Improving the First Selection Lemma in $\mathbb{R}^3$

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ABSTRACT

We present new bounds on the first selection lemma in $\mathbb{R}^3$. This makes progress on the open problems of Bukh, Matoušek and Nivash [6] and Boros-Füredi [4] for the three-dimensional case, improving the previously best result of Wagner [8]. While our results narrow the gap between the current best lower and upper bounds, they do not settle this question. However, they indicate that it is the current lower-bounds that are not tight, and we conjecture that the lower-bounds can be further improved to match the current upper bound.

Categories and Subject Descriptors

G.2 [**Combinatorics**]: Counting Problems

General Terms

Theory

Keywords

Centerpoints, First Selection Lemma, Hitting Simplices, Location Depth

1. INTRODUCTION

The so-called First Selection Lemma [7] states the following: given any set P of n points in $\mathbb{R}^d$, there exists a point in $\mathbb{R}^d$ contained in at least $c_d n^{d+1} - O(n^d)$ simplices spanned by P. It is a fundamental result that has had several applications in discrete geometry [7]. In particular, it was used to construct weak ϵ-nets [1], which were crucially used in the solution of the famous Hadwiger-Debrunner conjecture [2].

The first selection lemma is closely linked to the following useful property of point sets: given any set P of n points in $\mathbb{R}^d$, there always exists a point q such that any closed halfspace containing q contains at least $n/(d+1)$ points of P. Such a point is called a *centerpoint*. In general, a point p has *depth* t w.r.t. P if any closed halfspace containing p contains at least t points of P. The centerpoint theorem guarantees a point of depth $n/(d+1)$. The depth of P is defined as the maximum depth of any point $p \in \mathbb{R}^d$ w.r.t. P. Currently the best known bounds for c_d for $d \geq 3$ are achieved by using *any* centerpoint.

In the past several decades, the value of the constant c_d has been investigated in a series of papers. Unfortunately, there still remains a large gap between the current upper and lower bounds. Bárány [3] proved that $c_d \geq \frac{1}{d!(d+1)^{d+1}}$. For $d = 2$, this proves the existence of a point in $n^3/54$ of the triangles defined by any point set of size n in the plane. This was improved to $n^3/27$ in [4, 6] and was also shown to be optimal (see Bukh [5] for another proof which is simple and elegant).

Bárány's bound was improved to $\frac{d^2+1}{(d+1)!(d+1)^{d+1}}$ by Wagner [8], who in fact showed that any point of depth τn is contained in at least the following number of simplices:

$$\frac{(d+1)\tau^d - 2d\tau^{d+1}}{(d+1)!} \cdot n^{d+1} - O(n^d) \tag{1}$$

The recent work of Bukh, Matoušek and Nivash [6] is devoted to the investigation of upper bounds on the value of c_d. In particular, they showed that in $\mathbb{R}^d$, the centerpoint cannot give a better bound than the one above (set $\tau = 1/(d+1)$). Their main result is an elegant construction of a point set P so that no point in $\mathbb{R}^d$ is contained in more than $(n/(d+1))^{d+1}$ simplices defined by P. Furthermore, they conjecture that this is the right bound, and leave improving the lower bound as their main open problem.

Our Result

We make progress on the above problem, which was left as the main open problem in [6], by improving the bounds for the first selection lemma in $\mathbb{R}^3$ by a factor of 1.4, presented in Section 2. In particular, we prove the following:

THEOREM 1. *Let P be a set of n points in $\mathbb{R}^3$. Then there exists a point contained in at least $0.00227 \cdot n^4$ simplices spanned by P.*

The previous best result, which follows from (1), proves that given any P, there exists a point contained in at least $(5/12) \cdot (n/4)^4$ simplices (this is the centerpoint). While our result does not settle the question, it shows that it is the current lower-bounds which are not tight, and gives more strength to their conjecture.

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2. IMPROVING FIRST SELECTION LEMMA IN $\mathbb{R}^3$

The bound of Wagner [8] improves with the depth of the point set P. Our simple idea is to show that when the depth of P is low, one can also get a better bound. For example, when $depth(P) = n/(d+1)$, then the conjecture of [6] is in fact proven below. By combining the two, one gets an overall improvement. In particular, let $depth(P) = \tau n$. If $\tau \geq 0.3$, we will apply Wagner's result. We therefore concentrate on the case where $\tau < 0.3$, for which we present an improved bound below. We will use the following lemma, proven in [4].

LEMMA 1 (BOROS-FÜREDI [4]). *Given a set P of n points in $\mathbb{R}^d$, where $depth(P) = \tau n$, there exists a point p with depth τn, and a set $\mathcal{H}$ of $d+1$ halfspaces $\{h_1, \ldots, h_{d+1}\}$, such that i) $|h_i \cap P| = \tau n$, ii) p lies on the boundary plane of each h_i, and iii) $h_1 \cup \ldots \cup h_{d+1}$ cover the entire $\mathbb{R}^d$.*

We first need a technical lemma which can be seen as a generalization of Caratheodory's theorem:

LEMMA 2. *Let $P' = \{p_1, \ldots, p_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$. Then any point $x \in conv(P')$ lies in at least two d-simplices spanned by P'.*

PROOF. Take any point of P', say p_1, and consider the ray emanating from p_1 and passing through x. This ray, after passing through x, intersects $conv(P')$ in a $(d-1)$-simplex defined by d points, say P''. Then $P'' \cup p_1$ contains x, and has size $d+1$. Let p_i be the remaining point in $P \setminus (P'' \cup \{p_1\})$. Repeating the same procedure of shooting a ray from p_i through x results in another d-simplex, with p_i as one of its points, that contains x. □

Given a set P of n points in $\mathbb{R}^3$, with $depth(P) = \tau n$, use Lemma 1 to get the point p and a set of four halfspaces $\mathcal{H} = \{h_1, h_2, h_3, h_4\}$ satisfying the stated conditions. Define the following subsets of P:

$$\begin{aligned} A_i &= P \cap \Big(\bigcap_{l \neq i} \overline{h}_l\Big) \cap h_i, \\ B_{i,j} &= P \cap \Big(\bigcap_{l \neq i,j} \overline{h}_l\Big) \cap h_i \cap h_j, \\ C_i &= P \cap \Big(\bigcap_{l \neq i} h_l\Big) \cap \overline{h}_i \end{aligned} \tag{2}$$

Set $\alpha_i = |A_i|/n$, $\beta_{i,j} = |B_{i,j}|/n$, and $\gamma_i = |C_i|/n$. Our main lemma is the following.

LEMMA 3. *Let P be a set of n points in $\mathbb{R}^3$, with $depth(P) = \tau n$. Then, there exists a point contained in at least $g(P) \cdot n^4$ simplices spanned by P, where*

$$\begin{aligned} g(P) \;=\; & \left(\prod_i \alpha_i\right) + \left(\sum_{i,j} \beta_{i,j} \cdot \frac{\prod_l \alpha_l}{\max(\alpha_i, \alpha_j)}\right) \\ & + \left(\sum_i \gamma_i \cdot \frac{\prod_l \alpha_l}{\max_{l \neq i} \alpha_l}\right) \end{aligned} \tag{3}$$

PROOF. Let p be the point from Lemma 1, together with the four halfspaces $h_1, \ldots, h_4$. We first show that the simplex spanned by any four points, one from each of A_i, will always contain p.

Claim 1. Let $p_1, p_2, p_3, p_4 \in P$ be four points of P, such that $p_i \in A_i$. Then $p \in conv(\{p_1, p_2, p_3, p_4\})$.

PROOF. For $i \in \{1,2,3,4\}$, let r_i be the ray emanating from p and passing through p_i. Assume, for contradiction, that $conv(\{p_1, p_2, p_3, p_4\})$ does not contain p. Then there exists a hyperplane h that separates p from $conv(\{p_1, p_2, p_3, p_4\})$. For $i \in \{1,2,3,4\}$, define q_i to be the point $r_i \cap h$. By Radon's theorem, there exists disjoint sets Q_1, Q_2 and a point s so that $Q_1 \cup Q_2 = \{q_1, q_2, q_3, q_4\}$ and $s \in conv(Q_1) \cap conv(Q_2)$. For any $i \in \{1,2,3,4\}$, consider the halfspace h_i. By definition h_i passes through p, contains p_i and does not contain any other point p_j with $j \neq i$. Clearly then h_i contains q_i and does not contain any other point q_j with $j \neq i$. Suppose, without loss of generality, that $q_i \in Q_1$. Then, h_i does not contain any of the points in Q_2. Since, $s \in conv(Q_2)$, h_i does not contain s. This implies that none of the h_i's contain s, contradicting the assumption that union of the h_i's covers $\mathbb{R}^3$. □

The total number of such simplices is $n^4 \cdot \prod_i \alpha_i$, which is the first term in Equation (3). Call any such simplex a *basic simplex*, i.e., a simplex on $p_1, p_2, p_3, p_4 \in P$ is basic iff $p_i \in A_i$ for all i. All other simplices are called non-basic.
Now we use basic simplices, which always contain p, to prove the existence of several other simplices which must also contain p.

Claim 2. Let $P' = \{p_1, p_2, p_3, p_4, p_5\} \subset P$ be five points of P, such that $p_k \in A_k$, $k = 1 \ldots 4$, and $p_5 \in B_{i,j}$, for any $1 \leq i < j \leq 4$. Then either the simplex defined by $P' \setminus p_i$ or by $P' \setminus p_j$ contains p.

PROOF. By Claim 1, the basic simplex defined by p_1, p_2, p_3, p_4 contains p. Therefore, by Lemma 2, at least one other simplex spanned by P' must contain p. Note that this simplex must have p_5 as one of its points. Also, it must contain p_k, where $k \neq i, j$, since the plane h_k separates $P' \setminus p_k$ from p, as it follows from the definitions (2) that $p_l \in \overline{h}_k$ for all $l \neq k$, and $p_5 \in \overline{h}_k$ since $p_5 \in B_{i,j}$. So for this second simplex, the only possible choice is for the fourth vertex, which can be either p_i or p_j. □

For any fixed i, j, there are $n^5(\beta_{i,j} \cdot \prod_l \alpha_l)$ 5-tuples as in Claim 2, and each produces one d-simplex containing p. Each such d-simplex may be overcounted at most $n \cdot \max(\alpha_i, \alpha_j)$ times, so the total number of distinct d-simplices of the type in Claim 2 containing p are at least $n^4 \frac{\beta_{i,j} \cdot \prod_i \alpha_i}{\max(\alpha_i, \alpha_j)}$, which when summed over all i, j, forms the second term in (3).

Claim 3. Let $P' = \{p_1, p_2, p_3, p_4, p_5\} \subset P$ be five points of P, such that $p_k \in A_k$, $k = 1 \ldots 4$, and $p_5 \in C_i$, for any $1 \leq i \leq 4$. Then at least one of the three simplices defined by $\{p_5, p_i\} \cup P''$, $|P''| = 2$, contains p, where $P'' \subset P \setminus \{p_5, p_i\}$.

PROOF. As in Claim 2, at least one non-basic simplex spanned by P' must contain p, with p_5 as one of its points. Also, it must contain p_i: the plane h_i separates $P' \setminus p_i$ from p, as $P' \setminus p_i \subseteq \overline{h}_i$. The other two vertices of this second simplex must therefore be a subset of the remaining three vertices in P'. □

By similarly eliminating the overcounting, the d-simplices from Claim 3 form the third term of $g(P)$. Finally, note that no two simplices are counted twice in $g(P)$, since each contains exactly one point from a distinct region (one of $B_{i,j}$ or C_i). This concludes the proof of the lemma. □

It remains to show that regardless of the distribution of the points in A_i, $B_{i,j}$ and C_j, the quantity $g(P)$ is always bounded suitably from below.

LEMMA 4. *Let P be a set of n points in $\mathbb{R}^3$, with $depth(P) = \tau n$, and $g(P)$ as in Lemma 3. Then, if $\tau < 0.3$, $g(P) \geq \tau^2 \cdot (1-3\tau)^2 \cdot \frac{5\tau-1}{\tau}$.*

PROOF. Note the following for each $i = 1, \ldots, 4$:

$$\tau = |h_i \cap P| = \alpha_i + \sum_{j \neq i} \beta_{i,j} + \sum_{j \neq i} \gamma_j \tag{4}$$

Using the fact that $\alpha_i \leq \tau$, we get

$$\begin{aligned} g(P) &= \left(\prod_i \alpha_i\right) + \left(\sum_{i,j} \beta_{i,j} \cdot \frac{\prod_l \alpha_l}{\max(\alpha_i, \alpha_j)}\right) & (5) \\ &\quad + \left(\sum_i \gamma_i \cdot \frac{\prod_l \alpha_l}{\max_{l \neq i} \alpha_l}\right) \\ &\geq \left(\prod_i \alpha_i\right) + \left(\sum_{i,j} \beta_{i,j} \cdot \frac{\prod_l \alpha_l}{\tau}\right) & (6) \\ &\quad + \left(\sum_i \gamma_i \cdot \frac{\prod_l \alpha_l}{\tau}\right) \\ &= \left(\prod_i \alpha_i\right)\left(1 + \frac{\sum_{i,j} \beta_{i,j} + \sum_i \gamma_i}{\tau}\right). & (7) \end{aligned}$$

Since $\mathcal{H}$ covers the entire $\mathbb{R}^3$, we have $\cup_i (h_i \cap P) = P$, and so $\sum_i \alpha_i + \sum_{i,j} \beta_{i,j} + \sum_i \gamma_i = 1$. Summing up (4) for all four halfspaces, and subtracting the above from it, we get

$$\sum_{i,j} \beta_{i,j} + 2 \cdot \sum_i \gamma_i = 4 \cdot \tau - 1 \tag{8}$$

Therefore, $\sum_{i,j} \beta_{i,j} + \sum_i \gamma_i \leq 4\tau - 1$. This, together with Equation (4), implies that $(1-3\tau) \leq \alpha_i \leq \tau$ for $i = 1 \ldots 4$. Assuming $\tau < 0.3$, we can show the following:

Claim 4. The bound in equation (7) is minimized when $\sum_{i,j} \beta_{i,j} + \sum_i \gamma_i = 4\tau - 1$ or equivalently, when $\sum_i \gamma_i = 0$ (since $\sum_{i,j} \beta_{i,j} + 2\sum_i \gamma_i = 4\tau - 1$).

PROOF. Suppose that $\sum_i \gamma_i = \epsilon + \epsilon_1$, where $\gamma_1 = \epsilon_1 > 0$. We show that the variables $\alpha_i, \beta_{i,j}, \gamma_1$ can be re-adjusted to new values $\alpha_i', \beta_{i,j}', \gamma_1'$ to give a smaller value in Equation (7), while still satisfying all the constraints in Equation (4), and where $\gamma_1' = 0$. As long as $\sum_i \gamma_i > 0$, we can iteratively apply this procedure for all $\gamma_j > 0$ to make $\sum_i \gamma_i = 0$ without increasing the lower bound. Set $\gamma_1' = 0$, $\beta_{i,j}' = \beta_{i,j} + \frac{2\epsilon_1}{3}$ for all $i, j \neq 1$, and $\alpha_i' = \alpha_i - \epsilon_1/3$ for all $i \neq 1$. One can verify that Equation (4) still holds for each halfspace, and $\sum \alpha_i + \sum \beta_{i,j} + \sum \gamma_i = 1$. This further implies, as before, that $(1-3\tau) \leq \alpha_i' \leq \tau$; in particular, $\alpha_i' \geq 0.1$ as $\tau \leq 0.3$, We now have to show that the function can only decrease:

$$\left(\prod_i \alpha_i'\right)\left(1 + \frac{\sum \gamma_i' + \sum \beta_{i,j}'}{\tau}\right) \leq \left(\prod_i \alpha_i\right)\left(1 + \frac{\sum \gamma_i + \sum \beta_{i,j}}{\tau}\right)$$

Note that $\sum_{i,j} \beta_{i,j} + \sum_i \gamma_i = 4\tau - 1 - \epsilon - \epsilon_1$, and $\sum_{i,j} \beta_{i,j}' + \sum_i \gamma_i' = 4\tau - 1 - \epsilon$. So

$$\begin{aligned} \left(\prod_{i \neq 1} \alpha_i - \frac{\epsilon_1}{3}\right)\left(\frac{5\tau - 1 - \epsilon}{\tau}\right) &\leq \left(\prod_{i \neq 1} \alpha_i\right)\left(\frac{5\tau - 1 - \epsilon - \epsilon_1}{\tau}\right) \\ \prod_{i \neq 1}\left(1 - \frac{\epsilon_1}{3\alpha_i}\right) &\leq 1 - \frac{\epsilon_1}{5\tau - 1 - \epsilon} \end{aligned}$$

Since each α_i, $i \neq 1$, can be at most $\tau - \epsilon_1$ (from (4)), we have to prove

$$\left(1 - \frac{\epsilon_1}{3(\tau - \epsilon_1)}\right)^3 \leq 1 - \frac{\epsilon_1}{5\tau - 1 - \epsilon}$$

By expanding, dropping the negative cubic term and simplifying, it remains to show that

$$4\tau \geq 1 + \epsilon - \epsilon_1 + \frac{\epsilon_1(5\tau - 1 - \epsilon)}{3(\tau - \epsilon_1)}$$

Since $\sum_{i,j} \beta_{i,j} + 2\sum_i \gamma_i = 4\tau - 1$, we have $\sum \gamma_i = \epsilon + \epsilon_1 \leq (4\tau - 1)/2$. So $4\tau \geq 2\epsilon + 1$, and it remains to show that

$$\frac{\epsilon_1(5\tau - 1 - \epsilon)}{3(\tau - \epsilon_1)} \leq \epsilon_1 \quad \text{or equivalently,} \quad 2\tau \leq 1 + \epsilon - 3\epsilon_1$$

Now one can verify that $2\tau \leq 1 - 3\epsilon_1$, since $\epsilon_1 \leq (4\tau - 1)/2$, and $\tau \leq 0.3$. ☐

It now follows from Claim 4 that

$$g(P) \geq \left(\prod_i \alpha_i\right)\left(1 + \frac{4\tau - 1}{\tau}\right) = \left(\prod_i \alpha_i\right)\left(\frac{5\tau - 1}{\tau}\right),$$

where $\sum \alpha_i = 2 - 4\tau$. The product $\prod \alpha_i$ is minimized, under the constraint that $\sum \alpha_i = 2 - 4\tau$, when three α_i's are minimum, and one is maximum. However, that is not allowed by Equation (4), as $(1 - 3\tau) \leq \alpha_i \leq \tau$. It turns out that under those constraints, the product is minimized when two of them are maximum, say $\alpha_1 = \alpha_2 = \tau$, and the other two are equally small, $\alpha_3 = \alpha_4 = 1 - 3\tau$, and then $\beta_{3,4} = 4\tau - 1$.

Claim 5. $\prod \alpha_i$ is minimized when $\alpha_1 = \alpha_2 = \tau$, and $\alpha_3 = \alpha_4 = (1 - 3\tau)$.

PROOF. Recall that $(1-3\tau) \leq \alpha_i \leq \tau$. If each $\alpha_i > (1-3\tau)$, pick the smallest of them, say α_4, and set $\alpha_4' = \alpha_4 - \epsilon$, for a small enough $\epsilon > 0$, and add this excess to any other variable that is less than τ, say α_3 (there always exists another variable less than τ, else $\sum \alpha_i > (1 - 3\tau) + 3\tau = 1 \geq 2 - 4\tau$ for $\tau \geq 1/4$, a contradiction). Then $(\alpha_3 + \epsilon)(\alpha_4 - \epsilon) < \alpha_3\alpha_4$ since $\alpha_4 \leq \alpha_3$, minimizing the product further. Similarly, α_3 is also $(1 - 3\tau)$ in the configuration minimizing $\prod \alpha_i$. So we get that $\alpha_1 + \alpha_2 = (2 - 4\tau) - 2(1 - 3\tau) = 2\tau$. And as each α is at most τ, this forces $\alpha_1 = \alpha_2 = \tau$. ☐

It can be verified that all the constraints are satisfied, and so we get the required lower bound for $g(P)$:

$$g(P) \geq \left(\prod_i \alpha_i\right)\left(\frac{5\tau - 1}{\tau}\right) \geq \tau^2 \cdot (1 - 3\tau)^2 \cdot \frac{5\tau - 1}{\tau}$$

This concludes the proof of Lemma 4. ☐

PROOF OF THEOREM 1. Wagner [8] proved that any point of depth $\tau \cdot n$ in $\mathbb{R}^3$ is contained in at least $(4\tau^3 - 6\tau^4) \cdot \frac{n^4}{4!}$ simplices spanned by P. Setting this equal to the bound from Lemma 4, one gets the value $\tau' = 0.2889$. If $depth(P)$ is less than $\tau' n$, $g(P) \geq 0.00227$. Similarly, for $depth(P) \geq \tau' n$, Wagner's bound gives a point contained in at least $0.00227 \cdot n^4$ simplices. ☐

3. CONCLUSION

The conjecture of [6], that there always exists a point contained in at least $(n/(d+1))^{d+1}$ simplices spanned by any n points in $\mathbb{R}^d$, is an elegant one. So far, it has only been proven in $\mathbb{R}^2$ [4], and in this paper, we have made a step towards the optimal bound for $\mathbb{R}^3$. This indicates that the current lower-bounds are not tight, and gives more strength to their conjecture. The full conjecture, however, is still open.

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