Packing Rectangles into 2 OPT Bins using Rotations

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Abstract. We consider the problem of packing rectangles into bins that are unit squares, where the goal is to minimize the number of bins used. All rectangles can be rotated by 90 degrees and have to be packed non-overlapping and orthogonal, i.e., axis-parallel. We present an algorithm for this problem with an absolute worst-case ratio of 2, which is optimal provided $\mathcal{P} \neq \mathcal{NP}$.

Keywords: bin packing, rectangle packing, approximation algorithm, absolute worst-case ratio

1 Introduction

In the rectangle packing problem, a list $I = \{r_1, \ldots, r_n\}$ of rectangles of width $w_i \leq 1$ and height $h_i \leq 1$ is given. An unlimited supply of unit sized bins is available to pack all items from I such that no two items overlap and all items are packed axis-parallel into the bins. The goal is to minimize the number of bins used. The problem is also known as two-dimensional orthogonal bin packing problem and has many applications, for instance in stock-cutting or scheduling on partitionable resources. In many applications, rotations are not allowed because of the pattern of the cloth or the grain of the wood. However, in other applications, it might be possible to rotate the items. In the current paper, we consider the problem with rotations, i.e., items might be rotated by 90 degrees.

Most of the previous work on rectangle packing has focused on the *asymptotic* approximation ratio, i.e., the long term behavior of the algorithm, and on packing *without rotations*. Caprara was the first to present an algorithm with an asymptotic approximation ratio less than 2 for rectangle packing without rotations. Indeed, he considered 2-stage packing, in which the items must first be packed into shelves that are then packed into bins, and showed that the asymptotic worst case ratio between rectangle packing and 2-stage packing is $T_\infty = 1.691\ldots$. Therefore the asymptotic $\mathcal{FPTAS}$ for 2-stage packing from Caprara, Lodi and Monaci [3] achieves an approximation guarantee arbitrary close to T_∞.

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Recently, Bansal, Caprara & Sviridenko [1] presented a general framework to improve subset oblivious algorithms and obtained asymptotic approximation guarantees arbitrarily close to $1.525\ldots$ for packing with or without rotations. These are the currently best-known approximation ratios for these problems. For packing squares into square bins, Bansal, Correa, Kenyon & Sviridenko [2] gave an asymptotic $\mathcal{PTAS}$. On the other hand, the same paper showed the $\mathcal{APX}$-hardness of rectangle packing without rotations, thus no asymptotic $\mathcal{PTAS}$ exists unless $\mathcal{P} = \mathcal{NP}$. Chlebík & Chlebíková [4] were the first to give explicit lower bounds of $1 + 1/3792$ and $1 + 1/2196$ on the asymptotic approximability of rectangle packing with and without rotations, respectively.

In the current paper we consider the absolute worst-case ratio. Attaining a good absolute worst-case ratio is more difficult than attaining a good asymptotic worst-case ratio, because in the second case an algorithm is allowed to "waste" a constant number of bins, which allows e.g. the classification of items followed by a packing where each class is packed separately. Zhang [15] presented an approximation algorithm with an absolute approximation ratio of 3 for the problem without rotations. For the special case of packing squares, van Stee [14] showed that an absolute 2-approximation is possible.

A related two-dimensional packing problem is the strip packing problem, where the items have to be packed into a strip of unit basis and unlimited height such that the height is minimized. Steinberg [13] and Schiermeyer [12] presented absolute 2-approximation algorithms for strip packing without rotations. Kenyon & Rémila [9] and Jansen & van Stee [7] gave asymptotic $\mathcal{FPTAS}$'s for the problem without rotations and with rotations, respectively. The additive constant of these algorithms was recently improved from $\mathcal{O}(1/\varepsilon^2)$ to 1 by Jansen & Solis-Oba [6]. Thus, most versions of the strip packing problem are now closed.

Our contribution. We present an approximation algorithm for rectangle packing with rotations with an absolute approximation ratio of 2. As Leung et al. [10] showed that it is strongly $\mathcal{NP}$-complete to decide wether a set of *squares* can be packed into a given square, this is best possible unless $\mathcal{P} = \mathcal{NP}$. The algorithm is based on a separation of large and small items according to their area. It is very efficient for inputs consisting of small items but uses a less efficient subroutine to deal with large items. Our main lemma on the packability of certain sets of small items is of independent interest.

We started our investigation on the problem with an algorithm of Jansen & Solis-Oba [6] that finds a packing of profit $(1-\delta)\mathrm{OPT}$ into a bin of size $(1, 1+\delta)$, where OPT denotes the optimum for packing into a unit bin. Using the area of the items as their profit gives an algorithm that packs almost everything into an δ-augmented bin. The algorithm can easily be generalized to a constant number of bins.

An immediate idea to transform such a packing into a packing into 2OPT bins is to remove all items that intersect a strip of height δ at the top or bottom of each bin. These items and the items that were not packed by the algorithm

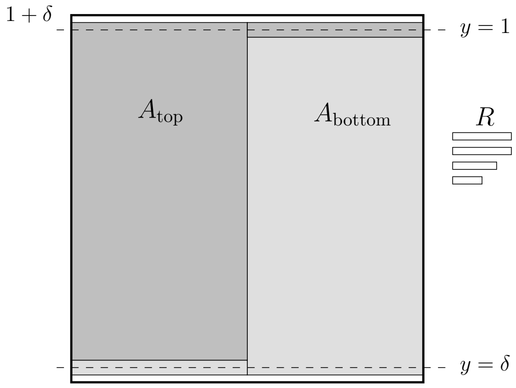

Fig. 1. Packing of Jansen & Solis-Oba's algorithm where it is not immediately clear how to derive a packing into 2 unit bins. The blocks in the packing might consist of several items and might contain small free spaces or items that are not in A_{top} or A_{bottom}. The items of A_{top} and the items of A_{bottom} have total area close to $1/2$. Thus adding the additional items R and packing everything with Steinberg's algorithm is not possible. Furthermore, it is not obvious how to rearrange A_{top} or A_{bottom} such that there is suitable free space to pack R.

would have to be packed separately. In Figure 1 we present an instance where it is not immediately clear how the removed items can be packed separately.

Organisation. The remainder of this article is organized as follows. In Section 2 we introduce notations and two algorithms for strip packing that we will use as subroutines for our rectangle packing algorithm: the algorithm of Steinberg and Next Fit Decreasing Height. We show that Steinberg's algorithm [13] yields an absolute 2-approximation for strip packing with rotations and an absolute 4-approximation for rectangle packing with rotations. Our main result is presented in Section 3. The algorithm is based on our main lemma that we prove in Section 4.

2 Steinberg's Algorithm and NFDH

We assume that all items are rotated such that $w_i \geq h_i$. Denote the total area of a given set T of items by $A(T) = \sum_{i \in T} w_i h_i$ and let $w_{\max} := \max_{r_i \in T} w_i$ and $h_{\max} := \max_{r_i \in T} h_i$. Steinberg [13] showed the following theorem.

Theorem 1 (Steinberg's algorithm [13]). *If the following inequalities hold,*

$$w_{\max} \leq a, \quad h_{\max} \leq b, \; \text{and} \quad 2A(T) \leq ab - (2w_{\max} - a)_+ (2h_{\max} - b)_+$$

where $x_+ = \max(x, 0)$*, then it is possible to pack all items from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$*.*

In our algorithm, we will repeatedly use the following direct corollary of this theorem.

Corollary 1. *If* $w_{\max} \leq a/2$ *and* $A(T) \leq ab/2$*, then it is possible to pack all items from* T *into* $R = (a, b)$ *in time* $\mathcal{O}((n \log^2 n)/\log\log n)$*.*

The following theorem was already mentioned in [6].

Theorem 2. *Steinberg's algorithm gives an absolute 2-approximation for strip packing with rotations.*

Proof. Rotate all items $r_i \in I$ such that $w_i \geq h_i$ and let $b := \max(2h_{\max}, 2A(I))$. Use Steinberg's algorithm to pack I into the rectangle $(1, b)$. This is possible since $2A(I) \leq b$ and $(2h_{\max} - b)_+ = 0$. The claim on the approximation ratio follows from $\mathrm{OPT} \geq \max(h_{\max}, A(I)) = b/2$. □

It is well-known that a strip packing algorithm with an approximation ratio of δ directly yields a rectangle packing algorithm with an approximation ratio of 2δ. To see this, cut the strip packing of height h into slices of height 1 so as to get $\lceil h \rceil$ bins of the required size. The rectangles that are split between two bins can be packed into $\lfloor h \rfloor$ additional bins. The strip packing gives a lower bound for rectangle packing. Thus if $h \leq \delta\ \mathrm{OPT}_{\mathrm{strip}}$, then $\lceil h \rceil + \lfloor h \rfloor \leq 2\delta \mathrm{OPT}_{\mathrm{bin}}$. Accordingly, we get the following theorem.

Theorem 3. *Steinberg's algorithm yields an absolute 4-approximation algorithm for rectangle packing with rotations.*

Jansen & Zhang [8] showed a corollary of Steinberg's theorem which reads as follows if $w_i \geq h_i$ for all items.

Corollary 2 ([8]). *If the total area of a set* T *of items is at most* $1/2$ *and there is at most one item of height* $h_i > 1/2$*, then the items of* T *can be packed into a bin of unit size in time* $\mathcal{O}((n \log^2 n)/\log\log n)$*.*

The NEXT-FIT-DECREASING-HEIGHT algorithm (NFDH) was introduced for squares by Meir & Moser [11] and generalized to rectangles by Coffman, Garey, Johnson & Tarjan [5]. It is given as follows. Sort the items by non-increasing order of height. Pack the items one by one into shelfs. The height of a shelf is defined by its first item, further items are added left-aligned until an item does not fit. In this case this item opens a new shelf. The algorithm stops if it runs out of items or a new shelf does not fit into the designated area. The running time of the algorithm is $\mathcal{O}(n \log n)$. The following lemma is an easy generalization of the result from Meir & Moser.

Lemma 1. *If a given set* T *of items is packed into a rectangle* $R = (a, b)$ *by* NFDH*, then either a total area of at least* $(a - w_{\max})(b - h_{\max})$ *is packed or the algorithm runs out of items, i.e., all items are packed.*

3 Our Algorithm: Overview

As the asymptotic approximation ratio of the algorithm from Bansal et al. [1] is less than 2, there exists a constant k such that for any instance with optimal value larger than k, the asymptotic algorithm gives a solution of value at most 2OPT. We address the problem of approximating rectangle packing with rotations within an absolute factor of 2, provided that the optimal value of the given instance is less than k. Combined with the algorithm from [1] we get an overall algorithm with an absolute approximation ratio of 2.

We begin by applying the asymptotic algorithm from [1]. If $\mathrm{OPT} > k$, then the algorithm outputs a solution of value $k' \leq 2\mathrm{OPT}$. Otherwise $\mathrm{OPT} \leq k$ and we apply the following algorithm.

Let $\varepsilon := 1/68$. We separate the given input according to the area of the items, so we get a set of large items $L = \{r_i \in I \mid w_i h_i \geq \varepsilon\}$ and a set of small items $S = \{r_i \in I \mid w_i h_i < \varepsilon\}$. Since the number of large items in each bin is bounded by $1/\varepsilon$ and their total area is at most k, we can enumerate all possible packings of the large items. Take an arbitrary packing of the large items into a minimum number $\ell \leq k$ of bins.

If there are bins that contain items with a total area less than $1/2 - \varepsilon$, we greedily add small items such that the total area of items assigned to each of these bins is in $(1/2 - \varepsilon, 1/2]$. We use Corollary 2 to repack these bins including the newly assigned small items. There is at most one item of height $h_i > 1/2$ since otherwise the total area exceeds $1/2$, because $w_i \geq h_i$. If we run out of items in this step, we found an optimal solution. Assume that there are still small items left and each bin used so far contains items of a total area of at least $1/2 - \varepsilon$. The following crucial lemma shows that we can pack the remaining small items well enough to achieve an absolute approximation ratio of 2.

Lemma 2. *Let $0 < \varepsilon \leq 1/68$. Given a set T of items that all have area at most ε such that for all $r \in T$ the total area of $T \setminus \{r\}$ is less than $1/2 + \varepsilon$. We can find a packing of T into a unit bin in time $\mathcal{O}((n \log^2 n)/\log\log n)$.*

The lemma is proved in the next section. To apply Lemma 2 we consider the following partition of the remaining items.

Let $r_1, \ldots, r_m$ be the list of remaining small items, sorted by non-increasing order of size. Partition these small items into sets $S_1 = \{r_{t_1}, \ldots, r_{t_2 - 1}\}, S_2 = \{r_{t_2}, \ldots, r_{t_3 - 1}\}, \ldots, S_s = \{r_{t_s}, \ldots, r_{t_{s+1} - 1}\}$ with $t_1 = 1$ and $t_{s+1} = m + 1$ such that

$$A(S_j \setminus \{r_{t_{j+1}-1}\}) < \frac{1}{2} + \varepsilon \qquad \text{and} \qquad A(S_j) \geq \frac{1}{2} + \varepsilon$$

for $j = 1, \ldots, s - 1$. Obviously, each set S_i satisfies the precondition of Lemma 2 and can therefore be packed into a single bin. Only S_s might have a total area of less than $1/2 + \varepsilon$. The overall algorithm is given in Algorithm 1.

Note that if no packing of L into at most k bins exists, then $\mathrm{OPT} \geq k$ and thus $k' \leq 2\mathrm{OPT}$ by definition of k.

Algorithm 1 Approximate rectangle packing with rotations

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apply the asymptotic algorithm from [1] to derive a packing $P'$ into $k'$ bins
if $k' \geq 2k$ then
    return $P'$
else
    let $\varepsilon := 1/68$
    partition $I$ into $L = \{r_i \in I \mid w_i h_i \geq \varepsilon\}$ and $S = \{r_i \in I \mid w_i h_i < \varepsilon\}$
    if $L$ cannot be packed in $k$ or less bins then
        return $P'$
    else
        let $P_\ell$ be a packing of $L$ into $\ell \leq k$ bins.
        while there exists a bin containing items of total area $< 1/2 - \varepsilon$ do
            assign small items to this bin until the total area exceeds $1/2 - \varepsilon$
            use Steinberg's algorithm (Corollary 2) to repack the bin
        order the remaining small items by non-increasing size
        greedily partition the remaining items into sets $S_1, \ldots, S_s$ such that
            $A(S_j \setminus \{r_{t_{j+1}-1}\}) < \frac{1}{2} + \varepsilon$  and  $A(S_j) \geq \frac{1}{2} + \varepsilon$  for $j = 1, \ldots, s-1$
        use the method described in the proof of Lemma 2 to pack each set $S_i$
        into a bin
        let $P$ be the resulting packing into $\ell + s$ bins
return the packing from $P, P'$ that uses the least amount of bins
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4 Packing sets of small items

In this section we prove Lemma 2. We will use the following partition of a set T of items of area at most ε in the remainder of this section. Let

$$T_1 := \{r_i \in T \mid 2/3 < w_i\} \qquad T_2 := \{r_i \in T \mid 1/2 < w_i \leq 2/3\}$$
$$T_3 := \{r_i \in T \mid 1/3 < w_i \leq 1/2\} \qquad T_4 := \{r_i \in T \mid w_i \leq 1/3\}.$$

Since $w_i h_i \leq \varepsilon$ and $w_i \geq h_i$, the heights of the items in each set are bounded as follows.

$$\begin{array}{llll} h_i \leq 3/2 \cdot \varepsilon & \text{for } r_i \in T_1, & h_i \leq 2 \cdot \varepsilon & \text{for } r_i \in T_2, \\ h_i \leq 3 \cdot \varepsilon & \text{for } r_i \in T_3 \text{ and} & h_i \leq \sqrt{\varepsilon} & \text{for } r_i \in T_4. \end{array}$$

It turns out that packing the items in T_2 involves the most difficulties. We will therefore consider different cases for packing items in T_2, according to the total height of these items. For all cases we need to pack $T_1 \cup T_3 \cup T_4$ afterwards, using the following lemma.

Lemma 3. *Given a rectangle $R = (1, h)$ and a set T of items that all have area at most ε such that $T_2 = \emptyset$. We can find a packing of a selection $T' \subseteq T$ into R in time $\mathcal{O}(n \log n)$ such that $T' = T$ or*

$$A(T') \geq \frac{2}{3}(h - \sqrt{\varepsilon}) - \varepsilon.$$

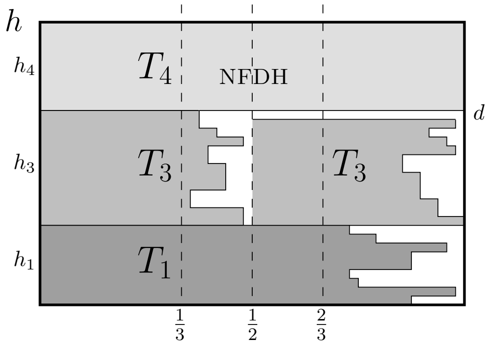

Fig. 2. Packing the sets T_1, T_3 and T_4 into a bin of width 1 and height h. The difference in height between the stacks of T_3 is denoted by d.

Proof. See Figure 2 for an illustration of the following packing. Stack the items of T_1 left-justified into the lower left corner of R. Stop if there is not sufficient space to accomodate the next item. In this case a total area of at least $A(T_1') \geq 2/3\,(h - 3/2 \cdot \varepsilon)$ is packed since $w_i > 2/3$ and $h_i \leq 3/2 \cdot \varepsilon$ for items in T_1.

Thus assume all items from T_1 are packed. Denote the height of the stack by h_1. Obviously, $A(T_1) \geq 2/3 \cdot h_1$.

Create two stacks of items from T_3 next to each other directly above the stack for T_1 by repeatedly assigning each item to the lower stack. Stop if an item does not fit into the rectangle. In this case both stacks have a height of at least $h - h_1 - 3\varepsilon$ as otherwise a further item could be packed. Therefore $A(T_1 \cup T_3') \geq 2/3(h - 3\varepsilon) \geq 2/3 \cdot (h - \sqrt{\varepsilon})$ since $3\varepsilon \leq \sqrt{\varepsilon}$ for $\varepsilon \leq 1/18$.

Otherwise denote the height of the higher stack by h_3 and the height difference by d. The total area of T_3 is at least $A(T_3) \geq 2/3(h_3 - d) + 1/3 \cdot d \geq 2/3 \cdot h_3 - 1/3 \cdot d \geq 2/3 \cdot h_3 - \varepsilon$ since $w_i \geq 1/3$ and $h_i \leq 3\varepsilon$ for $r_i \in T_3$.

Finally, let $h_4 := h - h_1 - h_3$ and add the items of T_4 by NFDH into the remaining rectangle of size $(1, h_4)$. Lemma 1 yields that either all items are packed, i.e., $T' = T$, or items $T_4' \subseteq T_4$ of total area at least $A(T_4') \geq 2/3\,(h_4 - \sqrt{\varepsilon})$ are packed. Thus the total area of the packed items T' is $A(T') \geq 2/3 \cdot h_1 + 2/3 \cdot h_3 - \varepsilon + 2/3(h_4 - \sqrt{\varepsilon}) \geq 2/3\,(h - \sqrt{\varepsilon}) - \varepsilon$.

The running time is dominated by the application of NFDH. □

If $T_4 = \emptyset$ then the last packing step is obsolete and the analysis above yields the following corollary.

Corollary 3. *Given a rectangle $R = (1, h)$ and a set T of items that all have area at most ε such that $T_2 \cup T_4 = \emptyset$. We can find a packing of a selection $T' \subseteq T$ into R in time $\mathcal{O}(n)$ such that $T' = T$ or*

$$A(T') \geq \frac{2}{3}h - 2\varepsilon.$$

The above packings are very efficient if there are no items of width within 1/2 and 2/3 as they essentially yield a width guarantee of 2/3 for the whole height, except for some wasted height that is suitably bounded. In order to pack items of T_2, we have to consider both possible orientations to achieve a total area of more than $1/2$ in a packing. The following main lemma shows how sets of items including items of width within $1/2$ and $2/3$ are being processed.

Lemma 2. *Let* $0 < \varepsilon \leq 1/68$. *Given a set* T *of items that all have area at most* ε *such that for all* $r \in T$ *the total area of* $T \setminus \{r\}$ *is less than* $1/2 + \varepsilon$. *We can find a packing of* T *into a unit bin in time* $\mathcal{O}((n \log^2 n)/\log\log n)$.

Proof. Let h_2 be the total height of items in T_2. We present three methods for packing T depending on h_2. For each method we give a lower bound on the total area of items that are packed. Afterwards we show that there cannot be any items that remain unpacked. Throughout the proof, we assume that we do not run out of items while packing the items in T. This will eventually lead to a contradiction in all three cases.

Case 1: $h_2 \leq 1/3$
Stack the items of T_2 left-justified into the lower left corner of the bin. Use Lemma 3 to pack $T_1 \cup T_3 \cup T_4$ into the rectangle $(1, 1 - h_2)$ above the stack–see Figure 3. We get an overall packed area of

$$\begin{aligned} A \geq \frac{h_2}{2} + \frac{2}{3}\left(1 - h_2 - \sqrt{\varepsilon}\right) - \varepsilon &= \frac{2}{3} - \frac{h_2}{6} - \varepsilon - \frac{2}{3}\sqrt{\varepsilon} \\ &\geq \frac{11}{18} - \varepsilon - \frac{2}{3}\sqrt{\varepsilon} \qquad \text{(since } h_2 \leq \frac{1}{3}\text{)}. \end{aligned}$$

Case 2: $h_2 \in (1/3, 2/3]$
Stack the items of T_2 left-justified into the lower left corner of the bin. Let $B = (1/3, h_2)$ be the free space to the right of the stack. We are going to pack items from $X = \{r_i \in T_3 \cup T_4 \mid w_i \leq h_2\}$ into B. Take an item from X and add it to X' as long as X is nonempty and $A(X') \leq h_2/6 - \varepsilon$. Rotate the items in X' and use Steinberg's algorithm (Corollary 1) to pack them into B. This is possible since the area of B is $h_2/3$, $A(X') \leq h_2/6$, and $h_i \leq h_2$ and $w_i \leq \sqrt{\varepsilon} \leq 1/6$ for $r_i \in X'$ (w_i and h_i are the rotated lengths of r_i). Use Lemma 3 to pack $(T_1 \cup T_3 \cup T_4) \setminus X'$ into the rectangle $(1, 1 - h_2)$ above the stack–see Figure 3. We distinguish two cases. If $A(X') \geq h_2/6 - \varepsilon$, then

$$A \geq \overbrace{\frac{h_2}{2}}^{T_2} + \overbrace{\frac{h_2}{6} - \varepsilon}^{X'} + \overbrace{\frac{2}{3}\left(1 - h_2 - \sqrt{\varepsilon}\right) - \varepsilon}^{(T_1 \cup T_3 \cup T_4) \setminus X'} = \frac{2}{3} - 2\varepsilon - \frac{2}{3}\sqrt{\varepsilon}.$$

Otherwise $A(X') < h_2/6 - \varepsilon$ and since no further item was added to X' we have $X' = X$. As $h_2 > 1/3$ we have $T_4 \subseteq X$ and we can apply Corollary 3 to get a total area of

$$\begin{aligned} A \geq \frac{h_2}{2} + \frac{2}{3}(1 - h_2) - 2\varepsilon &= \frac{2}{3} - \frac{h_2}{6} - 2\varepsilon \\ &\geq \frac{5}{9} - 2\varepsilon \qquad \text{(since } h_2 \leq \frac{2}{3}\text{)}. \end{aligned}$$

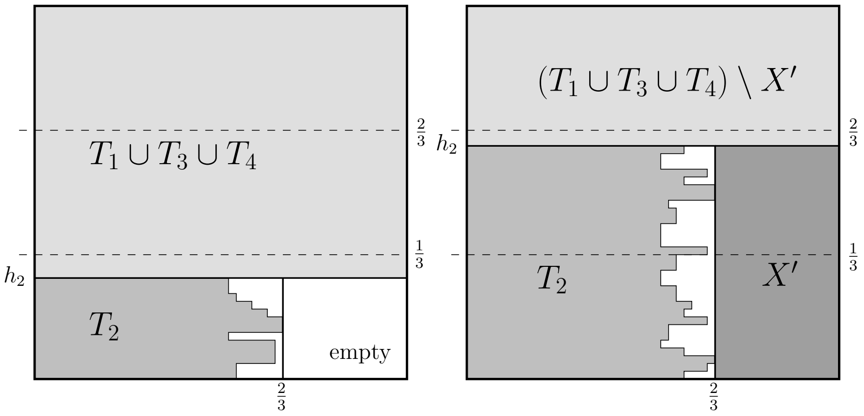

Fig. 3. Packing in Case 1 ($h_2 \leq 1/3$) and Case 2 ($1/3 < h_2 \leq 2/3$)

Case 3: $h_2 \in (2/3, 1 + 4\varepsilon]$
See Figure 4 for an illustration of the following packing and the notations. Order the items of T_2 by non-increasing order of width. Stack the items left-justified into the lower left corner of the bin while the current height h is less or equal to the width of the last item that was packed. In other words, the top right corner of the last item of this stack is above the line from $(1/2, 1/2)$ to $(2/3, 2/3)$, whereas the top right corners of all other items in the stack are below this line. Denote the height of the stack by h and the set of items that is packed into this stack by X_1. Let $r' = (w', h')$ be the last item on the stack. Clearly, $w_i \leq h$ for all items $r_i \in T_2 \setminus \ X_1$.

Consider the free space $B = (1/3, h)$ to the right of the stack. Rotate the items in $T_2 \setminus X_1$ and stack them horizontally, bottom-aligned into B. Stop if an item does not fit. We denote the items that are packed into B by X_2. Rotate the remaining items $T_2 \setminus (X_1 \cup X_2)$ back into their original orientation and stack them on top of the first stack X_1. Let this set of items be X_3 and the total height of the stack $X_1 \cup X_3$ be $\widehat{h}$. Use Lemma 3 to pack $T_1 \cup T_3 \cup T_4$ into the rectangle $(1, 1 - \widehat{h})$ above the stack $X_1 \cup X_3$.

Since $w_i \geq (h - h')$ for $r_i \in X_1 \setminus \{r'\}$ we have $A(X_1) \geq (h - h')^2 + h'/2$. Again we distinguish two cases for the analysis. If $X_3 = \emptyset$ (or equivalently $\widehat{h} \;= h$), then $A(X_2) \geq (h_2 - h)/2$ and therefore

$$\begin{aligned} A &\geq \overbrace{(h - h')^2 + \frac{h'}{2}}^{X_1} + \overbrace{\frac{h_2 - h}{2}}^{X_2} + \overbrace{\frac{2}{3}\left(1 - h - \sqrt{\varepsilon}\right) - \varepsilon}^{T_1 \cup T_3 \cup T_4} \\ &> (h - h')^2 + \frac{h'}{2} + \frac{1}{3} - \frac{h}{2} + \frac{2}{3}\left(1 - h - \sqrt{\varepsilon}\right) - \varepsilon =: A_1 \qquad (\text{since} \quad h_2 > \frac{2}{3}). \end{aligned}$$

To find a lower bound for the total packed area we consider the partial derivative of A_1 to h', which is $\frac{\partial A_1}{\partial h'} = 2h' - 2h + 1/2$. Since $2h' - 2h + 1/2 < 0$ for $h' \leq 2\varepsilon$ and $h \geq 1/2$, the total packed area is minimized for the maximal

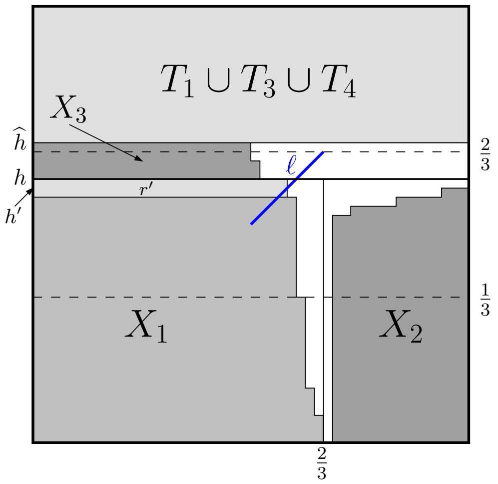

Fig. 4. Packing in Case 3 ($2/3 < h_2 \leq 1 + 4\varepsilon$). Item r' of height h' is dipicted larger than $\varepsilon \leq 1/68$ for the sake of visibility. The diagonal line ℓ shows the threshold at which the stack X_1 is discontinued.

value $h' = 2\varepsilon$ for any h in the domain. After inserting this value for h' we get $A_1 = (h - 2\varepsilon)^2 + \varepsilon + 1/3 - h/2 + 2/3(1 - h - \sqrt{\varepsilon}) - \varepsilon$ and $\frac{\partial A_1}{\partial h} = 2h - 7/6 - 4\varepsilon$. Thus the minimum is acquired for $h = 7/12 + 2\varepsilon$. We get

$$\begin{aligned} A_1 &\geq \left(\frac{7}{12}\right)^2 + \varepsilon + \frac{1}{3} - \frac{7}{24} - \varepsilon + \frac{2}{3}\left(\frac{5}{12} - 2\varepsilon - \sqrt{\varepsilon}\right) - \varepsilon \\ &= \frac{95}{144} - \frac{7}{3}\varepsilon - \frac{2}{3}\sqrt{\varepsilon}. \end{aligned}$$

Otherwise $X_3 \neq \emptyset$ (or equivalently $\widehat{h} > h$) and thus $A(X_2) \geq 1/2(1/3 - 2\varepsilon)$ as the stack X_2 leaves at most a width of 2ε of B unpacked. Furthermore, $\widehat{h} \leq 2/3 + 6\varepsilon$ since $h_2 \leq 1 + 4\varepsilon$ and a width of at least $1/3 - 2\varepsilon$ is packed into B. Since $A(X_3) \geq (\widehat{h} - h)/2$ and $\widehat{h} \leq 2/3 + 6\varepsilon$ we get

$$\begin{aligned} A &\geq \overbrace{(h - h')^2 + \frac{h'}{2}}^{X_1} + \overbrace{\frac{1}{2}\left(\frac{1}{3} - 2\varepsilon\right)}^{X_2} + \overbrace{\frac{\widehat{h} - h}{2}}^{X_3} + \overbrace{\frac{2}{3}\left(1 - \widehat{h} - \sqrt{\varepsilon}\right) - \varepsilon}^{T_1 \cup T_3 \cup T_4} \\ &\geq (h - h')^2 + \frac{h'}{2} + \frac{1}{2}\left(\frac{1}{3} - 2\varepsilon\right) - \frac{1}{9} - \varepsilon - \frac{h}{2} + \frac{2}{3}\left(1 - \sqrt{\varepsilon}\right) - \varepsilon := A_2. \end{aligned}$$

Again we consider the partial derivatives of A_2 to h and h', which are $\frac{\partial A_2}{\partial h} = 2h - 2h' - 1/2$ and $\frac{\partial A_2}{\partial h'} = 2h' - 2h + 1/2$. Since $h \geq 1/2$ and $h' \leq 2\varepsilon$, the first derivative is positive and the second is negative on the domain that we are interested in. Taking $h = 1/2$ and $h' = 2\varepsilon$, we find

$$\begin{aligned} A_2 &\geq \left(\frac{1}{2} - 2\varepsilon\right)^2 + \varepsilon + \frac{1}{6} - \varepsilon - \frac{1}{9} - \varepsilon - \frac{1}{4} + \frac{2}{3}\left(1 - \sqrt{\varepsilon}\right) - \varepsilon \\ &\geq \frac{13}{18} + 4\varepsilon^2 - 4\varepsilon - \frac{2}{3}\sqrt{\varepsilon}. \end{aligned}$$

If $h_2 > 1 + 4\varepsilon$ then $A(T_2) \geq 1/2 \cdot h_2 > 1/2 + 2\varepsilon$, which is a contradiction to the assumption of the lemma.Therefore the three cases cover all possibilities.

It is easy to verify that for $0 < \varepsilon \leq 1/68$ the following inequalities hold.

$$\begin{aligned} 11/18 - \varepsilon - 2/3\sqrt{\varepsilon} &\geq 1/2 + \varepsilon & 2/3 - 2\varepsilon - 2/3\sqrt{\varepsilon} &\geq 1/2 + \varepsilon \\ 5/9 - 2\varepsilon &\geq 1/2 + \varepsilon & 95/144 - 7/3\varepsilon - 2/3\sqrt{\varepsilon} &\geq 1/2 + \varepsilon \\ 13/18 + 4\varepsilon^2 - 4\varepsilon - 2/3\sqrt{\varepsilon} &\geq 1/2 + \varepsilon \end{aligned}$$

Now let us assume that we do not run out of items while packing a set T with the appropriate method above. Then the packed area is at least $1/2 + \varepsilon$ as the inequalities above show. The contradiction follows from the precondition that removing an arbitrary item form T yields a remaining total area of less than $1/2 + \varepsilon$. Thus all items are packed. □

5 The Approximation Ratio

Theorem 4. *There is an approximation algorithm for rectangle packing with rotations with an absolute worst case ratio of 2.*

Proof. Recall that we denote the number of bins used for an optimal packing of the large items by ℓ. Obviously $\ell \leq \mathrm{OPT}$. Let s be the number of bins used for packing only small items. If $s \leq \ell$, then the total number of bins is $\ell + s \leq 2\ell \leq 2\mathrm{OPT}$. If $s > \ell$, then at least one bin is used for small items and thus all bins for large items contain items with a total area of at least $1/2 - \varepsilon$. According to the partition of the remaining small items, all but the last bin for the small items contain items with a total area of at least $1/2 + \varepsilon$. Let A be the total area of all items and let $f > 0$ be the area of the items contained in the last bin. Then

$$\mathrm{OPT} \geq A \geq \ell \cdot \left(\frac{1}{2} - \varepsilon\right) + (s-1) \cdot \left(\frac{1}{2} + \varepsilon\right) + f > (s + \ell - 1) \cdot \frac{1}{2}.$$

Thus $s + \ell < 2\mathrm{OPT} + 1$ and we get $s + \ell \leq 2\mathrm{OPT}$ which proves the theorem. □

6 Conclusion and Future Work

The algorithm we presented depends on the asymptotic approximation algorithm from [1], in particular, the constant k that follows from this algorithm. It would be interesting to design an approximation algorithm for rectangle packing with rotations with asymptotic approximation ratio strictly less than 2 and small additive term. This could also improve the efficiency of our algorithm.

We conjecture that every set of items of height at most $1/2$ and total area at most $5/9$ can be packed into a unit bin using rotations. This would again improve the efficiency of our algorithm and might be useful for other packing problems as well. Other interesting open questions for further investigation include the following.

1. Does an approximation algorithm for rectangle packing *without rotations* with an absolute worst case ratio of 2 exist? As we pointed out in the introduction, the best-known approximation ratio for this problem is 3 [15].
2. Does an approximation algorithm for strip packing with or without rotations with an absolute worst case ratio less than 2 exist? An answer to this question for strip packing without rotations would narrow the gap between the lower bound of $3/2$ (as strip packing without rotations is a generalization of one-dimensional bin packing) and the upper bound of 2 from Steinberg's algorithm [13].

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