Improved Results on Geometric Hitting Set Problems

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Abstract

We consider the problem of computing minimum geometric hitting sets in which, given a set of geometric objects and a set of points, the goal is to compute the smallest subset of points that hit all geometric objects. The problem is known to be strongly NP-hard even for simple geometric objects like unit disks in the plane. Therefore, unless P=NP, it is not possible to get Fully Polynomial Time Approximation Algorithms (FPTAS) for such problems. We give the first PTAS for this problem when the geometric objects are half-spaces in $\mathbb{R}^3$ and when they are an r-admissible set regions in the plane (this includes pseudo-disks as they are 2-admissible). Quite surprisingly, our algorithm is a very simple local search algorithm which iterates over local improvements only.

1 Introduction

In the minimum hitting set problem, we are given a range space $\mathcal{R} = (P, \mathcal{D})$ consisting of a set P and a set $\mathcal{D}$ of subsets of P called the *ranges*, and the task is to compute the smallest subset $Y \subseteq P$ that has a non-empty intersection with each of the ranges in $\mathcal{D}$. This problem is equivalent to the set cover problem and is strongly NP-hard. If there are no restrictions on the set system $\mathcal{R}$, then it is known that it is NP-hard to approximate the minimum hitting within a factor of $c \log n$ of the optimal [17]. The problem is even NP-complete for the case where each point of P lies in at most two sets of $\mathcal{R}$ [6].

A natural occurrence of the hitting set problem occurs when the range space $\mathcal{R}$ is derived from geometry. For example, given a set P of n points in $\mathbb{R}^2$, and a set $\mathcal{D}$ of m convex polygons containing points of P, compute the minimum-sized subset of P that hits all the polygons in $\mathcal{D}$. Unfortunately, for many geometric range spaces, computing the minimum-sized hitting set remains NP-hard. For example, even the (relatively) simple case where $\mathcal{D}$ is a set of unit disks in the plane is strongly NP-hard [8]. In this paper, we will only be concerned with set systems where P is a set of points, and the ranges in $\mathcal{D}$ are induced by various geometric objects.

Since there is little hope of computing the minimum-sized hitting set for general geometric problems in polynomial time, effort has turned to approximating the optimal solution. In this regard, an interesting connection to the ϵ-net problem was made by Bronnimann and Goodrich [1]. Briefly, given a range space $(P, \mathcal{D})$, an ϵ-net is a subset $S \subseteq P$ such that $D \cap S \neq \emptyset$ for all $D \in \mathcal{D}$ with $|D| \geq \epsilon n$. The famous "ϵ-net theorem" of Haussler and Welzl [7] states that for range spaces with VC-dimension d, there exists an ϵ-net

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of size $O(d/\epsilon \log d/\epsilon)$ (this bound was later improved to $O(d/\epsilon \log 1/\epsilon)$, which was shown to be optimal in general. See [14, 10]). Sometimes, weighted versions of the problem are considered in which each $p \in P$ has some positive weight associated with it so that the total weight of all elements of P is 1. The weight of each range is the sum of the weights of the elements in it. The aim is to hit all ranges with weight more than ϵ. The condition of having finite VC-dimension is satisfied by many geometric set systems: disks, half-spaces, k-sided polytopes, r-admissible set of regions etc. in $\mathbb{R}^d$. However, for certain range spaces, one can even show the existence of ϵ-nets of size $O(1/\epsilon)$: half-spaces in $\mathbb{R}^3$ [11], pseudo-disks in $\mathbb{R}^2$ [16], unit cubes in $\mathbb{R}^3$ [4], and so on.

In 1994, Bronnimann and Goodrich [1] proved the following[1]: let $\mathcal{R} = (P, \mathcal{D})$ be a range-space for which we want to compute a minimum hitting set. If we can compute an ϵ-net of size c/ϵ for the weighted ϵ-net problem for $\mathcal{R}$ in polynomial time then we can compute a hitting set of size at most $c \cdot \text{OPT}$ for $\mathcal{R}$, where OPT is the size of the optimal (smallest) hitting set, in polynomial time. A shorter, simpler proof was given by Even *et al.* [5].

This connection between ϵ-nets and computing hitting sets implies that for the ranges mentioned above with $O(1/\epsilon)$-sized nets, there exist polynomial-time, constant-factor approximation algorithms for the corresponding hitting set problems. The constant in the approximation then depends on the constant in the size of the ϵ-nets, which are typically quite large. So for example, for $(P, \mathcal{H})$, where $\mathcal{H}$ is the set of half-spaces in $\mathbb{R}^3$, the current best size of ϵ-net is at least $20/\epsilon$ [16], yielding at best a 20-approximation factor. Furthermore, this is a fundamental limitation of the technique: it *cannot* give better than constant-factor approximations. The reason is the following: the technique reduces the problem of computing a minimum size hitting set to the problem of computing the minimum size ϵ-net and then uses a constant-factor approximation for the latter problem. It uses the fact that an ϵ-net of size c/ϵ can always be computed and that $1/\epsilon$ is a lower bound on the size of the ϵ-net to get the constant factor approximation. The Bronnimann-Goodrich technique therefore cannot give a PTAS even for relatively simple hitting set problems.

1.1 Our Contributions

$(1+\epsilon)$-approximations via local search. We present a new general technique for approximating geometric hitting sets that avoids the limitation of the Bronnimann-Goodrich technique: we give the first polynomial-time approximation schemes for the minimum geometric hitting set problem for a wide class of geometric range spaces. All these problems are strongly NP-complete and hence, unless P=NP, there is no FPTAS for these problem. Specifically, we show that:

- Given a set P of n points, and a set $\mathcal{H}$ of m half-spaces in $\mathbb{R}^3$, one can compute a $(1+\epsilon)$-approximation to the smallest subset of P that hits all the half-spaces in $\mathcal{H}$ in $O(mn^{O(\epsilon^{-2})})$ time.
- Given a set P of n points in $\mathbb{R}^2$, and a set of r-admissible regions $\mathcal{D}$, one can compute a $(1+\epsilon)$-approximation to the smallest subset of P that hits all the regions in $\mathcal{D}$ in $O(mn^{O(\epsilon^{-2})})$ time. This includes pseudo-disks (they are 2-admissible), same-height rectangles, circular disks, translates of convex objects etc. See Definition 2.1 for the definition of an r-admissible set of regions.

The above results should be contrasted with the fact that even for relatively simple range spaces like those induced by unit disks in the plane, the previous best known approximation algorithm is due to a recent paper

[1]They actually proved a more general statement, but the following is more relevant for our purposes.

of Carmi *et. al.* [2] which gives a 38-approximation algorithm improving the earlier best known factor of 72 [13].

Our algorithm for both the problems is the following simple local search algorithm: start with any hitting set $S \subseteq P$ (e.g., take all the points of P), and iterate local-improvement steps of the following kind: If any k points of S can be replaced by $k-1$ points of P such that the resulting set is still a hitting set, then perform the swap to get a smaller hitting set. Halt if no such local improvement is possible. We will call this a *k-level local search algorithm*. Then our main result is the following:

Theorem 1.1. *Let P be a set of n points in $\mathbb{R}^3$ (resp. $\mathbb{R}^2$), and let $\mathcal{H}$ (resp. $\mathcal{D}$) be the geometric objects as above. Then a $(c/\epsilon)^2$-level local search algorithm returns a hitting set of size at most $(1+\epsilon) \cdot \textsc{opt}$, where $\textsc{opt}$ is the size of an optimal (smallest) hitting set.*

Note that, for any fixed k, the naive implementation of the k-level local search algorithm takes polynomial time: start the algorithm with the entire set P as (the most likely sub-optimal) hitting set P'. The size of P' decreases by at least one at each local-improvement step. Hence, there can be at most n steps of local improvement, where there are at most $\binom{n}{k} \cdot \binom{n}{k-1} \leq n^{2k-1}$ different local improvements to verify. Checking whether a certain local improvement is possible takes $O(nm)$ time. Hence the overall running time of the algorithm is $O(mn^{2k+1})$. By using data-structuring techniques, this bound can be improved by polynomial factors; however that is not the goal of this paper.

Combinatorial bounds on ϵ-nets via Local Search. As a side result, we show that the local search technique can also be used to prove the existence of small size ϵ-nets. Specifically, we show that for the case where we have points in the plane and ranges consist unit squares in the plane, a simple local-search method gives the optimal bound of $O(1/\epsilon)$ for the size of the ϵ-net. It is quite easy to prove the same result using other techniques but it is interesting that the local search technique can be used to prove this. This kind of result is currently known only for half-spaces in $\mathbb{R}^2$ and is implied by the proof of the existence of $O(1/\epsilon)$ size ϵ-nets by Pach and Woeginger [15]. It is not at all clear that the same holds for half-spaces in $\mathbb{R}^3$. We conjecture that this holds for more general range spaces defined by a set of points and an r-admissible set of regions in the plane – we leave this as an open problem.

Organization. In Section 2 we present the proof of Theorem 1.1. The alternate proof for the existence $O(1/\epsilon)$ size ϵ-nets for unit squares in the plane is given in Section 3.

2 A $(1+\epsilon)$-approximation scheme for hitting geometric sets

Let $\mathcal{R} = (P, \mathcal{D})$ be a range space where P is the *ground set* and $\mathcal{D} \subseteq 2^P$ is the set of ranges. A minimum hitting set for $\mathcal{R}$ is a subset $Q \subseteq P$ of the smallest size such that $Q \cap D \neq \emptyset$, for all $D \in \mathcal{D}$. In this section we will show that given any parameter $\epsilon > 0$, a $O(\epsilon^{-2})$-level local search returns a hitting set whose size is at most $(1+\epsilon)$ times the size of the minimum hitting set for range spaces that satisfy the following *locality condition*.

Locality Condition. A range space $\mathcal{R} = (P, \mathcal{D})$ satisfies the locality condition if for any two disjoint subsets $R, B \subseteq P$, it is possible to construct a planar bipartite graph $G = (R, B, E)$ with all edges going between R and B such that for any $D \in \mathcal{D}$, if $D \cap R \neq \emptyset$ and $D \cap B \neq \emptyset$, then there exist two vertices $u \in D \cap R$ and $v \in D \cap B$ such that $(u, v) \in E$.

For example, if P is a set of points in the plane and $\mathcal{D}$ is defined by intersecting P with a set of circular disks, then $\mathcal{R} = (P, \mathcal{D})$ satisfies the locality condition. To see this consider, for any given R and B, the Delaunay triangulation G of $R \cup B$. Removing the non red-blue edges from the triangulation gives the required bipartite planar graph since for each disk D in the plane, the vertices in $(R \cup B) \cap D$ induce a connected subgraph of G and hence there must be an edge between a vertex in $D \cap R$ and a vertex in $D \cap B$ whenever both the intersections are non-empty.

Let us now return to the hitting set problem. For any vertex v in a graph G, denote by $N_G(v)$ the set of neighbors of v. Similarly, for any subset of the vertices W of G, let $N_G(W)$ denote the set of all neighbors of the vertices in W, i.e., $N_G(W) = \bigcup_{v \in W} N_G(v)$. Our basic theorem is the following:

Theorem 2.1. *Let $\mathcal{R} = (P, \mathcal{D})$ be a range space satisfying the locality condition. Let $R \subseteq P$ be an optimal hitting set for $\mathcal{D}$, and $B \subseteq \mathcal{D}$ be the hitting set returned by a k-level local search. Furthermore, assume $R \cap B = \emptyset$. Then there exists a planar bipartite graph $G = (R, B, E)$ such that for every subset $B' \subseteq B$ of size at most k, $|N_G(B')| \geq |B'|$.*

Proof. Let $\mathcal{R} = (P, \mathcal{D})$ be a range space satisfying the locality condition where P is set of size n and $\mathcal{D}$ is a set of m subsets of P. From now on, we will call R and B the red points and the blue points respectively. Since no local improvement in possible in B, we can conclude that no k blue points can be replaced by $k-1$ or fewer non-blue points. In particular, no k blue points can be replaced by $k-1$ or fewer red points.

Let G be the bipartite planar graph between R and B, given by the locality condition for $\mathcal{R}$. Since both R and B are hitting sets for $\mathcal{R}$, we know that each range in $\mathcal{D}$ has both red and blue points.

Claim 2.2. *For any $B' \subseteq B$, $(B \setminus B') \cup N_G(B')$ is a hitting set for $\mathcal{R}$.*

Proof. If there is range $D \in \mathcal{D}$ which is only hit by the blue points in B', then one of those blue points has a red neighbor that hits D and therefore $N_G(B')$ hits D. Otherwise, D is hit by some point in $B \setminus B'$. □

This finishes the proof, since the above claim implies that if $B' \subseteq B$ is a set of at-most k blue points, then $|N_G(B')| \geq |B'|$ since otherwise a local improvement would be possible in B. □

Note that we can always assume, without loss of generality, that $B \cap R = \emptyset$. If not, let $I = B \cap R$, $P' = P \setminus I$, $B' = B \setminus I$, $R' = R \setminus I$ and let $\mathcal{D}'$ be the set of ranges that are not hit by the points in I. B' and R' are disjoint. Also, R' is a hitting set of minimum size for the hitting set problem with points P' and the ranges in $\mathcal{D}'$. If we can show that $|B'|$ is approximately equal to $|R'|$, we can conclude that $|B|$ is approximately equal to $|R|$.

Now, the following lemma (also proved independently in Har-Peled and Chan [3]) implies that given any parameter ϵ, a k-level local search with $k = c^2\epsilon^{-2}$ gives a $(1+\epsilon)$-approximation to the minimum hitting set problem for $\mathcal{R}$.

Lemma 2.3. *Let $G = (R, B, E)$ be a bipartite planar graph on red and blue vertex sets R and B, $|R| \geq 2$, such that for every subset $B' \subseteq B$ of size at most k, where k is a large enough number, $|N_G(B')| \geq |B'|$. Then $|B| \leq (1 + c/\sqrt{k})\ |R|$, where c is a constant.*

The above lemma follows directly from the following planar graph partition theorem of Koutis and Miller [9].

Theorem 2.4 (Planar graph partition with small boundary size [9])**.** *Given a planar graph H with n vertices and a parameter t, the vertices of H can be divided into groups of size at most t so that the total number of vertices of a group shared with other groups, summed over all groups, is at most $\gamma n/\sqrt{t}$, where γ is a fixed constant.*

Note that some vertices belong to more than one group – these vertices are called *boundary* vertices. Furthermore, each non-boundary vertex has edges only to members of its own group (which could include some boundary vertices).

Proof of Lemma 2.3. Let $r = |R|$ and $b = |B|$. Consider the groups of G formed according to Theorem 2.4 with the parameter $t = k$. Each group has at most k vertices. Consider the i^{th} group and let r_i^∂ and b_i^∂ be the number of red and blue boundary vertices respectively in the group. Similarly, let b_i^{int} and r_i^{int} be the number of red and blue interior (non-boundary) vertices in this group. Theorem 2.4 guarantees that $\sum_i r_i^\partial + b_i^\partial \le \gamma(r+b)/\sqrt{k}$. Since there are at most k interior blue vertices in the group, by the expansion condition of the theorem, their neighborhood must be at least as large as their own number, i.e., $b_i^{int} \le r_i^{int} + r_i^\partial$. Adding b_i^∂ to both sides and summing over all i we have

$$\begin{aligned} b \le \sum_i (b_i^{int} + b_i^\partial) &\le \sum_i r_i^{int} + \sum_i (r_i^\partial + b_i^\partial) \\ &\le r + \gamma(r+b)/\sqrt{k} \end{aligned}$$

Let us assume that $k \ge 4\gamma^2$ and set $c = 4\gamma$. Then,

$$\begin{aligned} b \le r\frac{1+\gamma/\sqrt{k}}{1-\gamma/\sqrt{k}} &= r(1+\gamma/\sqrt{k})(1+(\gamma/\sqrt{k})+(\gamma/\sqrt{k})^2+\cdots) \\ &\le r(1+\gamma/\sqrt{k})(1+2\gamma/\sqrt{k}) \quad \text{(since } \gamma/\sqrt{k} \le 1/2 \text{)} \\ &= r(1+3\gamma/\sqrt{k}+2\gamma^2/k) \\ &\le r(1+4\gamma/\sqrt{k}) \quad \text{(since } 2\gamma^2/k \le \gamma/\sqrt{k} \text{)} \\ &= r(1+c/\sqrt{k}). \end{aligned}$$

□

Remark. In the preliminary version of this paper [12], there is a technical gap in the proof of Lemma 2.3, as there are cases where the graph $G_{B'}$ can be a multi-graph. This has been corrected above by applying Theorem 2.4 to the entire graph, instead of the one induced by the blue vertices.

PTAS for an r-admissible set of regions. It turns out that the locality condition, by a more complicated construction of the planar graph G [16], also holds for an r-admissible set of regions, for any r, in the plane. This yields a PTAS for the minimum hitting set problem with an r-admissible set of regions in the plane. The definition of an r-admissible set of regions is as follows:

Definition 2.1. *A set of regions in $\mathbb{R}^2$, each of which is bounded by a closed Jordan curve, is called r-*admissible *(for r even), if for any two s_1, s_2 of the regions, the Jordan curves bounding them cross in $l \le r$ points, (for some even l), and both $s_1 \setminus s_2$ and $s_2 \setminus s_1$ are connected regions.*

As mentioned earlier, this includes pseudo-disks (they are 2-admissible), same-height rectangles, circular disks, translates of convex objects etc.

PTAS for half-spaces in $\mathbb{R}^3$. Given a set of half-spaces and a set of points in $\mathbb{R}^3$, we first pick one of the points o and add it to our hitting set. We then ignore o and all half-spaces containing it. Let $\mathcal{R} = (P, \mathcal{D})$ be the range space defined by the remaining set of points and the remaining set of half-spaces. A PTAS for $\mathcal{R}$ gives a PTAS for the original problem. We will show that $\mathcal{R}$ satisfies the locality condition. Let R and B be disjoint red and blue subsets of P.

We construct the required graph G on the vertices $R \cup B$ in two stages and prove its planarity by giving its embedding on the boundary $\partial\mathcal{C}$ of the convex hull $\mathcal{C}$ of $R \cup B$. In the first stage, we add all red-blue edges (1-faces) of $\mathcal{C}$ to G. In the second stage we map each red or blue point p lying in the interior $\mathcal{C}$ to a triangular face $\Delta(p)$ of $\mathcal{C}$ that intersects the ray op emanating from o and passing through p.[2] Let Q be the set of points mapped to a triangle Δ. We will construct a planar bipartite graph on Q and the corners of Δ and embed it so that the edges lie inside Δ. If Δ has two red corners and one blue corner, we add an edge between each red point in Q to the blue corner of Δ and each blue point of Q to the two red corners of Δ. It it quite easy to see that this can be done so that the graph remains planar. The case when Δ has two blue corners and one red corner is handled similarly. Consider now the case when all corners of Δ are red and let r_1, r_2 and r_3 be the corners. In this case we will connect at most one blue point of Q to all three corners of Δ and we will connect the rest of the blue points to two of the corners of Δ. Again, it is clear that this can be done while keeping the graph planar. For each blue point $b \in Q$, we try to find one corner c of Δ such that there is no half-space $h \in \mathbb{R}^3$ with the following properties: i) the only blue point in h is b, ii) h contains exactly one of the corners of Δ. If we can find such a corner c, then we put an edge between b and the two corners of Δ other than c. There can be at most one blue point in Q for which we cannot find such a corner and we will connect that blue point to all three corners of Δ. For contradiction, assume that there are two points b_1 and b_2 in Q such that for each pair of red and blue points in $F = \{r_1, r_2, r_3, b_1, b_2\}$ there is a half-space in $\mathbb{R}^3$ containing exactly those two points of F. This means that each $r_i b_j$ is an edge in the convex hull of F and therefore F is in convex position. The Radon partition [10] of F is then a $(3, 2)$-partition. Since the blue points lie on the same side of the plane containing Δ, the partition with two points has one red point and one blue point and there cannot be a half-space containing exactly these two among the points of F, contradicting our assumption. The case when Δ has three blue corners is handled similarly. The construction of G is complete.

We now show that for any half-space $h \in \mathbb{R}^3$ that does not contain o and contains both red and blue points, there is an edge in G between a red point and blue point both of which lie in h. If h contains both red and blue points which lie on $\partial\mathcal{C}$ then there is a red-blue edge among two of those due to the edges added in the first stage. Otherwise assume, without loss of generality, that only the red points in h lie on $\partial\mathcal{C}$. Consider the half-space h' parallel to and contained in h which contains the smallest number of points and still contains both red and blue points. Clearly, h' contains exactly one blue point b. Since h, and hence h', does not contain o, h' must contain one of the corners of the triangle Δ that b is mapped to. If b is connected to all three corners of Δ in G, we are trivially done. Also, if h contains two of the corners of Δ, then we are done since b is connected to at least one of those corners. If h' contains exactly one corner c of Δ, then b must be connected to c since it cannot be the case that we connected b to the other two corners of Δ. Hence, in all cases, b is connected to one of the red points in h'.

[2]Here we are assuming that each face of $\mathcal{C}$ is a triangle, since one can always triangulate the faces.

3 Combinatorial Bounds on ϵ-nets via Local Search

Consider the range space $\mathcal{R} = (P, \mathcal{D})$ in which P is a set of points in the plane and $\mathcal{D}$ is defined by intersecting P with a set of unit squares in the plane. Construct an ϵ-net for $\mathcal{R}$, say Y, using the 3-level local search: starting with $Y = P$, keep improving Y as long as there exists a subset of size at most three of Y that can be swapped to get a smaller set. We now argue that $|Y| = O(1/\epsilon)$.

For the argument we will consider an equivalent problem. We will replace each of the squares by a point at its center and each of the points with a unit square centered at it. The task now is to pick the smallest subset of the squares which cover all points which are covered by more than an ϵ fraction of the squares. Let the number of squares be n and the number of points be m. We will refer to the set of squares corresponding to points in P by S and the set of squares corresponding to the points in Y by M.

First some definitions. Call the squares in M the "ϵ-net squares" and the squares in $S \setminus M$ as "normal squares". A point $p \in \mathbb{R}^2$ is *dense* if it is covered by more than ϵn squares in S. Each $s \in M$ must have a *personal dense point*, i.e., a dense point which no other square in M covers. Fix any unit griding of the plane, and call a grid point p *active* if at least one of the four cells touching it contains a dense point. Denote the set of active grid points by A. The following claim is easy to show.

Claim 3.1. $|A| = O(1/\epsilon)$.

Proof. By a packing argument, each active point has ϵn unit squares intersecting one of its four adjacent squares. These squares contribute a constant number of active points, and there can be only $O(1/\epsilon)$ such sets. □

Each unit square $s \in S$ contains exactly one of the grid points, and for the squares in M, this grid point belongs to A. For each active grid point $p \in A$, label the four cells around it as $C_1(p)$, $C_2(p), C_3(p)$ and $C_4(p)$ in counter-clockwise order. For each cell $C_i(p)$, refer to its opposite cell as $C'_i(p)$ (e.g., $C_1(p)$ is the opposite cell to $C_3(p)$). Denote the set of squares in M that contain the grid point p by $M(p)$, and among these, those that have a personal dense point in $C_i(p)$ as $M_i(p)$. Each square of M containing p must belong to at least one of the four $M_i(p)$'s. Each set $M_i(p)$ forms a *cascade* and there is a natural linear order on them. Call the squares which are not the first or the last in this order the *middle* squares of $M_i(p)$. Each square $s \in M_i(p)$ has some region in $C_i(p)$ which is not covered by other squares in $M_i(p)$ and we denote this region by $R_i(s)$ (see Figure 1). This square s also has a region in $C'_i(p)$ which is not covered by other squares in $M_i(p)$, denoted by $R'_i(s)$. For a normal square r and an ϵ-net square $s \in M_i(p)$ we say that "r stabs s in $C_i(p)$" if r intersects the region $R_i(s)$ and we say that "r covers s in $C_i(p)$" if r contains the region $R_i(s)$. Note that if r covers s then r also stabs s.

Lemma 3.2. *No three middle squares in $M_i(p)$ have a common coverer in both $C_i(p)$ and $C'_i(p)$. Furthermore, no five squares in $M_i(p)$ are stabbed by a common square in both $C_i(p)$ and $C'_i(p)$.*

Proof. If three middle squares in $M_i(p)$ have a common coverer r in $C_i(p)$ and a common coverer r' in $C'_i(p)$, then a local improvement is possible by replacing the three squares by two squares r and r' in the ϵ-net. Similarly, if five squares are stabbed by a common square r (resp. r') in $C_i(p)$ (resp. $C'_i(p)$), then the three middle squares among them are covered by r (resp. r'), which is not possible by the first statement. □

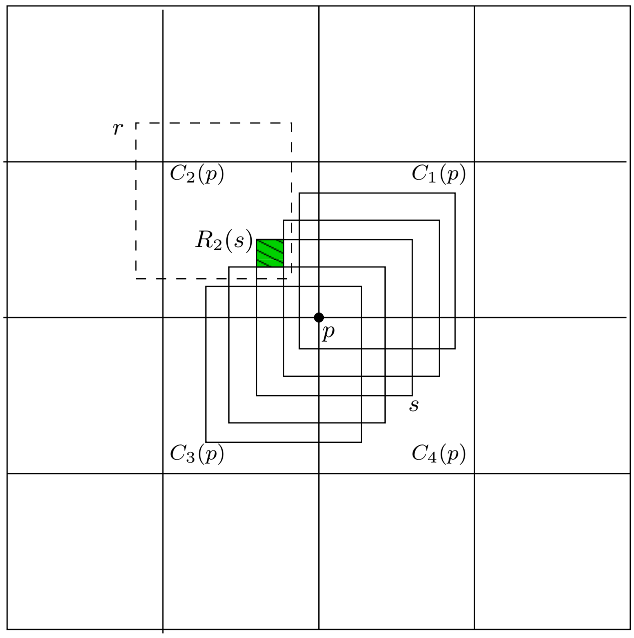

Figure 1: The normal square r covers the ϵ-net square s and stabs its neighbors (in the cascade $M_2(p)$) in the cell $C_2(p)$.

For any square $s \in M$, let $N(s)$ be the set of normal squares intersecting s. Also, let $Z(p) = \cup_{s \in M(p)} N(s)$ be the neighborhood of $M(p)$ and $Z_i(p) = \cup_{s \in M(p)} N(s)$ be the neighborhood of $M_i(p)$.

Claim 3.3. *For any $p \in A$ and any i, $|Z_i(p)| \geq \frac{|M_i(p)|}{15} \cdot \epsilon n$. Furthermore, $|M(p)| \leq 60 \frac{|Z(p)|}{\epsilon n}$.*

Proof. First notice that the second statement in the claim easily follows from the first since for some j, $M_j(p) \geq M(p)/4$ and therefore $|Z(p)| \geq |Z_j(p)| \geq |M_j(p)|/15 \geq |M(p)|/60$. We will now prove the first statement.

Partition the squares in $M_i(p)$ into two types: those that have personal dense points in $C_i(p)$ only, or in both $C_i(p)$ and $C'_i(p)$. If the former set has size at least $|M_i(p)|/3$, we are done: each such square has $N(s) \geq \epsilon n$ (due to the personal dense point), and by Lemma 3.2, each normal square is double-counted at most five times when summing up $N(s)$ for squares in this set. Therefore $|Z_i(p)| \geq (|M_i(p)|/15)\epsilon n$.

Otherwise, assume that there are at least $2|M_i(p)|/3$ squares, say set M', which have personal dense points in both $C_i(p)$ and $C'_i(p)$. Let $t = |M'|$ and let $s_1, s_2, \cdots, s_t$ be the squares of M' along the cascade defined by them. For each square s_j, define its *red (blue) successor* to be the square s_k with the smallest index $k > j$ such that s_j and s_k are not stabbed by a common square in $C_i(p)$ ($C'_i(p)$). Note that a square may not have a red or blue successor. Let us also say that a red or blue successor of a square s_i is *far* if the successor is s_j with $j - i \geq 5$ and *near* otherwise. If some square s_i has a red (blue) successor s_j that is far then s_i the squares of M' between s_i and s_{j-1}, of which there are at least 5, are stabbed by a common square in $C_i(p)$ ($C'_i(p)$). Lemma 3.2 therefore implies that both red and blue successors of a square cannot be far. At least one of them has to be near. Assume, without loss of generality, that at least half of the squares in M' have a red successor that is near. Let M'' be the set of such squares. Let M''' be the set of squares in which we take every fifth square of M'' starting with the first in the cascade defined by them. Clearly no two squares in

M''' are stabbed by a common square in $C_i(p)$ since otherwise one of them would have a far red successor. Now, since $|M'''| \geq |M_i(p)|/15$ and each normal square can contain the personal dense point of at most one of the squares of M''' in $C_i(p)$, we have $|Z_i(p)| \geq (|M_i(p)|/15)\epsilon n$.

□

A square can belong to the neighborhood of at most nine active points, i.e., $\sum_{p \in A} |Z(p)| \leq 9n$. Summing the second inequality in Claim 3.3 over all $p \in A$ and using Claim 3.1, one gets the required statement: $|M| = \sum_{p \in A} |M_p| = O(1/\epsilon)$.

4 Future Work

We gave a PTAS for some geometric hitting set problems and proved a theorem about bipartite planar graphs in the process. We believe that the theorem about bipartite planar graphs may be true for a more general class of graphs. This may allow us to get PTAS for other geometric hitting set problems. It is also worth exploring whether there is a PTAS with a running time $O(mn^{O(\epsilon^{-1})})$ instead of $O(mn^{O(\epsilon^{-2})})$ for the problems we considered.

We also believe that the local search technique can be used to find alternative proofs of the existence of small ϵ-nets for many other geometric range spaces including those induced by half-spaces in $\mathbb{R}^3$ and by an r-admissible set of regions in the plane. Currently, however, it is not even clear how to use it to prove $O(1/\epsilon)$ size ϵ-nets for range spaces induced by different sized squares in the plane.

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