

Centerpoints and Tverberg's Technique

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Abstract

Using a technique that Tverberg and Vrećica [TV93] discovered to give a surprisingly simple proof of Tverberg's theorem, we show the following extension of the centerpoint theorem: given any set P of n points in the plane, and a parameter $1/3 \leq c \leq 1$, one can always find a disk D such that any closed half-space containing D contains at least cn points of P , and furthermore, D contains at most $(3c - 1)n/2$ points of P (the case $c = 1$ is trivial – take any D containing P ; the case $c = 1/3$ is the centerpoint theorem!). We also show that, for all c , this bound is near-optimal.

1 Introduction

The centerpoint theorem is one of the fundamental theorems in discrete geometry, and it states the following: given any set P of n points in \mathbb{R}^d , there exists a point q such that any closed half-space containing q contains at least $n/(d + 1)$ points of P . The point q need not be a point of P – indeed, the statement is false if q is restricted to be in P , as shown by any set of points in convex position. Furthermore, this bound is tight: place $n/(d + 1)$ points each in $(d + 1)$ small disks centered around the vertices of an equilateral simplex in \mathbb{R}^d . This theorem has found several applications in combinatorial geometry, statistics, geometric algorithms and related areas [CEM⁺96, MTTV97, Yao83, MTTV93, DG94, MR08].

An even more fundamental theorem, encompassing the centerpoint theorem, was first proven by Tverberg in 1966: given any set P of n points in \mathbb{R}^d , one can partition P into (roughly) $\frac{n}{d+1}$ sets, each of $d + 1$ points, such that the simplices spanned by the sets have a non-empty intersection. More precisely:

Theorem 1.1 (Tverberg's Theorem [Tve66]). *Any set P of $(d + 1)(r - 1) + 1$ points in \mathbb{R}^d can be partitioned into r subsets A_1, \dots, A_r such that $\bigcap \text{conv}(A_i) \neq \emptyset$.*

The original proof given by Tverberg was fairly complicated, but in 1993, Tverberg and Vrećica [TV93] gave an ingenious proof by considering the natural generalization of the question to balls: partition P into r sets such that there exists a ball $D \subseteq \mathbb{R}^d$ intersecting the convex-hull of all r sets. Such a ball/partition is easy to find (just pick any ball containing P , and pick any partition). It is then shown that by adjusting the

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partitions, one can always locally translate and shrink D to finally get a point – all the while maintaining the intersection invariant.

The essence of this clever technique can be summarized as follows: if one wants to prove that a point with a certain property exists, start with a ball satisfying the analogue of the property, and then show that the ball can be shrunk smoothly to a point. Tverberg and Vrećica proved this for Tverberg’s theorem [TV93]. Corollary 2.1 proves this for the centerpoint theorem: start with any ball B s.t. any closed halfspace containing B contains at least $n/(d + 1)$ points of P . Then this ball can be shrunk to a point – the centerpoint. Helly’s theorem has a similar proof, by starting with any ball that intersects all convex objects.

We use this technique to come up with following generalization of the centerpoint theorem.

Our Contributions. In this paper we generalize the centerpoint theorem to a new theorem about “center-disks”, such that the centerpoint theorem becomes a special case of this more general statement. Specifically, we ask if there exists a disk D such that any half-space containing D contains a larger fraction of points of P than $n/(d + 1)$ (see [BMN08] for another extension of the centerpoint theorem by looking at k -flats in \mathbb{R}^d). For example, given a set P of n points in the plane, does there exist a disk D in the plane such that $|D \cap P| = \emptyset$, and any closed half-space containing D contains at least $n/2$ points of P ? The answer is ‘no’, as a variant of the simplex lower-bound shows. So we ask that D contain as few points of P as possible. We prove the following:

Theorem 1.2. *Given a set P of n points in the plane, there exists a disk D containing at most $(3c - 1)n/2$ points of P , $1/3 \leq c \leq 1$, such that any closed half-space containing D contains at least cn points of P .*

Theorem 1.3. *For any fixed c , there exists a set P of n points in the plane such that for any disk D , if each closed half-space containing D contains at least cn points of P , then D must contain at least $(3c - 1)n/6$ points of P .*

2 Proof of Theorem 1.2

Preliminaries. A direction in \mathbb{R}^d can be represented by a unit vector $u \in \mathbb{S}^{d-1}$. Given a ball $D \subset \mathbb{R}^d$ with center s and a direction $u \in \mathbb{S}^{d-1}$, let \hat{u}_D be the intersection of the boundary of D with the ray $s\vec{u}$. And let $h(D, u)$ (called the *tangent half-space*) be the closed half-space containing D that is tangent to D at \hat{u}_D . Let $\bar{h}(D, u)$ be the complement open half-space of $h(D, u)$. See Figure 1(a). When the ball D is clear from the context, we drop the subscript D from the notation.

Claim 2.1. *Given a ball D in \mathbb{R}^d and unit vectors $u, u' \in \mathbb{R}^d$ such that $\langle u, u' \rangle > 0$, let D' be the ball obtained by translating D in the direction u' by a small distance $\epsilon > 0$. Then $h(D, u) \subset h(D', u)$.*

Proof. By rotating, and translating D to the origin, one can assume $u = (u_1 = 0, \dots, u_{d-1} = 0, u_d = 1)$, and let $u' = (u'_1, \dots, u'_d)$. So $x \in h(D, u)$ iff the d -th coordinate of x , x_d , is at most 1. Then $\hat{u}_{D'} = \hat{u}_D + \epsilon u' = (\epsilon u'_1, \dots, \epsilon u'_{d-1}, \epsilon u'_d + u_d)$, and since $\langle u, u' \rangle > 0$, we have $u'_d > 0$, and so the d -th coordinate of \hat{u}_D is less than the d -coordinate of $\hat{u}_{D'}$. Then $\hat{u}_D \in h(D', u)$, and the claim follows. See Figure 1(b). \square

Let $P = \{p_1, \dots, p_n\}$ be a set of n points in the plane. For any ball D , let $|D|$ denote the number of points of P contained in D .

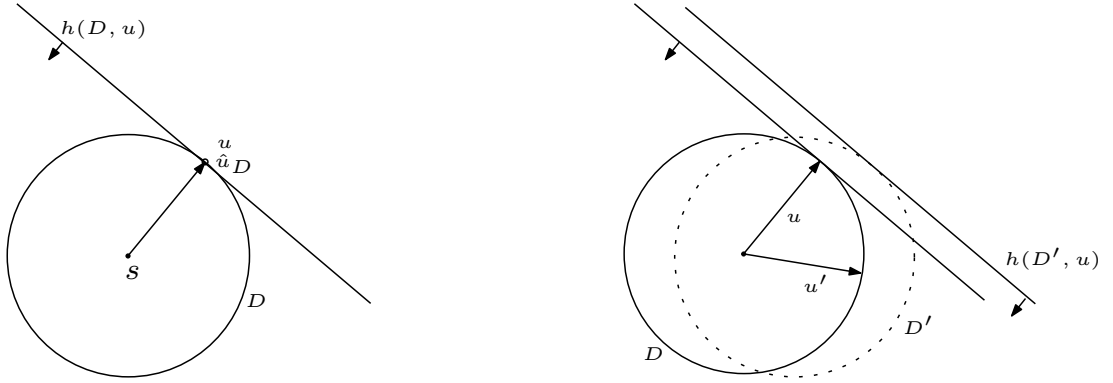


Figure 1: (a) Definition of \hat{u}_D and $h(D, u)$ in \mathbb{R}^2 , (b) Illustration of Claim 2.1 in \mathbb{R}^2 , where D' is the ball obtained by shifting D in the direction u' , where $\langle u, u' \rangle > 0$.

Call a tangent half-space, say $h(D, u)$, to D *tight* if it contains exactly cn points of P and by shrinking D infinitesimally to get D' , the half-space $h(D', u)$ contains less than cn points. Note that then $h(D, u)$ must contain at least one point of P on its defining line.

Consider any disk D satisfying the following property: any closed half-space containing D contains at least cn points of P . First note that such a D clearly exists, e.g., take D to be any disk containing at least cn points of P . By doing two operations iteratively – shrinking D while keeping its center fixed, and translating D – we show that ultimately one gets the required disk:

Theorem 2.2. *Consider the smallest radius disk D such that any closed half-space containing D contains at least cn points of P . Then D contains at most $\frac{3c-1}{2}n$ points of P .*

Proof. Let s be the center and r be the radius of D , and let U be the set of directions such that the corresponding tangent half-spaces $h(D, u)$, $u \in U$, are tight. The claim is that $\text{conv}(U)$ contains the center s . Otherwise, by the Separation Theorem [Mat02], there exists a line l passing through s such that all points $\hat{u}_D, u \in U$ lie in the same closed half-space defined by l . Then the direction u' perpendicular to l satisfies $\langle u', u \rangle \geq 0$ for all $u \in U$. There are two cases:

Case 1: If any two tight tangent half-spaces are parallel, say $h(D, u_1)$ and $h(D, u_2)$ where $u_1 = -u_2$, then the number of points contained in D is at most $|(h(D, u_1) \cap P) \cap (h(D, u_2) \cap P)| = |h(D, u_1) \cap P| + |h(D, u_2) \cap P| - n = (2c - 1)n$, which is less than $(3c - 1)n/2, \forall c < 1$.

Case 2: Otherwise, if no two tight tangent half-spaces are parallel, there exists a separating line l such that the direction u' perpendicular to l satisfies $\langle u', u \rangle > 0$ for all $u \in U$. Translate D slightly in the direction u' to get D' . By Claim 2.1, all the tangent half-spaces $h(D', u)$, $u \in U$, are no longer tight. Since U contained all the tight tangent half-space directions, the disk D' has no tight tangent half-spaces, and by shrinking it slightly, one gets a contradiction to the fact that D was the smallest-radius disk.

Since $s \in \text{conv}(U)$, by Caratheodory's theorem [Mat02], there exist three points $\hat{u}_1, \hat{u}_2, \hat{u}_3$ such that the triangle defined by them contains s . See Figure 2. Let a_1n, \dots, a_7n be the number of points in each spatial partition induced by the *closed* half-spaces $h(D, u_1), h(D, u_2)$ and $h(D, u_3)$. Since $h(D, u_1), h(D, u_2), h(D, u_3)$

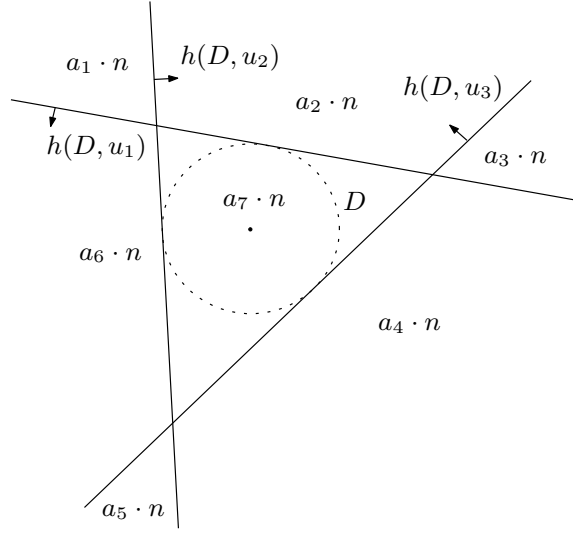


Figure 2: The 7 partitions induced by the three tight tangent half-spaces.

were tight tangent half-spaces, we have:

$$a_1 + a_2 + a_3 = 1 - c \quad (1)$$

$$a_3 + a_4 + a_5 = 1 - c \quad (2)$$

$$a_1 + a_5 + a_6 = 1 - c \quad (3)$$

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 - a_7 \quad (4)$$

Combining the first three equations,

$$2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) \geq 2a_1 + a_2 + 2a_3 + a_4 + 2a_5 + a_6 = 3 - 3c$$

Together with equation (4), this implies $2(1 - a_7) \geq 3 - 3c$, and so $a_7 \leq \frac{3c-1}{2}$. \square

Corollary 2.1 (The Centerpoint Theorem). *There exists a point p such that any closed half-space containing p contains at least $n/3$ points of P .*

Proof. By setting $c = 1/3$ in Theorem 2.2, there exists a disk D , and three tight tangent half-spaces such that $a_7 \leq (3c-1)/2 = 0$. Also note that $a_2 + a_4 + a_6 = 0$, since otherwise $2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) > 2a_1 + a_2 + 2a_3 + a_4 + 2a_5 + a_6 = 2$ from Equations (1), (2), (3), while $2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) = 2(1 - a_7) = 2$ from Equation (4), a contradiction. Therefore, $a_2 = a_4 = a_6 = a_7 = 0$. Since $h(D, u_j), j = 1, 2, 3$ were tight tangent half-spaces, each of these three closed half-spaces must contain a point on its boundary. But this gives a contradiction, since any point on the boundary of any of these three half-spaces must count in one of a_2, a_4, a_6 or a_7 . Therefore the half-spaces cannot be tight and the disk's radius can be further decreased, till it shrinks to a point. \square

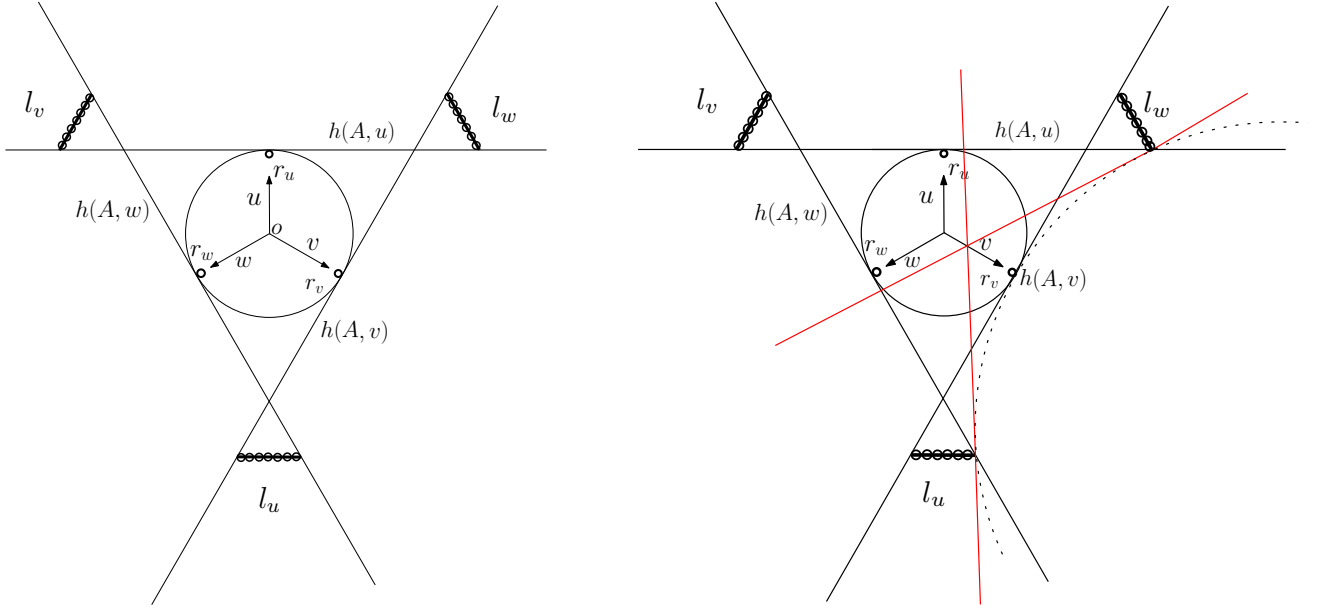


Figure 3: (a) The point set used for the lower-bound construction, (b) separating tangent half-space used in Claim 3.4

3 Near-Optimality of the Centerdisk bound

We now show that the bound derived in Theorem 1.2 is tight within a factor of three, by proving the following: for any value of $c > 1/3$, there exists a set P of n points such that for any disk D containing less than $(3c - 1)n/6$ points of P , there exists a half-space containing D with less than cn points. This proves Theorem 1.3.

Construction. Take a unit circle A centered at the origin o and let u, v and w be three distinct unit vectors such that $\langle u, v \rangle = \langle v, w \rangle = \langle u, w \rangle = 0.5$.

Define the regions $R_x, x \in \{u, v, w\}$ to be identical circular regions, with some arbitrarily small radius ϵ , centered on the ray $\vec{o}x$ such that the tangent to A at \hat{x}_A is also a tangent to R_x . Place $(3c - 1)n/6$ points of P uniformly on the boundary of each R_x and let $r_x = R_x \cap P$. See Figure 3(a).

Define L_u to be line segment perpendicular to the ray $\vec{o}u$ lying in the intersection of the two halfspaces $\bar{h}(A, v)$ and $\bar{h}(A, w)$. Define L_v, L_w similarly. Place $(1 - c)n/2$ points uniformly on each segment L_x and let $l_x = L_x \cap P$. The segments L_x are placed at an equal distance from o such that the following condition is satisfied: for every disk D containing points *partially* from only one of the sets r_x , say r_v , and partially from both the sets l_u and l_w , the half-space defined by the tangent to D at the point where it intersects L_u or L_w will contain points only from r_v, l_u and l_w . Note that by placing L_u and L_w sufficiently close to o , the above condition can always be guaranteed: as we move L_x closer to the origin o the radius of D increases to be arbitrarily large, and the tangent to D where it intersects L_u or L_w converges to the line defining $h(A, v)$ – for which the tangents contain points from only the sets r_v, l_u and l_w . As the convergence is continuous,

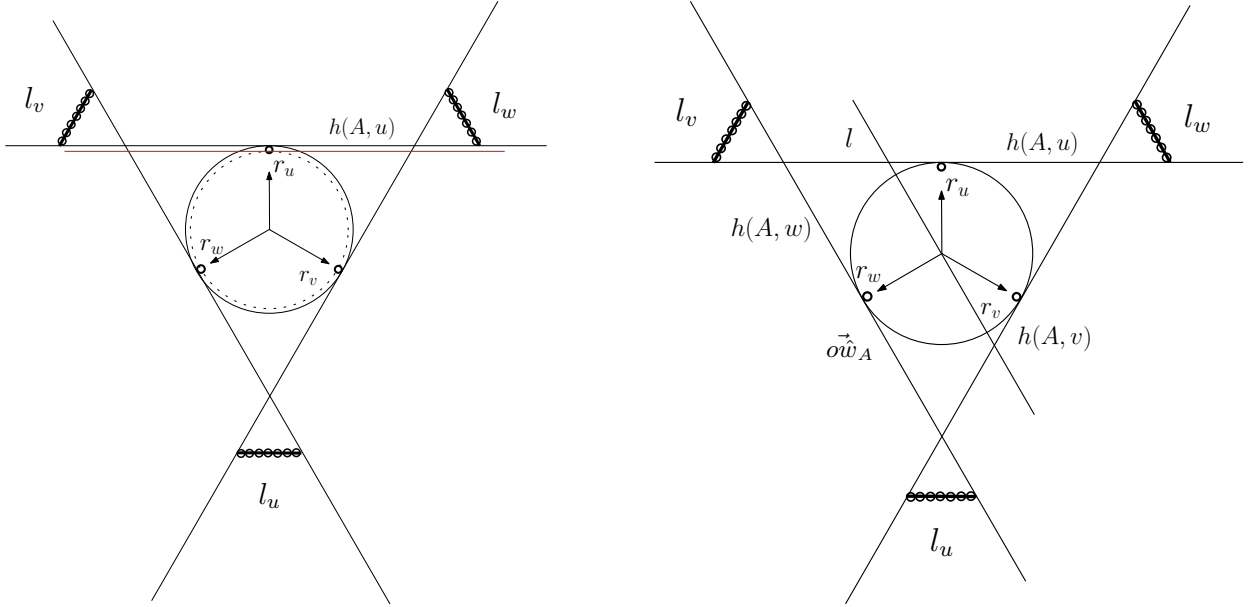


Figure 4: (a) $D \subset A$, (b) D intersecting r_u and r_v

there exists the required positioning. See Figure 3(b).

Note that each of the halfspaces $h(A, x)$ is tight and contains $(1 - c)n/2 + 3 \cdot (3c - 1)n/6 = cn$ points.

We now show that every disk that contains less than $(3c - 1)n/6$ points of P must have a tangent half-plane that contains less than cn points. For the rest of the proof, let D be any such disk. There are three cases that we consider, depending on the number of r_x sets that D intersects. For each, we exhibit a tangent half-space containing less than cn points.

First note the following facts:

Fact 3.1. D cannot lie in the interior of any of the half-spaces $h(A, x)$, $x \in \{u, v, w\}$.

Assume D lies in the interior of, say $h(A, v)$. Since $h(A, v)$ contains exactly cn points, the tangent half-plane to D parallel to the line defining $h(A, v)$ contains less than cn points.

Fact 3.2. D cannot contain any of r_x , $x \in \{u, v, w\}$ completely.

Otherwise D already contains at least $(3c - 1)n/6$ points.

Claim 3.3. If D contains points from at least two of r_x , $x \in \{u, v, w\}$, then there exists a half-space containing D and containing less than cn points.

Proof. Suppose D contains points from each of r_x without completely containing any one, then $D \subset A$ (since A is the smallest disk containing all three sets completely). Also note that since A is touching the defining lines of all three half-spaces $h(A, u)$, $h(A, v)$ and $h(A, w)$, D will lie in the interior of at least one of the half-spaces. Then it follows from Fact 3.1 that D is not a valid centerdisk. If $D \not\subset A$ then D must contain at least one of r_x completely and is therefore not a valid disk (Fact 3.2). See Figure 4(a).

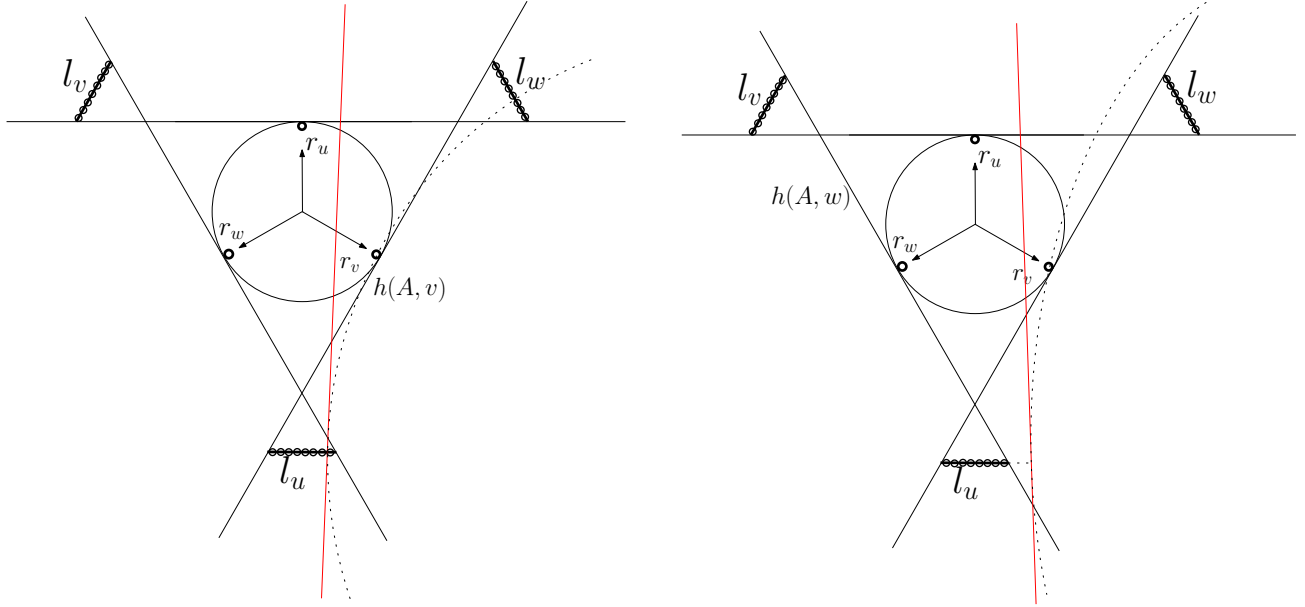


Figure 5: (a) D touching r_v and l_u (b) D touching l_u and l_w

Now let D be such that it contains at least one point from both r_u and r_v but not from r_w . Let the point d be the center of D , and let l be the line parallel to $h(A, w)$ and passing through the center of A . See Figure 4(b). First, note that d cannot lie on the same side of l as r_w : otherwise, as D intersects both r_u and r_v , D will have to intersect r_w as well, a contradiction.

So d lies on the other side of l . Assume first that d lies inside A . By Fact 3.1, D must intersect the line defining $h(A, w)$. So the radius of D is larger than one (recall that A had unit radius), and so D must completely contain either r_u or r_v . This is a contradiction to Fact 3.2.

Finally, the remaining case is when d lies outside A such that D intersects both r_u and r_v , and also intersects the line defining $h(A, w)$. In this case, take the half-space defined by the tangent to D at the point where it intersects r_u or r_v . This half-space will contain points from only r_u, r_v and l_w – at most $2 \cdot (3c - 1)n/6 + (1 - c)n/2$ points, which is less than cn , $\forall c > 1/3$.

□

Claim 3.4. *If D contains points from only one set r_x , $x \in \{u, v, w\}$, then there exists a half-space containing D and containing less than cn points.*

Proof. Without loss of generality, suppose D contains points from r_v . Note that $D \cap l_v = \emptyset$. Let $t_u = |D \cap l_u|$ and $t_w = |D \cap l_w|$. If $t_u, t_w \neq 0$, since $|D| < (3c - 1)n/6$, $\min(t_u, t_w) \leq (3c - 1)n/12$. Say t_u is smaller, then, by construction, the tangent to D at its intersection with l_u contains points from only sets r_v, l_u and l_w – at most $(3c - 1)n/12 + (3c - 1)n/6 + (1 - c)n/2$ points, which is less than cn , $\forall c > 1/3$. See Figure 5(a).

Consider the case where $t_u = 0$ or $t_w = 0$, say $t_u = 0$. Let p be the closest point of D to l_u . As D must intersect the line defining $h(A, w)$ (Fact 3.1), p must be in the interior of $h(A, w)$. Now the tangent at p

must intersect $h(A, w)$ below l_u , and therefore doesn't contain r_w and l_v . So the tangent to D at point p contains at most $2 * (3c - 1)n/6 + (1 - c)n/2 < cn$ points. See Figure 5(b). \square

Claim 3.5. *If $D \cap r_x = \emptyset$ for all $x \in \{u, v, w\}$, then there exists a half-space, containing D , with less than cn points.*

Proof. First, note that D cannot contain points from all three of l_u, l_v and l_w , otherwise D would contain all of r_u, r_v , and r_w . Suppose D contain points from l_u and l_w only. The proof for this claim follows similarly to the proof of the previous claim. \square

4 Conclusion

A number of theorems in discrete geometry – Helly's theorem, Centerpoint theorem, Tverberg's Theorem – where one has to find a point satisfying certain property, can be proven by Tverberg's technique. In this paper, we proved near-optimal bounds on centerdisks in the plane using this technique.

We suspect that this technique can be used to simplify several other proofs. We conjecture that the recent result of Fulek *et. al.* [FHP08] has an elementary proof based on Tverberg's technique. The result states that given any set of disjoint convex objects in \mathbb{R}^d , there exists a point q such that any half-infinite ray emanating from q intersects at most $dn/d + 1$ objects. Similarly, the theorem on regression depth [ABET00] might have an elementary proof.

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