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An optimal extension of the centerpoint theorem

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Abstract

We prove an optimal extension of the centerpoint theorem: given a set P of n points in the plane, there exist two points (not necessarily among input points) that hit all convex sets containing more than $\frac{4}{7}n$ points of P. We further prove that this bound is tight. We get this bound as part of a more general procedure for finding small number of points hitting convex sets over P, yielding several improvements over previous results.

Keywords: Combinatorial geometry; Discrete geometry; Weak ϵ-nets; Centerpoint theorem; Extremal methods; Hitting convex sets

1. Introduction

The centerpoint theorem is one of the fundamental combinatorial results in discrete geometry, with applications in geometric algorithms [4,7,9], large-scale computing [10], multivariate data analysis [11] and several others. It states the following:

Centerpoint Theorem. *(See [8,13].) Given a set P of n points in the plane, there exists a point c (not necessarily in P) such that any convex set containing more than $\frac{2}{3}n$ points of P contains c.*[1] *Furthermore, this bound is tight.*

In this paper we look at a generalization of the above theorem to more than one point. For example, is it possible to find two points c_1 and c_2 in the plane such that any convex set containing at least $n/2$ points must contain either c_1 or c_2? We present a general procedure that gives the following results: one can hit all convex sets containing more than $\frac{4}{7}n$ points with 2 points. Furthermore, we prove that this bound is tight. Similar results are derived for larger number of points. In particular, we show that if each convex set contains more than $\frac{20}{41}n$ points, then five points suffice. This improves a natural way of adding five points [2] which gives the worse $n/2$-bound: find two lines (using the ham-sandwich theorem [8]) which partition the point set into four regions with $n/4$ points in each. Add the intersection

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[1] This theorem can be equivalently stated as: there exists a point c such that any halfspace containing c contains at least $n/3$ points of P.

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point x of the lines along with the centerpoints of the four regions. Since any set avoiding the centerpoints of the four regions can contain only $\frac{2}{3}rd$ of the points in any of the regions and must avoid one of the regions completely if it avoids x, these 5 points form a $\frac{1}{2}$-net.

1.1. Previous results

Aronov et al. [2] proved that given a set P of n points in the plane, all convex sets containing greater than $\frac{5}{8}n$ points of P can be hit by two points. They also construct inputs where regardless of how one picks the two points, there exists a convex set containing at least $\frac{5}{9}n$ points that is not hit. In this paper, we improve both their results to get the optimal result of $\frac{4}{7}n$. We similarly improve their results for other small numbers of points (see Section 3 for specific improvements).

Our problem is related to two other areas of research. In the weak ϵ-net problem [1,3,12], given a parameter $\epsilon > 0$ and a point set P, one would like to compute a small set of (not necessarily input) points that hit all convex sets containing at least ϵn points of P. The concept of weak ϵ-nets with respect to convex ranges was introduced by Haussler and Welzl [6]. Alon et al. [1] proved that for any ϵ, d, there exist a weak ϵ-net of size $O(1/\epsilon^{d+1-\delta_d})$, where δ_d tends to zero with $d \to \infty$. This result was improved by Chazelle et al. [3] to $O(1/\epsilon^d \operatorname{polylog}(1/\epsilon))$. More recently, Matousek and Wagner [12] gave an elegant algorithm that computes weak ϵ-nets in $\mathbb{R}^d$. Clearly for $\epsilon > \frac{2}{3}$, the centerpoint is the desired weak ϵ-net in the plane. Our work can be seen as constructing small weak ϵ-nets.

The other related area of research is the so-called *Gallai-type* problems [8] which ask whether certain families of geometric shapes can be "pierced" by a small number of points. An example of such a problem is the following: Given a set of closed disks in the plane such that every pair intersects, what is the smallest number of points needed to hit all these disks? In this case, the answer which is both necessary and sufficient, is four [5]. In our problem, we are looking to hit considerably more general objects (convex sets), with the added constraint that one first fixes n input points, and each convex set contains a constant proportion of these points.

2. Main theorem

We first present some definitions. Given a set $P = \{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}^d$ and a finite set $Q \subset \mathbb{R}^d$, define the following:

$$\epsilon(P, Q) = \min\bigl\{\epsilon \mid |\mathcal{C} \cap Q| \neq \emptyset \ \forall \text{ convex sets } \mathcal{C} \text{ s.t. } |\mathcal{C} \cap P| > \epsilon n\bigr\}$$

and let $\epsilon_i^d(P) = \min_{Q,|Q|=i} \epsilon(P, Q)$. Set $\epsilon_i^d = \sup_P \epsilon_i^d(P)$. In other words, given any P, the set of all convex sets containing $\epsilon_i^d n$ points of P can be hit by i points. These i points are said to form a *weak* ϵ_i^d-net for P. The centerpoint theorem in d dimensions states that $\epsilon_1^d = \frac{d}{d+1}$.

We fix a direction $\vec{u} \in \mathbb{R}^d$ which we call the *upward* direction. For a point $p \in \mathbb{R}^d$, let $f_u(p) = \langle u, p\rangle$ denote the *height* of the point p in the upward direction ($\langle u, p\rangle$ denotes the inner product of u and p). For a convex set $\mathcal{C}$, let $f_u(\mathcal{C})$ denote the height of the lowest point in $\mathcal{C}$, i.e. $f_u(\mathcal{C}) = \inf_{p\in\mathcal{C}} f_u(p)$.
We now present our main result.

Theorem 2.1. *Given a set P of n points in $\mathbb{R}^d$, and two integers $r \geqslant 0$ and $s \geqslant 0$,*

$$\epsilon_{r+ds+1}^d \leqslant \frac{\epsilon_r^d \cdot (1 + (d-1)\epsilon_s^d)}{1 + \epsilon_r^d \cdot (1 + (d-1)\epsilon_s^d)},$$

where we define $\epsilon_0^d = 1$.

Construction. Let $a, b \in [0, 1]$ be two reals to be fixed later.

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be the set of all closed halfspaces which contain at least an points of P and whose bounding hyperplane passes through d points in P. Define $\mathcal{H}^d = \{(h_{i_1}, h_{i_2}, \ldots, h_{i_d}) \mid |P \cap (h_{i_1} \cap h_{i_2} \cdots \cap h_{i_d})| \geqslant bn$, where $h_{i_1}, h_{i_2}, \ldots, h_{i_d} \in \mathcal{H}\}$ to be the set of all d-tuples of halfspaces in $\mathcal{H}$ whose intersection contains at least bn points of P. Consider the d-tuple, say $(h_{l_1}, \ldots, h_{l_d})$, such that

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1. $(h_{l_1}, \ldots, h_{l_d}) \in \mathcal{H}^d$;
2. $(h_{l_1} \cap \cdots \cap h_{l_d})$ has the highest lowest-intersection point among the d-tuples of halfspaces in $\mathcal{H}^d$, i.e., $f_u(h_{l_1} \cap \cdots \cap h_{l_d}) = \max_{(h_{i_1},\ldots,h_{i_d})\in\mathcal{H}^d} f_u(h_{i_1} \cap \cdots \cap h_{i_d})$.

We choose the upward direction $\vec{u}$ so that the d-tuple $(h_{l_1}, \ldots, h_{l_d})$ is well defined. Note that $f_u(h_{i_1} \cap \cdots \cap h_{i_d}) = -\infty$ iff $h_{i_1} \cap \cdots \cap h_{i_d}$ is unbounded in the *downward* direction $-\vec{u}$. Let $\mathcal{P}$ be the convex hull of P and let $h_{j_1}, \ldots, h_{jd}$ be d halfspaces defining a vertex v of $\mathcal{P}$ and containing P. Choose the upward direction $\vec{u}$ so that the vertex v is the unique lowest vertex of the polyhedron $\mathcal{P}' = h_{j_1} \cap \cdots \cap h_{jd}$ in the upward direction and each of the points $p \in P$ get a unique height. Such a choice of $\vec{u}$ ensures that the bounding hyperplane of no halfspace in $\mathcal{H}$ has a normal parallel to the upward direction u and there is at least one d-tuple of halfspaces in $\mathcal{H}^d$ whose intersection is bounded in the downward direction $-\vec{u}$. Therefore, $(h_{l_1}, \ldots, h_{l_d})$ is well defined and the lowest point in $h_{l_1} \cap \cdots \cap h_{l_d}$ is unique.

Let $\mathcal{R}$ be the polyhedron $\mathcal{R} = \{h_{l_1} \cap \cdots \cap h_{l_d}\}$. Without loss of generality, we can assume that $\mathcal{P}$ is full dimensional and hence $\mathcal{R}$ is full dimensional. Let $\mathcal{R}_{l_i}$ be the intersection of the halfspaces in $\{h_{l_1}, \ldots, h_{l_d}\}$ except l_i i.e., $\mathcal{R}_{l_i} = \bigcap_{k\in[1,d],k\neq i} h_{l_k}$. Since each of the halfspaces contain at least an points from P, $|P \cap \mathcal{R}_{l_i}| \geqslant (d-1)an - (d-2)n$. Construct and return the set $Q = \{x\} \cup Q' \cup Q_{l_1} \cup \cdots \cup Q_{l_d}$, where

1. x is the unique lowest point in $h_{l_1} \cap \cdots \cap h_{l_d}$;
2. Q' is an ϵ_r^d-net for the point set $P \setminus (P \cap h_{l_1} \cap \cdots \cap h_{l_d})$ using r points;
3. Q_{l_i} is an ϵ_s^d-net for the point set $P \setminus (P \cap R_{l_i})$ using s points.

Lemma 2.1. *Q is an a-net for P, and has size $r + ds + 1$.*

Proof. The size of Q is obvious from the construction, and we show that it is an a-net for the value required in the statement of the theorem. We first need the following crucial fact.

Claim 2.1. *Let $\mathcal{C}'$ be a convex set containing at least an points of P which does not contain x and contains points from $P \cap h_{l_1} \cap \cdots \cap h_{l_d}$. Then, $|P \cap C' \cap R_{l_i}| < bn$ for some $i \in [1, d]$.*

Proof. For contradiction, assume that $\mathcal{C}'$ intersects all R_{l_i} at least bn points of P. Let $\mathcal{R}'$ be the convex hull of $P \cap \mathcal{C}'$. Then, $\mathcal{R}'$ does not contain x, and therefore there exists a halfspace h' defining a facet of $\mathcal{R}'$ such that $\mathcal{R}' \subseteq h'$, and h' does not contain x. Since $\mathcal{R}$ intersects $h_{l_1} \cap \cdots \cap h_{l_d}$, (i) h' intersects $h_{l_1} \cap \cdots \cap h_{l_d}$, and (ii) h' contains at least an points of P (since $\mathcal{R}' \subseteq h'$), and (iii) $|P \cap h' \cap R_{l_i}| \geqslant bn\ \forall i \in [1, d]$.

Now, the lowest point z in $\mathcal{R} \cap h'$ is strictly higher than x (since h' does not contain x) and is defined by exactly d halfspaces from $\mathcal{H}$ since $\mathcal{R}$ is full dimensional and is defined by exactly d halfspaces from $\mathcal{H}$. Furthermore, the set of halfspaces defining z is $\{h'\} \cup \{h_{l_1}, \ldots, h_{l_d}\} \setminus h_{l_i}$ for some $i \in [1, d]$ and since $|P \cap C' \cap R_{l_i}| \geqslant bn\ \forall i \in [1, d]$, their intersection contains at least bn points from P. This is a contradiction to the assumption that $(h_{l_1}, \ldots, h_{l_d})$ has the highest lowest-intersection point among the d-tuples in $\mathcal{H}^d$. See Fig. 1 for an example in $\mathbb{R}^2$. □

We now show that any convex set $\mathcal{C}'$ containing an points must contain a point of Q by one of the following cases:

1. $\mathcal{C}'$ contains x, so is hit by Q.
2. $\mathcal{C}'$ does not contain points from $\mathcal{R}$. Since $|P \cap \mathcal{R}| \geqslant bn$, $\mathcal{C}'$ contains an points from the remaining set $P \setminus (P \cap \mathcal{R})$, whose size is at most $(1-b)n$. If $an \geqslant \epsilon_r^d(1-b)n$, then $\mathcal{C}'$ is hit by Q'.
3. $\mathcal{C}'$ does not contain x and yet contains points from $\mathcal{R}$. Then, by Claim 2.1, $\mathcal{C}' \cap R_{l_i} \leqslant bn$ for some $i \in [1, d]$. Then it must contain at least $an - bn$ points from $P \setminus (P \cap R_{l_i})$. If $an - bn \geqslant \epsilon_s^d(1 - ((d-1)a - (d-2)))n$, then $\mathcal{C}'$ is hit by Q_{l_i}.

Therefore, if

$$an \geqslant \epsilon_r^d(1-b)n \quad \text{and} \quad an - bn \geqslant \epsilon_s^d(d-1)(1-a)n \tag{1}$$

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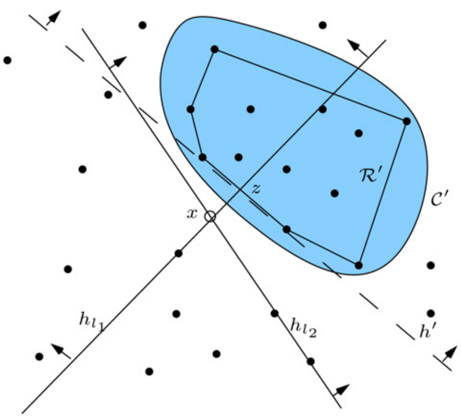

Fig. 1. Illustration of Theorem 2.1.

then $\mathcal{C}'$ is hit by Q. Maximizing a while satisfying (1) yields

$$\epsilon^d_{r+ds+1} \leqslant a = \frac{\epsilon^d_r \cdot (1 + (d-1)\epsilon^d_s)}{1 + \epsilon^d_r \cdot (1 + (d-1)\epsilon^d_s)},$$

completing the proof of Lemma 2.1 and hence Theorem 2.1. □

Remark. The above method actually gives another elementary proof of the centerpoint theorem in any dimension. The proof for two dimensions, as in the method of Theorem 2.1 is: consider all halfspaces containing more than $\frac{2}{3}n$ points, and take the pair with the highest lowest-intersection point x. This is the required point, since any convex set not containing this point cannot intersect the intersection of the halfspaces (Claim 2.1), which contains more than $n/3$ points of P. Hence, such a convex set can only contain the remaining points of P, of which there are fewer than $\frac{2}{3}n$. This follows from Theorem 2.1 by setting $r = s = 0$ and $d = 2$ to get $\epsilon^2_1 = \frac{2}{3}$! The proof for d-dimensions is exactly the same: consider sets of d halfspaces, each of which contains more than $\frac{d}{d+1}n$ points and choose the set with the highest lowest-intersection point (w.r.t. any dimension).

3. Consequences of main theorem

Improving upon previous work [2], we completely resolve the 2-point case in the plane.

Proposition 3.1. *Given a set P of n points in $\mathbb{R}^2$, the set of all convex sets which contain more than $\frac{4}{7}n$ points of P can be hit by* two *points (i.e., $\epsilon^2_2 \leqslant \frac{4}{7}$). Furthermore, there exist arbitrarily large point sets such that the set of all convex sets containing $\frac{4}{7}n$ points cannot be hit by two points.*

Proof. The upper bound follows from Theorem 2.1 by setting $r = 1, s = 0$ and $d = 2$.

Our lower bound construction is similar to the lower bound construction in [2]. We construct a set of P of 7 points such that for any two given points p and q in the plane there is a convex set which avoids both the points and contains 4 of the points in P. By replacing each of the points of P by a set of $n/7$ points (for arbitrary n) contained in a sufficiently small disk, one gets a set Q of size n such that no two points in the plane hit all the convex sets containing at least $\frac{4}{7}n$ points of Q.

Our set P is the set of vertices of regular heptagon. Let us name the vertices a, b, c, d, e, f and g in clockwise order. If one of the points p or q is identical to one of the 7 points, say a, then the other point cannot hit the convex sets $bcde$, $defg$ and $fgbc$ simultaneously since they don't have a common intersection. On the other hand, if neither p nor q is identical to any of the 7 points, then one of the closed halfspaces defined by the line passing through p and q contains 4 of the points of P whose convex hull is not hit by either p or q. □

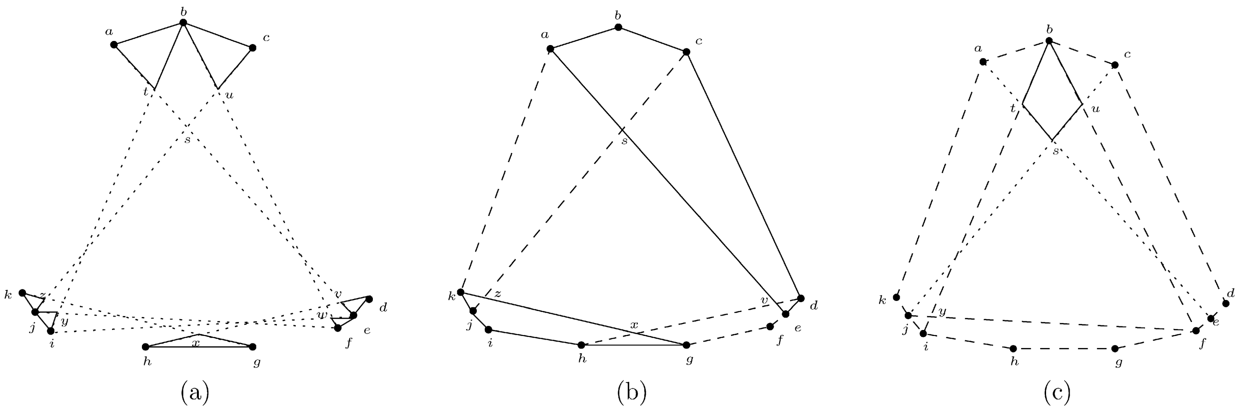

Fig. 2. (a) One of the seven (bold) triangles contains a point of the weak ϵ-net. (b) One of the four triangles jzk, gxh, dve or asc contains a point of the weak ϵ-net. (c) jyi contains a point of the weak ϵ-net.

Proposition 3.2. *Given P, the set of all convex sets which contain more than $\frac{8}{15}n$ points of P can be hit by* three *points (i.e., $\epsilon_3^2 \leqslant \frac{8}{15}$). Furthermore, there exist arbitrarily large point sets such that the set of all convex sets containing $\frac{5}{11}n$ points cannot be hit by three points.*

Proof. The upper bound follows from Theorem 2.1 by setting $r = 2$, $s = 0$ and $d = 2$.

The lower bound construction is as follows. We construct a set of 11 points such that for any three given points p, q and t in the plane there is a convex set containing 5 points from P and avoids all the three points. As in the proof of Proposition 3.1, one can replace each of these points with a set of $n/11$ points (for arbitrary n) contained in a sufficiently small disk and obtain a set Q of points such that no three points in the plane hits all the convex sets containing at least $\frac{5}{11}n$

Our set P is shown in Fig. 2(a). Assume that there are three points which hit all convex sets containing $\frac{5}{11}n$ points of P. We first show that none of these points can be identical to any of the 11 sets in the point set. Observe that if all the three points are identical to one of the 11 sets in the point set, then they cannot hit the convex hull of the remaining points, of which there are at least 8. Also, if two of the points p, q and t are identical to one of the points, then the remaining points, of which there are at least 9, can be used to define two convex sets containing 5 points each and sharing only one of the 11 points. A single point hitting both these sets should be identical to the shared point implying that all the three points are identical to one of the points. If only one of the points, say p, is identical to one of the 11 points, say the point k, then consider the convex sets $defgh$, $fghij$ and $jabcd$. Since q and t hit all the three sets, one of the points should be contained in the region $hvfg$, where v is intersection point of the segments fj and dh. Now, consider the sets $hijab$ and $bcdef$. The third point must hit both these sets and therefore must be identical to b.

Assuming that none of the points is identical to one of the 11 points, we show that if there exists a set of three points which hits all convex sets containing 5 points from P then one of those points is contained in one of the bold triangles shown in Fig. 2(a). Consider the four convex sets $jkabc$, $abcde$, $defgh$ and $ghijk$ (see Fig. 2(b)) containing 5 points each. In order to hit all the four sets, one of the three points must be in one the four triangles jzk, gxh, dve or asc. If there is a point in one of the triangles jzk, gxh or dve, we are done. So, assume that there is a point in the triangle asc. There cannot be two points in this region since then the remaining one point cannot hit the disjoint regions $ahijk$ and $cdefg$ simultaneously.

If the point in asc is in one of the triangles atb or buc (see Fig. 2(c)), we are done again. So, we assume that it is in the region $stbu$ but does not lie on bt or bu. Then, the regions $abijk$, $fghij$ and $bcdef$ must be hit by the other two points, and one of those must be in the triangle jyi (see Fig. 2(c)) since we have assumed that none of the points is identical to f.

Hence, one of the bold triangles shown in Fig. 2(a) must contain one of three weak ϵ-net points.

Assume that the triangle hxg contains one of the points (the other cases are analogous). Since the regions $abcdk$, $efijk$ and $defij$ must be hit by two points, the region $efijr$ must contain one of the points (see Fig. 3(a)). Now, since the regions $abcjk$ and $abcde$ must be hit by one point (see Fig. 2(a)), the region $abcs$ contains a point.

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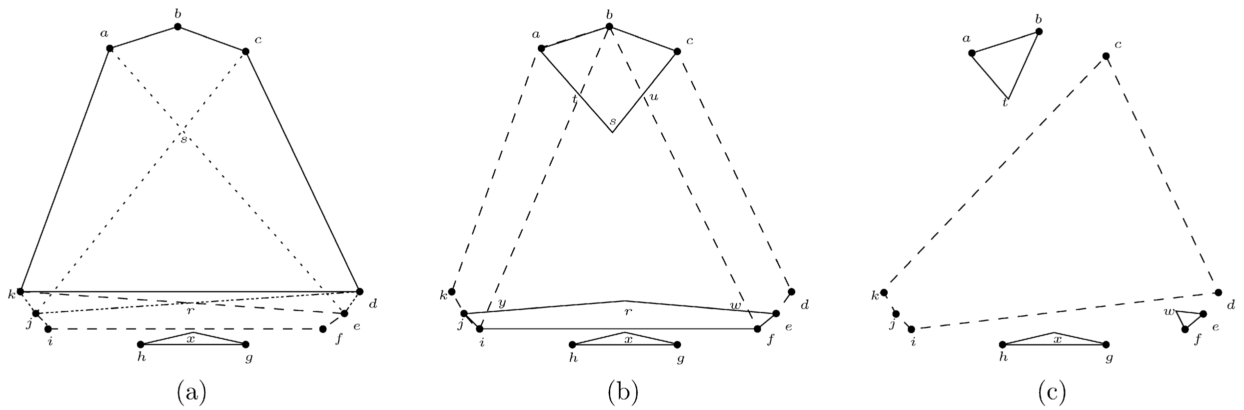

Fig. 3. (a) $efijr$ contains a point of the weak ϵ-net. (b) Either abt and efw contain one point each or buc and ijy contain one point each. (c) abt, efw and hxg contain one point each. Hence $cdijk$ cannot be hit.

Also, since the regions $abijk$ and $bcdef$ must be hit (see Fig. 3(b)), either the regions abt and efw contain one point each or the regions buc and ijy contain one point each. Since the cases are symmetric, let us assume that the regions abt and efw contain one point each.

But then, the region $cdijk$ does not contain any point (see Fig. 3(c)) although it contains 5 points of P. Hence, it is not possible to hit all the convex regions containing 5 points of P using 3 points. □

Aronov et al. [2] proved that $\epsilon_4^2 \leqslant \frac{4}{7}$. We actually are able to hit sets containing $\frac{4}{7}n$ points by just two points (Proposition 3.1). Theorem 2.1 yields $\epsilon_4^2 \leqslant \frac{16}{31}$, again improving upon Aronov et al.'s result. Improving upon a result of Alon et al. [1], Aronov et al. [2] showed that if each convex set contains $n/2$ points, then they can be hit by five points. Theorem 2.1 yields an improvement (set $r = 2$, $s = 1$, and $d = 2$).

Corollary 3.1. $\epsilon_4^2 \leqslant \frac{16}{31}$.

Corollary 3.2. $\epsilon_5^2 \leqslant \frac{20}{41}$.

4. Conclusions

We presented a general technique for constructing small number of points that hit all convex sets containing certain fractions of points of P. This then gives an optimal extension of the centerpoint to two points and improves the previous bounds for larger number of points. One intriguing open problem is whether the bound can be closed for the three-point case. Our work leaves a gap ($\frac{5}{11} \leqslant \epsilon_3^2 \leqslant \frac{8}{15}$), and it would be nice to get an optimal bound there.

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