Problem Set 4

Approximation Algorithms

WS 2008/09

S. Angelopoulos, N. Megow, and R. Raman

Problem 1. **1 point**

Let A be an $n \times n$ symmetric matrix. Then prove that the following are equivalent.

1. $x^T Ax \geq 0 \ \ \forall x \in \mathbb{R}^n$
2. All the eigenvalues of A are non-negative.
3. There exists a matrix B, such that $A = B^T B$.

Problem 2. **2 points**

Let $A, B \succeq 0$. Show that $A \bullet B = 0 \Leftrightarrow AB = 0$, where $A \bullet B$ is the matrix inner product: $A \bullet B = \sum_{i,j} A_{ij} B_{ij}$. (**Hint:** Use the fact that A can be decomposed as $\text{Q}\Lambda Q^T$, where Q is an orthogonal matrix of eigenvectors, and Λ is the diagonal matrix of eigenvalues.)

Problem 3. **1 point**

A semidefinite program in standard form is defined as follows.

$$\begin{array}{lll} \min & C \bullet X & \\ \text{s.t.} & & \\ & A_i \bullet X & = b_i, \quad i = 1, \cdots, m. \\ & X & \succeq 0 \end{array}$$

1. Using the previous exercise, show how to derive the dual of a semidefinite program.
2. Prove weak duality.
3. Derive the dual of the SDP for Max-Cut.

Problem 4. **1 point**

A *unit distance representation* of a graph $G = (V, E)$ is a mapping $u : V \to \mathbb{R}^d$, for some $d \geq 1$ such that $|u_i - u_j| = 1 \ \forall \{i, j\} \in E$. Consider the problem of determining the smallest radius R of a sphere that contains a unit distance representation of G. Give a semidefinite programming formuation for this problem.

Problem 5. **1 point**

Show that for C_5 the integrality for MAX-CUT is ≈ 0.884

Problem 6. **2 point**

A hypergraph $G = (V, E)$ is k-uniform if all hyperedges contain exactly 3 vertices. A 2-coloring of a 3-uniform hypergraph is an assignment of 2 colors to V such that no hyperedge $\{a, b, c\}$ is monochromatic. The objective is to maximize the number of non-monochromatic edges.

1. Show that this problem can be modeled as an SDP.
2. Show how to round this SDP using the Goemans-Williamson rounding procedure.

Problem 7. **1 point**

Give an approximation algorithm to color a 4-colorable graph using $O(n^{2/3})$ colors. (**Hint:** Use an extension of Wigderson's technique).

Problem 8. **1 point**

Show that the vector chromatic number of a graph lies between the clique and chromatic numbers.