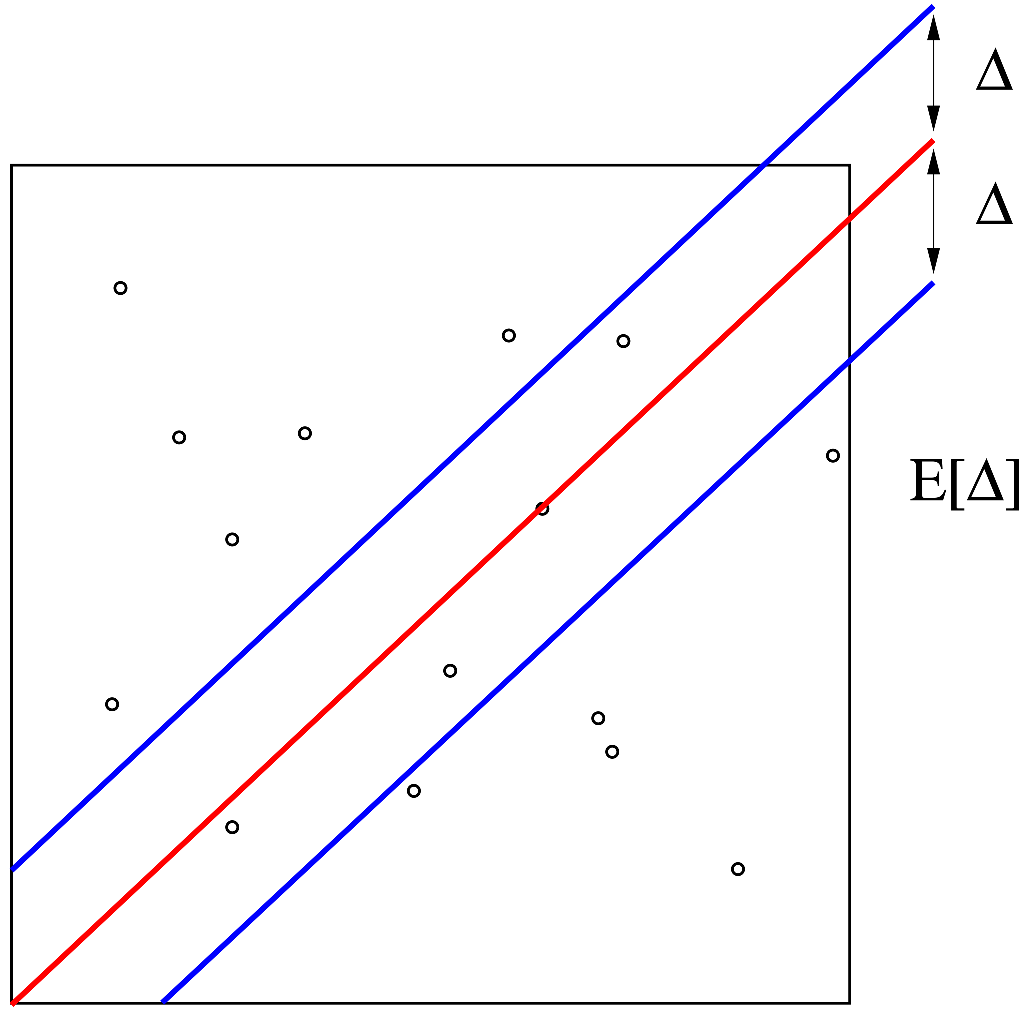
core algorithms – integrality gap



core algorithms – loss of an element

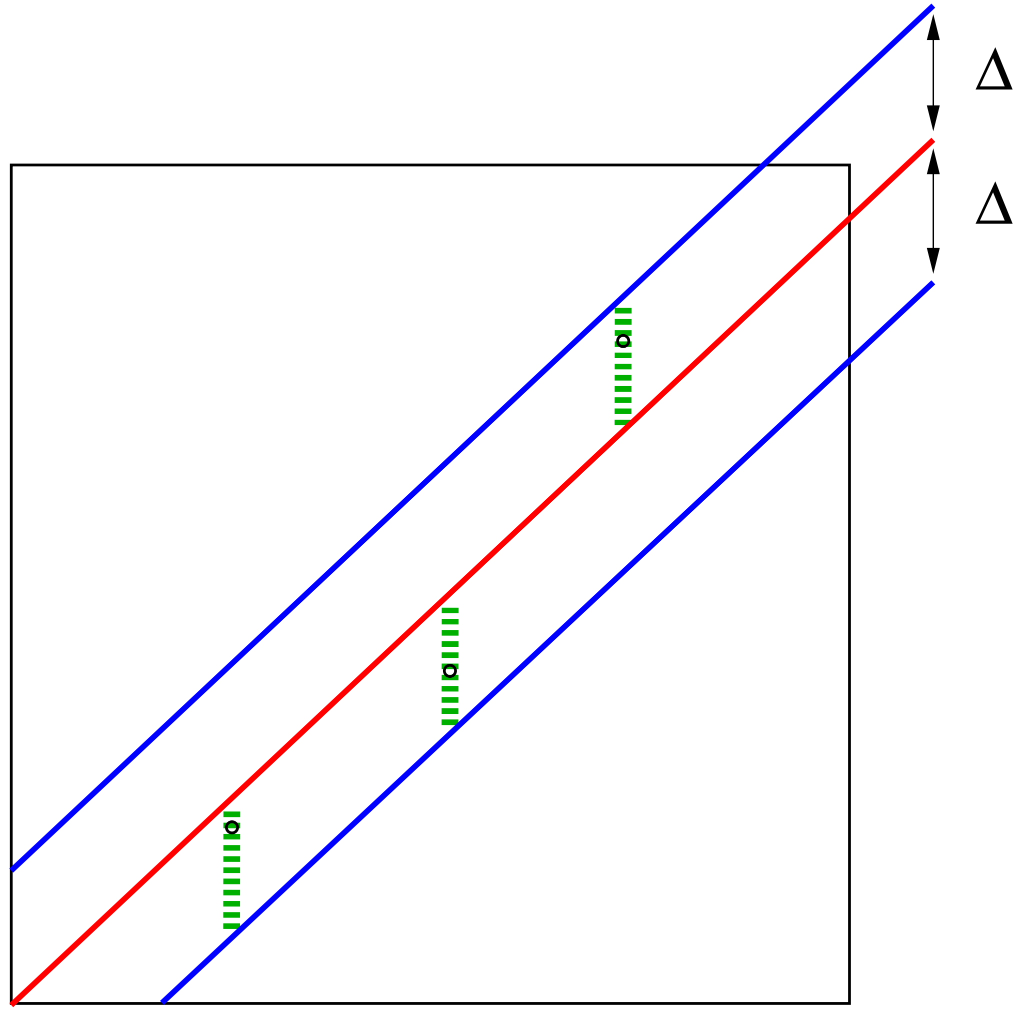


Lueker's result (1982)



$$E[\Delta] = \Theta\left(\frac{\log^2 n}{n}\right)$$

implications for core algorithms



implications for core algorithms

- Apply the Nemhauser/Ullmann algorithm to core elements
- Profits follow a uniform distribution with density $1/\Delta$
- $k \Rightarrow$ number of core elements
- expected running time of Nemhauser/Ullmann is $O(k^5/\Delta)$

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$$\Rightarrow O(n \,\text{polylog}(n))$$

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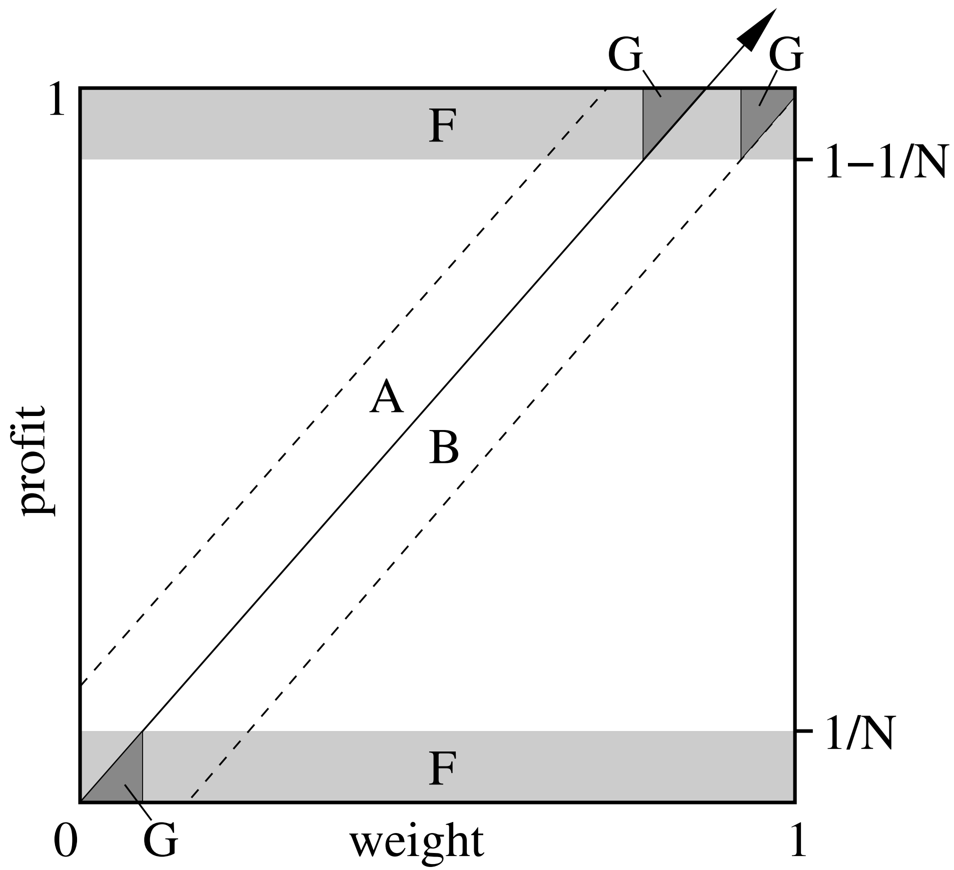

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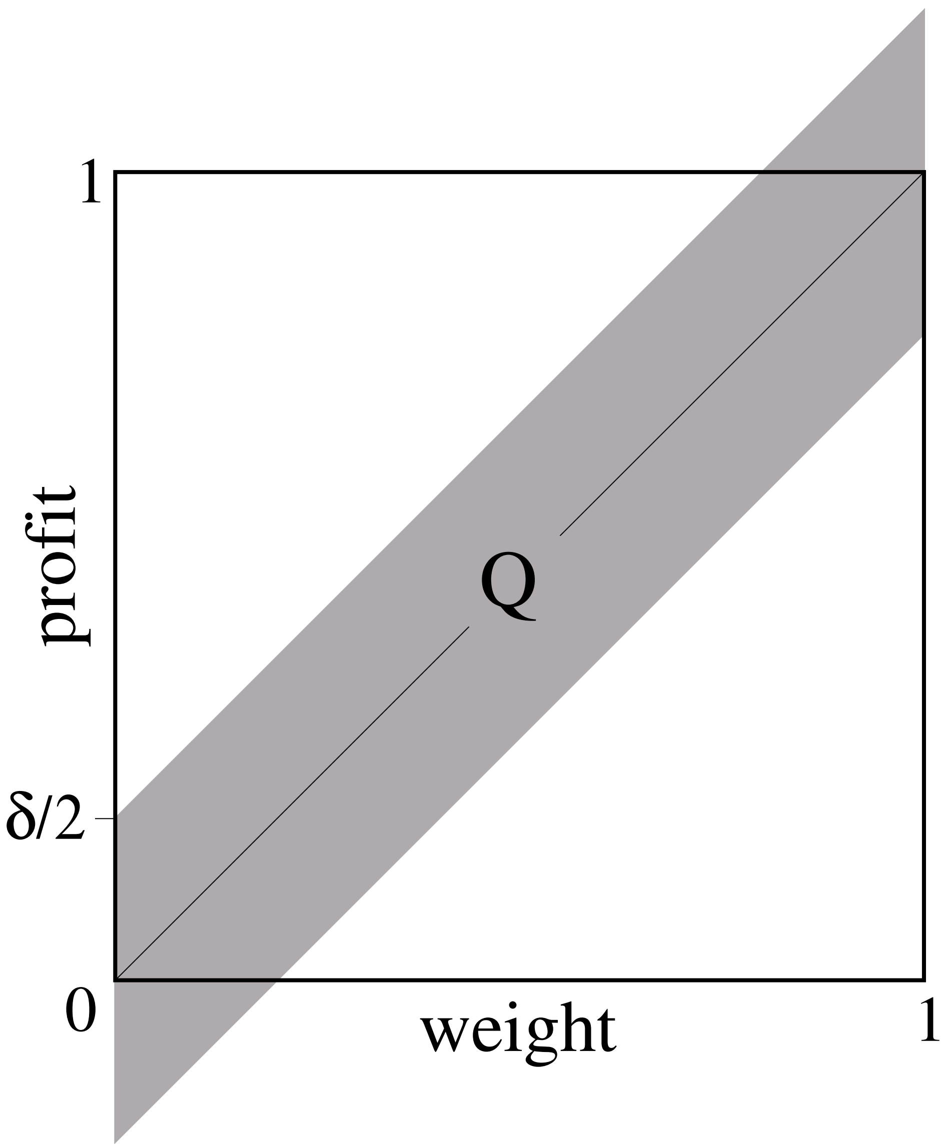
- not all core items have sufficient randomness
- number of core items
- integrality gap is not known
  - Choose large core region s.t. P [*failure*] $\leq n^{-5}$
  - in case of *failure*:
    - compute second list of dominating sets for remaining items
    - Combine two lists in linear time (Horowitz and Sahni ’74)

δ-correlated instances



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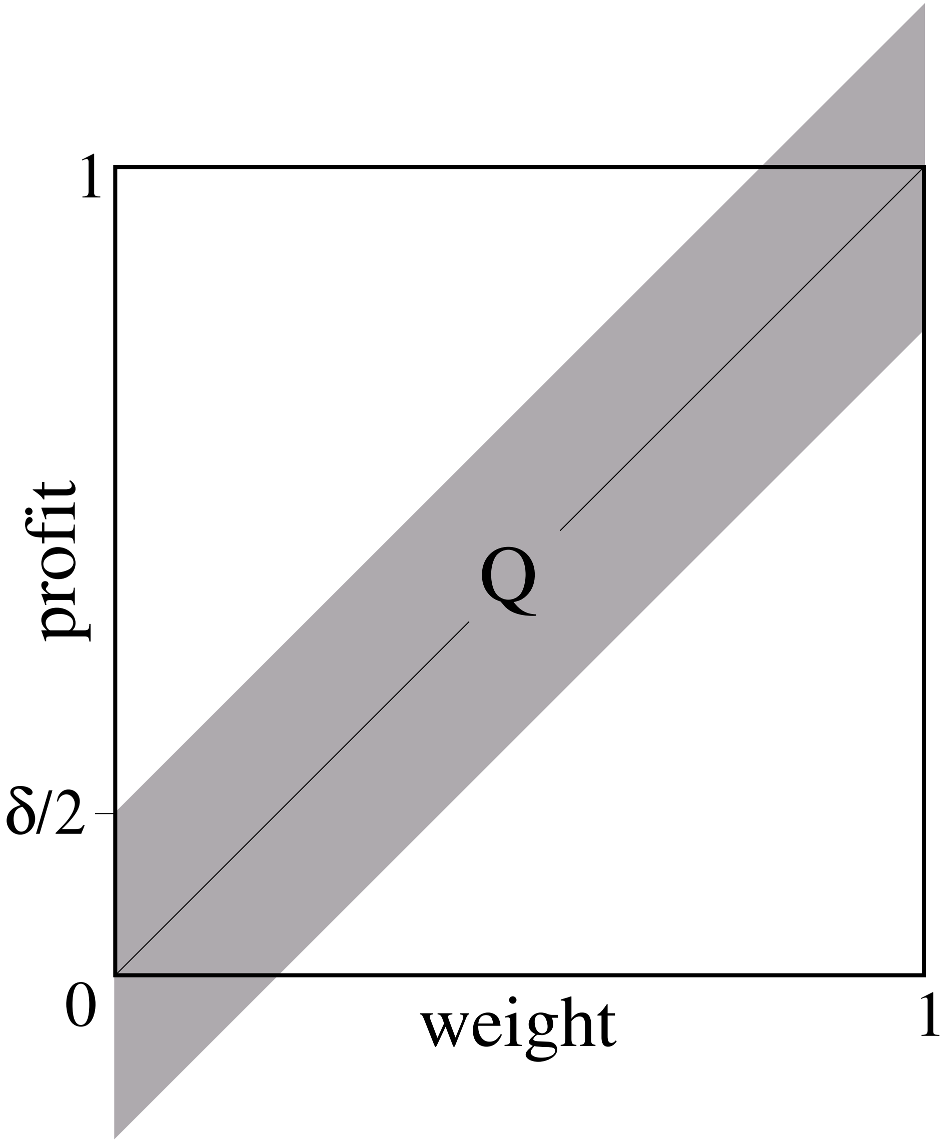

$\mathbf{E}[\Delta] = O(\frac{\delta}{n} \log^2 \frac{n}{\delta})$

running time: $O(\frac{n}{\delta} \text{ polylog} \frac{n}{\delta})$