Improved Absolute Approximation Ratios for Two-Dimensional Packing Problems

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Abstract. We consider the two-dimensional bin packing and strip packing problem, where a list of rectangles has to be packed into a minimal number of rectangular bins or a strip of minimal height, respectively. All packings have to be non-overlapping and orthogonal, i.e., axis-parallel. Our algorithm for strip packing has an absolute approximation ratio of 1.9396 and is the first algorithm to break the approximation ratio of 2 which was established more than a decade ago. Moreover, we present ~~a polynomial time approximation scheme ($\mathcal{PTAS}$) for strip packing where rotations by 90 degrees are permitted and~~[1] an algorithm for two-dimensional bin packing with an absolute worst-case ratio of 2, which is optimal provided $\mathcal{P} \neq \mathcal{NP}$.

Keywords: two-dimensional bin packing, strip packing, rectangle packing, approximation algorithm, absolute worst-case ratio

1 Introduction

In the two-dimensional bin packing problem, a list $I = \{r_1, \ldots, r_n\}$ of rectangles of width $w_i \leq 1$ and height $h_i \leq 1$ is given. An unlimited supply of equally-sized, rectangular bins is available to pack all items from I such that no two items overlap and all items are packed axis-parallel into the bins. The goal is to minimize the number of bins used. We assume that the bins have unit size, which can be achieved by scaling the items appropriately. For the strip packing problem, the given items have to be packed into a strip of unit width and minimal height.

Both problems have many applications, for instance in stock-cutting or scheduling on partitionable resources. In many applications, rotations are not allowed because of the pattern of the cloth or the grain of the

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[1] *Note of the authors:* The $\mathcal{PTAS}$ for strip packing with rotations turned out to be wrong. See [9] for the proof of a 3/2 lower bound on the approximability of this problem. All further references to this wrong result have been removed from this version of the paper.

wood. This is the case that we consider in this paper. Note that the assumption of unit-sized bins is a restriction in the case where rotations are permitted.

Most of the previous work on two-dimensional packing problems has focused on the *asymptotic* approximation ratio, i.e., the behavior of the algorithm on instances with large optimal value. The asymptotic approximation ratio is defined as follows. Let $\mathrm{ALG}(I)$ be the value, i.e., the height of the strip or the number of bins, of a packing produced by algorithm ALG on input I. Denote the optimal algorithm by OPT. The asymptotic approximation ratio of packing algorithm ALG is defined to be

$$\limsup_{n\to\infty} \sup_{I} \left\{ \frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)} \;\middle|\; \mathrm{OPT}(I) = n \right\}.$$

Kenyon & Rémila [12] and Jansen & van Stee [10] gave asymptotic fully polynomial approximation schemes ($\mathcal{FPTAS}$'s) for strip packing without rotations and with rotations, respectively. The additive constant was recently improved from $\mathcal{O}(1/\varepsilon^2)$ to 1 by Jansen & Solis-Oba [9] at the cost of a higher running time.

Caprara [5] was the first to present an algorithm with an asymptotic approximation ratio less than 2 for two-dimensional bin packing. Indeed, he considered 2-stage packing, in which the items must first be packed into shelves that are then packed into bins, and showed that the asymptotic worst case ratio between two-dimensional bin packing and 2-stage packing is $T_\infty = 1.691\ldots$. Therefore the asymptotic $\mathcal{FPTAS}$ for 2-stage packing by Caprara, Lodi & Monaci [6] achieves an asymptotic approximation guarantee arbitrarily close to T_∞.

Recently, Bansal, Caprara & Sviridenko [2] presented a general framework to improve subset oblivious algorithms and obtained asymptotic approximation guarantees arbitrarily close to $1.525\ldots$ for packing with rotations of 90 degrees or without rotations. These are the currently best-known asymptotic approximation ratios for general two-dimensional bin packing problems. For packing squares into square bins, Bansal, Correa, Kenyon & Sviridenko [4] gave an asymptotic $\mathcal{PTAS}$. On the other hand, the same paper showed the $\mathcal{APX}$-hardness of two-dimensional bin packing without rotations, thus no asymptotic $\mathcal{PTAS}$ exists unless $\mathcal{P} = \mathcal{NP}$. Chlebík & Chlebíková [7] were the first to give explicit lower bounds of $1+1/3792$ and $1+1/2196$ on the asymptotic approximability of rectangle packing with and without rotations, respectively.

It should be noted that for the positive results for bin packing mentioned above, the approximation ratio only gets close to the stated value for very large inputs. In particular, the 1.525-approximation by Bansal et al. [2] has an additive constant which is not made explicit in the paper but which the authors believe is extremely large [1]. Thus, for any reasonable input, the actual (absolute) approximation ratio of their algorithm is much larger than 1.525, and it therefore makes sense to consider alternative algorithms and in particular, an alternative performance measure.

In the current paper, we consider the absolute approximation ratio. This is defined simply as $\sup_I \text{ALG}(I)/\text{OPT}(I)$, where the supremum is taken over all inputs. Proving a bound on the absolute approximation gives us a performance guarantee for all inputs, not just for (very) large ones.

Steinberg [15] and Schiermeyer [14] presented absolute 2-approximation algorithms for strip packing. Especially Steinberg's algorithm has been used in many subsequent bin packing and strip packing papers as subroutines. Since one-dimensional bin packing is a natural subproblem of strip packing, there exists no $(3/2 - \varepsilon)$-approximation for any $\varepsilon > 0$. Jansen & Solis-Oba [9] showed an absolute $\mathcal{PTAS}$ for strip packing with rotations on instances with optimal height at least 1.

For the bin packing problem, Zhang [17] presented an absolute 3-approximation algorithm. For the special case of packing squares into bins, van Stee [16] showed that an absolute 2-approximation is possible. Moreover, Harren & van Stee [8] gave an absolute 2-approximation for bin packing with rotations. They also showed that the algorithm Hybrid First Fit has an absolute approximation ratio of 3 for packing without rotations, as conjectured by Zhang [17].

Our contribution. We present an approximation algorithm for strip packing with an absolute approximation ratio of 1.9396. Although Schiermeyer [14] already expected in his work in 1994 that this bound can be reduced below 2, this is the first improvement on the absolute approximability of strip packing since Schiermeyer's work.

Moreover, we present an approximation algorithm for two-dimensional bin packing with an absolute approximation ratio of 2. As Leung et al. [13] showed that it is strongly $\mathcal{NP}$-complete to decide whether a set of squares can be packed into a given square, this is best possible unless $\mathcal{P} = \mathcal{NP}$.

2 Important tools and preparations

Let $I = \{r_1, \ldots, r_n\}$ be the set of given rectangles, where $r_i = (w_i, h_i)$. For $\delta \leq 1/2$, let $W_\delta = \{r_i \mid w_i > 1-\delta\}$ be the set of so-called δ-*wide* items and let $H_\delta = \{r_i \mid h_i > 1-\delta\}$ be the set of δ-*high* items. To simplify the presentation, we denote the 1/2-wide items as *wide* items and the 1/2-high items as *high* items. Let W and H be the sets of wide and high items, respectively. The set of *small* items, i.e., items r_i with $w_i \le 1/2$ and $h_i \le 1/2$, is denoted by S. Finally, we call items that are wide and high at the same time *big*.

For a set T of items, let $\mathcal{A}(T) = \sum_{i \in T} w_i h_i$ be the total area and let $h(T) = \sum_{r_i \in T} h_i$ and $w(T) = \sum_{r_i \in T} w_i$ be the total height and total width, respectively. Finally, let $w_{\max}(T) = \max_{r_i \in T} w_i$ and $h_{\max}(T) = \max_{r_i \in T} h_i$.

Steinberg [15] proved the following theorem for his algorithm that we use as a subroutine.

Theorem 1 (Steinberg's algorithm). *If the following inequalities hold,*

$$w_{\max}(T) \le a, \quad h_{\max}(T) \le b, \qquad \textit{and}$$
$$2\mathcal{A}(T) \le ab - (2w_{\max}(T) - a)_+ (2h_{\max}(T) - b)_+$$

where $x_+ = \max(x, 0)$*, then it is possible to pack all items from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/ \log\log n)$*.*

Bansal, Caprara & Sviridenko [3] considered the two-dimensional knapsack problem in which each item $r_i \in I$ has an associated profit p_i and the goal is to maximize the total profit that is packed into a unit-sized bin. Using a very technical *Structural Lemma* they derived an algorithm that we call BCS algorithm in this paper. We use the following corollary of their analysis for the case where we want to maximize the total packed area, i.e., $p_i = w_i h_i$ for all items $r_i \in I$. Let $\mathrm{OPT}_{(a,b)}(T)$ denote the maximum area of items from T that can be packed into the rectangle (a, b), where individual items in T do not necessarily fit in (a, b).

Corollary 1. *For any fixed* $\varepsilon > 0$*, the BCS algorithm returns a packing of* $I' \subseteq I$ *in a rectangle of width* $a \le 1$ *and height* $b \le 1$ *such that* $\mathcal{A}(I') \ge \mathrm{OPT}_{(a,b)}(I) - \varepsilon$*.*

3 Strip packing

An important link between strip packing and two-dimensional bin packing is the interpretation of a strip of height 1 as a bin of unit size. This link is especially crucial as handling instances that fit into one bin turns out to be a major challenge for bin packing. Moreover, strip packing can essentially be reduced to the packing of instances with optimal value at most 1 as the following lemma shows.

Lemma 1. *Let $0 < \varepsilon < 1/4$. If there exists a polynomial-time algorithm for strip packing that packs any instance I with optimal value at most 1 into a strip of height h, then there also exists a polynomial-time algorithm for strip packing with absolute approximation ratio at most $h + \varepsilon$.*

Proof. Let ALG be the algorithm that packs any instance I with optimal value at most 1 into a strip of height h and assume that $h \leq 2$ by otherwise applying Steinberg's algorithm. Let ε' be the maximal value with $\varepsilon' \leq \varepsilon/(4h)$ such that $1/\varepsilon'$ is integer. We guess the optimal value approximately and apply ALG on an appropriately scaled instance. To do this, we first apply Steinberg's algorithm on I to get a packing into height $h' \leq 2\,\mathrm{OPT}(I)$. We split the interval $J = [h'/2, h']$ into $1/\varepsilon'$ subintervals $J_i = [(1+\varepsilon'(i-1))h'/2, (1+\varepsilon' i)h'/2]$ for $i = 1, \ldots, 1/\varepsilon'$. Then we iterate over $i = 1, \ldots, 1/\varepsilon'$, scale the heights of all items by $2/((1+\varepsilon' i)h')$ and apply the algorithm ALG on the scaled instance I'. Convert the packing to a packing of the unscaled instance I and finally output the minimal packing that was derived. We eventually consider $i^* \in \{1, \ldots, 1/\varepsilon'\}$ with $\mathrm{OPT}(I) \in J_{i^*}$. Then we have

$$1 - 2\varepsilon' < 1 - \frac{\varepsilon' h}{1+\varepsilon' i^* h} = \frac{1+\varepsilon'(i^*-1)h}{1+\varepsilon' i^* h} \leq \mathrm{OPT}(I') \leq \frac{1+\varepsilon' i^* h}{1+\varepsilon' i^* h} = 1$$

and thus

$$\frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)} = \frac{\mathrm{ALG}(I')}{\mathrm{OPT}(I')} < \frac{h}{1-2\varepsilon'} = h + \frac{2\varepsilon' h}{1-2\varepsilon'} \leq h + 4\varepsilon' h \leq h + \varepsilon. \quad \square$$

Thus we concentrate on approximating instances that fit into a strip of height 1 and therefore assume $\mathrm{OPT}(I) \leq 1$ for the remainder of this section. The overall approach for our algorithm for strip packing consists of two parts. First, we use the BCS algorithm to pack instances where the total height of the δ-wide items is small relative to δ into a strip of height $2-x$ for some positive value x. Second, we derive an area guarantee for instances that could not be packed in the previous step and use this guarantee to successfully pack the instance into a strip of height $2-x$.

Finally, we will show that x can be chosen as large as $(1-\ln 2)/(3 + 3\ln 2) - \varepsilon$ and with Lemma 1 we get the following theorem for any $0 < \varepsilon < 10^{-5}/2$.

Theorem 2. *There exists a polynomial-time approximation algorithm for strip packing with absolute approximation ratio*

$$2 - x + \varepsilon = \frac{5 + 7\ln 2}{3 + 3\ln 2} + 2\varepsilon < 1.9396.$$

Assume that we have a fixed $x \in [0, 1/6 - 5/3\,\varepsilon)$ and $0 < \varepsilon \leq 10^{-5}/2$.

3.1 Small total height of the δ-wide items

In the following we describe an important subroutine that is used by our algorithms for strip and bin packing. We consider the case that the total height of the δ-wide items is small relative to some δ, i.e.,

$$h(W_\delta) \le \frac{\delta(1-x) - 2x - \varepsilon}{1+2\delta} =: f(\delta)$$

for some $\delta \in ((2x+\varepsilon)/(1-x), 1/2]$ (the lower bound is required as otherwise $f(\delta) < 0$). We want to derive a packing of I into two bins such that only a height of $1-x$ is used in the second bin. For strip packing this directly gives a height of $2-x$ by putting the second bin on top of the first. And for bin packing we get a feasible solution for all $x \ge 0$.

Let $\gamma := f(\delta) + x = (\delta(1+x) - x - \varepsilon)/(1+2\delta) < 1/2$. In the first step, we show that a packing of almost all items into a unit bin and with a special structure exists. This special structure consists of a part of width $w(H_\gamma)$ for the γ-high items and a part of width $1 - w(H_\gamma)$ for the other items. The following lemma shows that almost all other items can be packed.

Lemma 2. *We have* $\mathrm{OPT}_{(1-w(H_\gamma),1)}(I \setminus H_\gamma) \ge \mathcal{A}(I \setminus H_\gamma) - 2\gamma$.

Proof. Consider an optimal packing of I into a bin. Remove all items that are completely contained in the top or bottom γ-margin. After this step there is no item directly above or below any item of $H_\gamma = \{r_i \mid h_i > 1-\gamma\}$. Thus we can cut the remaining packing at the left and right side of any item from H_γ. These cuts partition the packing into parts which can be swapped without losing any further items. Move all items of H_γ to the left of the bin and move all other parts of the packing to the right. The total area of the removed items is at most 2γ and thus a total area of at least $\mathcal{A}(I \setminus H_\gamma) - 2\gamma$ fits into the rectangle of size $(1 - w(H_\gamma), 1)$ to the right of H_γ. ☐

In the second step, we actually derive a feasible packing that is based on the structure described above (see Figure 1(a)). First, pack H_γ into a stack of width $w(H_\gamma)$ at the left side of the first bin. Note that $w(H_\gamma) \le 1$. This leaves an empty space of width $1 - w(H_\gamma)$ and height 1 at the right. We therefore apply the BCS algorithm on $I \setminus H_\gamma$ and a rectangle of size $(1 - w(H_\gamma), 1)$ using an accuracy of ε. Lemma 2 and Corollary 1 yield that at least a total area of $\mathcal{A}(I \setminus H_\gamma) - 2\gamma - \varepsilon$ is packed by the algorithm.

Let T be the set of remaining items with $\mathcal{A}(T) \le 2\gamma + \varepsilon$. Pack the remaining δ-wide items, i.e., the items of $T \cap W_\delta$, in a stack at the bottom of the second bin. The total area of the remaining items $T \setminus W_\delta$ is

$$\mathcal{A}(T \setminus W_\delta) \leq \mathcal{A}(T) - (1-\delta)h(T \cap W_\delta) \leq 2\gamma + \varepsilon - (1-\delta)h(T \cap W_\delta).$$

We pack these items into the free rectangle of size (a, b) with $a = 1$ and $b = 1 - h(T \cap W_\delta) - x$ above the stack of $T \cap W_\delta$ in the second bin. A short calculation shows that Steinberg's algorithm is applicable.

So far we assumed the knowledge of $\delta \in ((2x+\varepsilon)/(1-x), 1/2]$ for which $h(W_\delta) \leq f(\delta)$. It is easy to see that this value can be computed by calculating $h(W_\delta)$ for $\delta = 1 - w_i$ for all $r_i = (w_i, h_i)$ with $w_i > 1/2$. As $h(W_\delta)$ changes only for these values of δ, we will necessarily find a suitable δ if one exists. We therefore have the following lemma.

Lemma 3. *For any fixed $\varepsilon > 0$, there exists a polynomial-time algorithm that, given an instance I with $\mathrm{OPT}(I) = 1$ and $h(W_\delta) \leq f(\delta)$ for some $\delta \in ((2x+\varepsilon)/(1-x), 1/2]$, returns a packing of I into two bins such that only a height of $1 - x$ is used in the second bin.*

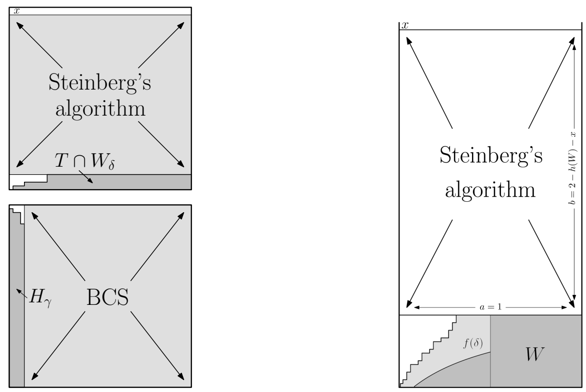
3.2 Using an area guarantee for the wide items

In this section we describe how to use a guarantee on the total area of the wide items for the instances that cannot be packed into a strip of height $2 - x$ by Lemma 3. Consider a strip with the lower left corner at the origin of a cartesian coordinate system and consider the stack of wide items ordered by non-increasing width and aligned with the lower right corner of the strip. If there exists a $\delta \in ((2x+\varepsilon)/(1-x), 1/2]$ such that $h(W_\delta) \leq f(\delta)$ then we use the algorithm of Lemma 3 to pack the instance into a strip of height $2 - x$ (see Figure 1(a)). Otherwise the stack of wide items exceeds the function $f(\delta)$ for *all* $\delta \in ((2x+\varepsilon)/(1-x), 1/2]$ (see Figure 1(b)). Then we have

$$\mathcal{A}(W) > \int_{\frac{2x+\varepsilon}{1-x}}^{1/2} \frac{\delta(1-x) - 2x - \varepsilon}{1+2\delta} d\delta + \frac{h(W)}{2} > \xi(x) + \frac{h(W)}{2} \qquad (1)$$

for $\xi(x) := \frac{1}{4}(1 - \ln 2) - \frac{1}{4}x(1 + 3\ln 2) - \frac{1}{2}\varepsilon \ln 2$ (this function corresponds to a lower bound of the area in darker shade below $f(\delta)$ in Figure 1(b))—see full version for the calculation.

We use this lower bound for the area of W to derive a packing into a strip of height $2 - x$. Assume that $2 - h(W) - x \geq 1$. Stack the wide items in the bottom of the strip and use Steinberg's algorithm to pack $I \setminus W$ above this stack into a rectangle of size (a, b) with $a = 1$ and



(a) Packing into two bins.

(b) Using the area guarantee of $\xi(x) + h(W)/2$ to pack into height $2 - x$.

Fig. 1. Main cases for strip packing.

$b = 2 - h(W) - x$. Then we have $h_{\max}(I \setminus W) \leq 1 \leq b$, $w_{\max}(I \setminus W) \leq 1/2$ and for

$$\begin{aligned} 2\mathcal{A}(I \setminus W) \leq 2 - 2\xi(x) - h(W) &\leq 2 - h(W) - x \\ &= ab = ab - (2w_{\max} - a)_+(2h_{\max} - b)_+ \end{aligned}$$

we require $x \leq 2\,\xi(x)$. This is satisfied for

$$x \leq \frac{1 - \ln 2}{3 + 3\ln 2} - \varepsilon \leq \frac{1 - \ln 2 - 2\varepsilon \ln 2}{3 + 3\ln 2}.$$

We give a simple algorithm that also has the requirement $x \leq 2\,\xi(x)$ for the other case in the full version. Thus we can choose $x = (1 - \ln 2)/(3 + 3\ln 2) - \varepsilon$ and together with Lemmas 1 and 3 we proved Theorem 2.

4 Two-dimensional bin packing

As the asymptotic approximation ratio of the algorithm by Bansal, Caprara & Sviridenko [2] is arbitrarily close to $1.525\ldots$, there exists a constant k such that for any instance I with optimal value larger than k, their algorithm gives a solution of value at most $2\,\mathrm{OPT}(I)$. This constant k is not explicitly known as we already mentioned in the introduction. We show how to approximate the problem within an absolute factor of 2, provided that the optimal value of the given instance is less than k.

Combined with the algorithm by Bansal et al., this proves the existence of an algorithm with an absolute approximation ratio of 2.

Our approach for packing instances I with $\mathrm{OPT}(I) < k$ consists of two parts. First, we give an algorithm that is able to pack instances I with $\mathrm{OPT}(I) = 1$ in two bins in Section 4.1 and second, we show how to approximate instances with $1 < \mathrm{OPT}(I) < k$ within a factor of 2 in Section 4.2. This at first glance surprising distinction is due to the inherent difficulty of packing wide and high items together into a single bin. In the case $\mathrm{OPT}(I) = 1$ we cannot ensure a separation of the wide and high items into easily feasible sets whereas for $\mathrm{OPT}(I) > 1$ this is possible in many cases.

The approach to solve instances with optimal value greater than some constant k with an asypmotic algorithm is similar to the 2-approximation for two-dimensional bin packing with rotations in [8] but the methods we use here to handle the instances with smaller optimal value are much more involved. The reason for this is that we cannot use rotations to avoid the necessity to combine wide and high items in a bin. Our approach for solving instances I with $1 < \mathrm{OPT}(I) < k$ is comparable to the main algorithm in [8] as it is also based on an enumeration of the large items. However, a new ingredient in this paper is a separation of the wide and high items after this enumeration. Another crucial novelty in our algorithm is the use of the BCS algorithm to ensure a good area guarantee for at least one bin. In total we show the following theorem.

Theorem 3. *There exists a polynomial-time approximation algorithm for two-dimensional bin packing with absolute approximation ratio 2.*

4.1 Packing instances that fit into one bin

Throughout this section we assume that the given instance I can be packed into a single bin, i.e., $\mathrm{OPT}(I) = 1$. At first glance it seems surprising that packing such an instance into two bins is difficult. However, we need to carefully analyse different cases to be able to give a polynomial-time algorithm that solves this problem.

Let $\varepsilon := 1/52$. In a first step we consider instances I that satisfy the requirements of Lemma 3 for $x = 0$, i.e., we have $h(W_\delta) \leq f(\delta) = (\delta - \varepsilon)/(1 + 2\delta)$ for some $\delta \in (\varepsilon, 1/2]$. Obviously, we can apply Lemma 3 to the high items instead of the wide items as well. We get the following Lemma from Inequality (1) for $\xi = 0.075 < \xi(0)$.

Lemma 4. *For any input which cannot be packed in two bins by the methods of Lemma 3, we have*

$$\mathcal{A}(W \cup H) \geq 2\,\xi + \frac{w(H) + h(W)}{2}.$$

It is crucial for our work that we get this additional area guarantee of $2\,\xi = 0.15$ on top of the trivial guarantee of $w(H)/2 + h(W)/2$ here. We use this area guarantee to give different methods to pack the input, depending on the total height of the wide items. To do this, we assume that we have $h(W) \geq w(H)$ by otherwise rotating the whole instance and apply different methods for $w(H) > 1/2$ and $w(H) \leq 1/2$. In all cases we are able to pack the input into at most two bins. Before we show how to solve both cases above we need the following lemma that allows us to pack *all* wide items and high items of almost half of their total width (see full version for a proof of this lemma).

Lemma 5. *For any fixed $\varepsilon > 0$, there exists a polynomial-time algorithm that, given sets W and H of wide and high items with $\mathrm{OPT}(W \cup H) = 1$, returns a packing of $W \cup H'$ into a bin with $H' \subseteq H$ and $w(H') > w(H)/2 - \varepsilon$.*

With these preparations, the following lemma is easy to show.

Lemma 6. *Let $\varepsilon > 0$ and let I be an instance with $\mathrm{OPT}(I) = 1$, $h(W) \geq w(H) > 1/2$, and $h(W_\delta) > f(\delta)$ and $w(H_\delta) > f(\delta)$ for all $\delta \in (\varepsilon, 1/2]$. There exists a polynomial-time algorithm that returns a packing of I into two bins.*

Proof. Use Lemma 5 to pack $W \cup H'$ with $H' \subseteq H$ and $w(H') > w(H)/2 - \varepsilon$ in the first bin. Build a stack of the remaining high items $H \setminus H'$ and align it with the left side of the second bin. The width of this stack is $w(H \setminus H') < w(H)/2 + \varepsilon$. Note that $w(H \setminus H') \leq 1/2$, as otherwise $h(W) \geq w(H) \geq 1 - 2\varepsilon$ and $\mathcal{A}(W \cup H) \geq 2\xi + (w(H) + h(W))/2 \geq 2\xi + 1 - 2\varepsilon > 1$ (by Lemma 4) which is a contradiction to $\mathrm{OPT}(I) = 1$. Pack the remaining items T with Steinberg's algorithm in the free rectangle of size (a, b) with $a = 1 - w(H \setminus H')$ and $b = 1$ next to the stack of $H \setminus H'$. This is possible since $w_{\max}(T) \leq 1/2 \leq 1 - w(H \setminus H')$, $h_{\max}(T) \leq 1/2$ and with Lemma 4 we have (see full version)

$$\begin{aligned} 2\mathcal{A}(T) \leq 2\Big(1 - 2\xi - \frac{w(H) + h(W)}{2}\Big) &< 1 - w(H \setminus H') \\ &= ab - (2w_{\max} - a)_+(2h_{\max} - b)_+. \end{aligned}$$

□

In the following we assume that $w(H) \leq 1/2$ as otherwise we could pack the instance into two bins with the algorithms of Lemma 3 or Lemma 6. Furthermore, we still have our initial assumption $h(W) \geq w(H)$. Using Steinberg's algorithm it is straightforward to prove the following lemma.

Lemma 7. *Any set $T = \{r_1, \ldots, r_m\}$ where $r_i = (w_i, h_i)$ with $w_i \leq 1/2$, $h_i \leq 1 - h(W)$ for $i = 1, \ldots, m$ and total area $\mathcal{A}(T) \leq 1/2 - h(W)/2$ can be packed together with W.*

Obviously, Lemma 7 can also be formulated such that we pack the high items together with a set of small items of total area at most $1/2 - w(H)/2$ (in this case we do not need a condition like $h_i \leq 1 - h(W)$, as $w(H) \leq 1/2$ and thus all remaining items fit into the free rectangle next to the stack of H). This suggests partitioning the small items into sets with these area bounds in order to pack them with the wide and high items. This is possible in all but two special cases, which we deal with separately. In the end, we find the following lemma. Details are in the full version.

Lemma 8. *Let $\varepsilon > 0$ and let I be an instance with $\mathrm{OPT}(I) = 1$, $w(H) \leq 1/2$, and $h(W_\delta) > f(\delta)$ and $w(H_\delta) > f(\delta)$ for all $\delta \in (\varepsilon, 1/2]$. There exists a polynomial-time algorithm that returns a packing of I into two bins.*

This concludes our algorithm for instances I with $\mathrm{OPT}(I) = 1$ as the Lemmas 3, 6 and 8 cover all the cases.

4.2 Packing instances that fit into a constant number of bins

In the following we give a brief description of our algorithm that packs the instances I with $2 \leq \mathrm{OPT}(I) < k$ into $2\,\mathrm{OPT}(I)$ bins.

Let $\varepsilon := 1/(20k^3 + 2)$. Let $L = \{r_i \mid w_i h_i > \varepsilon\}$ be the set of *large* items and let $T = \{r_i \mid w_i h_i \leq \varepsilon\}$ be the set of *tiny* items. As defined in Section 2 we refer to items as wide (W), high (H), small (S) and big, according to their side lengths. Note that the terms *large* and *tiny* refer to the area of the items whereas *big*, *wide*, *high* and *small* refer to their widths and heights. Also note that, e.g., an item can be tiny and high, or wide and big at the same time.

We guess $\ell = \mathrm{OPT}(I) < k$ and open 2ℓ bins that we denote by $B_1, \ldots, B_\ell$ and $C_1, \ldots, C_\ell$. By *guessing* we mean that we iterate over all possible values for ℓ and apply the remainder of this algorithm on every value. As there are only a constant number of values, this is possible in polynomial time. We assume that we know the correct value of ℓ as we eventually consider this value in an iteration. For the ease of presentation, we also denote the sets of items that are associated with the bins by $B_1, \ldots, B_\ell$ and $C_1, \ldots, C_\ell$. We will ensure that the set of items that is associated with a bin is feasible and a packing is known or can be computed in polynomial time. To do this we use the following corollary from Theorem 1 for some of these sets.

Corollary 2 (Jansen & Zhang [11]). *If the total area of a set T of items is at most $1/2$ and there are no wide items (except a possible big item) then the items in T can be packed into a bin.*

Obviously, this corollary also holds for the case that there are no high items (except a possible big item). This corollary is an improvement upon Theorem 1 if there is a big item in T as in this case Theorem 1 would give a worse area bound.

Let I_i^* be the set of items in the i-th bin in an optimal solution. We assume w.l.o.g. that $\mathcal{A}(I_i^*) \geq \mathcal{A}(I_j^*)$ for $i < j$. Then we have

$$\mathcal{A}(I) = \mathcal{A}(I_1^*) + \cdots + \mathcal{A}(I_\ell^*) \leq \ell \cdot \mathcal{A}(I_1^*). \tag{2}$$

In a first step, we guess the assignment of the large items to bins. Using this assignment and the BCS algorithm we pack a total area of at least $\mathcal{A}(I_1^*) - \varepsilon$ into B_1 and keep C_1 empty. This step has the purpose of providing a good area bound for the first bin and leaving a free bin for later use. We ensure that the large items that are assigned to B_1 are actually packed. For all other bins we reserve B_i for the wide and small items (except the big items) and C_i for the high and big items for $i = 2, \ldots, \ell$. This separation enables us to use Steinberg's algorithm (Corollary 2) to pack up to half of the bins' area. In detail, the first part of the algorithm works as follows.

1. Guess $L_i = I_i^* \cap L$ for $i = 1, \ldots, \ell$.
2. Apply the BCS algorithm on $L_1 \cup T$ while ensuring that L_1 is actually packed (see full version for the details). Assign the output to bin B_1 and keep an empty bin C_1.
3. For $i = 2, \ldots, \ell$, assign the wide and small items of L_i to B_i (omitting big items) and assign the high and big items of L_i to C_i. That is, $B_i = L_i \setminus H$ and $C_i = L_i \cap H$.
4. For $i = 2, \ldots, \ell$, greedily add tiny wide items from $T \cap W$ by non-increasing order of width to B_i as long as $\mathcal{A}(B_i) \leq 1/2$ and greedily add tiny high items from $T \cap H$ by non-increasing order of height to C_i as long as $w(C_i) \leq 1$.

Corollary 2 shows that using Steinberg's algorithm the bins $B_2, \ldots, B_\ell$ can be packed as there are no wide items and the total area is at most $1/2$. The bins $C_2, \ldots, C_\ell$ can be packed with a simple stack as they contain only high items of total width at most 1. Observe that in Step 4 we only add to a new bin B_i if the previous bins contain items of total area at least $1/2 - \varepsilon$ and we only add to a new bin C_i if the previous bins contain items of total width at least $1 - 2\varepsilon$ (as the width of the tiny high items is at most 2ε) and thus of total area at least $1/2 \cdot (1 - 2\varepsilon) = 1/2 - \varepsilon$. After the application of this first part of the algorithm, some tiny items $T' \subseteq T$ might remain unpacked. Note that if $\mathcal{A}(B_\ell) < 1/2 - \varepsilon$, then there are no wide items in T' and if $\mathcal{A}(C_\ell) < 1/2 - \varepsilon$ then there are no high items in T' (as these items would have been packed in Step 4).

In the full version we prove that

$$\mathcal{A}(B_1) \geq \mathcal{A}(I_1^*) - \varepsilon. \tag{3}$$

We distinguish different cases to continue the packing according to the filling of the last bins B_ℓ and C_ℓ.

Exemplarily assume that $\mathcal{A}(B_\ell) < 1/2 - \varepsilon$ and $\mathcal{A}(C_\ell) < 1/2 - \varepsilon$. In this case T' does not contain any wide or high items as these items would have been packed to B_ℓ or C_ℓ. Greedily add items from T' into all bins except B_1 as long as the bins contain items of total area at most $1/2$. This process packs all remaining items as otherwise we had a packed area of at least $\mathcal{A}(I_1^*) - \varepsilon + (2\ell - 1)(1/2 - \varepsilon) > \ell \mathcal{A}(I_1^*)$ by Inequality (3) (see full version for the calculation) which is a contradiction to Inequality (2).

All other cases are more complex and we refer to the full version of the paper for a detailed description. In total we showed the following lemma which concludes our presentation of the 2-approximation algorithm for two-dimensional bin packing.

Lemma 9. *There exists a polynomial-time algorithm that, given an instances I with $1 < \mathrm{OPT}(I) < k$, returns a packing in $2\,\mathrm{OPT}(I)$ bins.*

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