Problem Set 4

Discrete Geometry

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Problem 1. **10 points**

1. Prove the Centerpoint theorem using Tverberg's theorem.
2. Give a proof of Radon's theorem in three dimensions by case analysis.

Problem 2. **10 points**

Tverberg's theorem says that a set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into r subsets so that the convex hulls of the subsets have a common intersection. Prove the following weaker version using the centerpoint theorem: a set of $r(d+1)^2$ points in $\mathbb{R}^d$ can be partitioned into r subsets whose convex hulls have a common intersection.

Problem 3. **20 points**

An equivalent formulation of the Borsuk-Ulam theorem is the following:

Theorem 1. Let $F_1, \ldots, F_{d+1}$ be $d+1$ closed sets that cover the ball B^n in d-dimensions. Then, there is some set F_i that contains a pair of antipodal points. i.e., $F_i \cap -F_i \neq \emptyset$.

A similar theorem equivalent to Brouwer's fixed point theorem is the KKM lemma.
Prove the KKM Lemma using the above theorem.

Problem 4. **10 points**

1. Let R and B be a set of n red, and n blue points respectively in the plane. Show that we can find n non-intersecting line segments such that each line segment has one red and one blue end-point (and no red or blue points in the interior).
2. Let $S_1, \ldots, S_d$ be d sets in R^d, each with n points in general position. Show that $S_1 \cup S_2 \cup \ldots \cup S_d$ can be partitioned into d-tuples, where each d-tuple contains exactly one point from each A_i, $i = 1, \ldots, d$ such that the convex hulls of the d-tuples are disjoint.