Representation theorems and the semantics of (semi)lattice based logics

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Overview

- Motivation
- Connection between different classes of models
- Representation theorems
- Examples
- Decidability results
- Automated theorem proving
- Conclusions

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Motivation

Logical consequence

provability relation	logical connective
$\vdash$	$\longrightarrow$

Residuation condition

$p, q \vdash r$ if and only if $p \vdash q \longrightarrow r$

Motivation. Premise combination

Structural rules

$$\frac{\Gamma, \Delta \vdash A}{\Gamma, Y, \Delta \vdash A} \qquad \frac{\Gamma, \Delta \vdash A}{\Delta, \Gamma \vdash A} \qquad \frac{\Gamma, X, X, \Delta \vdash A}{\Gamma, X, \Delta \vdash A}$$

(Weakening) (Exchange) (Contraction)

Motivation. Premise combination

Structural rules

$$\frac{\Gamma, \Delta \vdash A}{\Gamma, Y, \Delta \vdash A}$$

(Weakening)

$$\frac{\Gamma, \Delta \vdash A}{\Delta, \Gamma \vdash A}$$

(Exchange)

$$\frac{\Gamma, X, X, \Delta \vdash A}{\Gamma, X, \Delta \vdash A}$$

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(Weakening) (Exchange) (Contraction)

Examples

- Relevant logic weakening may not hold

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(Weakening) (Exchange) (Contraction)

Examples

– Relevant logic	weakening may not hold
– Linear logic	weakening, contraction do not hold

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(Weakening) (Exchange) (Contraction)

Examples

– Relevant logic	weakening may not hold
– Linear logic	weakening, contraction do not hold
– Lambek calculus	contraction, exchange do not hold

Motivation. Premise combination

Logical consequence

provability relation	logical connective
$\vdash$	$\rightarrow$

Residuation condition

$\phi, \psi \vdash \gamma$ if and only if $\phi \vdash \psi \rightarrow \gamma$

Motivation. Premise combination

Logical consequence

provability relation	logical connective
$\vdash$	$\longrightarrow$
$\leq$	$\rightarrow$

Residuation condition

$\phi, \psi \vdash \gamma$ if and only if $\phi \vdash \psi \longrightarrow \gamma$

$[\phi] \circ [\psi] \leq [\gamma]$ $\quad$ $[\phi] \leq [\psi] \rightarrow [\gamma]$

Motivation. Premise combination

Structural rules

$$\frac{\Gamma, \phi, \Delta \vdash A}{\Gamma, \psi, \phi, \Delta \vdash A}$$

(Weakening)

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$[\phi] \circ [\psi] \leq [\psi] \circ [\phi]$

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(Contraction)

$[\phi] \leq [\phi] \circ [\phi]$

$$\frac{(\phi_1, \phi_2), \phi_3 \vdash A}{\phi_1, (\phi_2, \phi_3) \vdash A}$$

(Regrouping)

associativity of $\circ$

$$\frac{\Gamma \vdash A \qquad \Delta, A, \Delta' \vdash B}{\Delta, \Gamma, \Delta' \vdash B}$$

(Cut)

$\leq$ partial order; $\circ$ monotone

Definitions

$(M, \leq)$ poset; $\circ, \rightarrow$: $M^2 \rightarrow M$

$\rightarrow$ is the left residuation associated with $\circ$ if $a \circ b \leq c$ iff $a \leq b \rightarrow c$.

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$(M, \leq, \circ, \rightarrow)$ left residuated semigroup if

- $(M, \circ)$ semigroup; $\circ$ monotone in all arguments
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$(M, \leq, \circ, \rightarrow, 1)$ left residuated monoid if

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BCC-algebras

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BCC-algebras

$(M, \vee, \circ, \rightarrow)$ left residuated semilattice if

- $(M, \vee)$ semilattice; $\circ$ join-hemimorphism in both arguments
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Definitions

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Commutative:

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Integral: $x \leq 1$ $\forall x \in M$

BCC-algebras

$(M, \vee, \wedge, \circ, \rightarrow)$ left residuated lattice if

- $(M, \vee, \wedge)$ lattice; $\circ$ join-hemimorphism in both arguments
- $\rightarrow$ left residuation associated with $\circ$.

Examples

Positive logics [Goldblatt 1974, Dunn 1995] Binary logics

- no implication in the language $\phi \vdash \psi$ $[\phi] \leq [\psi]$
- algebraic models: lattices with operators

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Logics based on Heyting algebras Post-style

- algebraic models: Heyting algebras with operators

$p \wedge q \leq r$ iff $p \leq (q \to r)$

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Logics based on residuated (semi)lattices Łukasiewicz-style

- algebraic models: residuated (semi)lattices with operators

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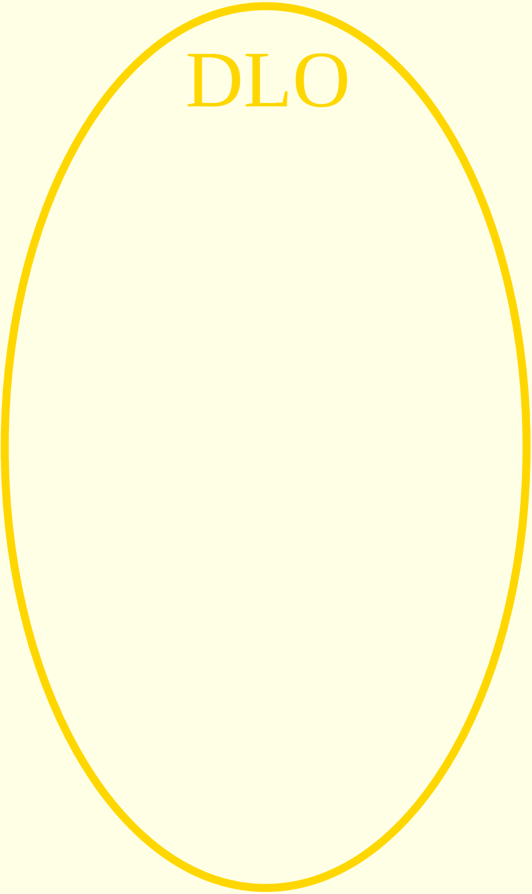
Examples

- positive logics [Dunn 1995]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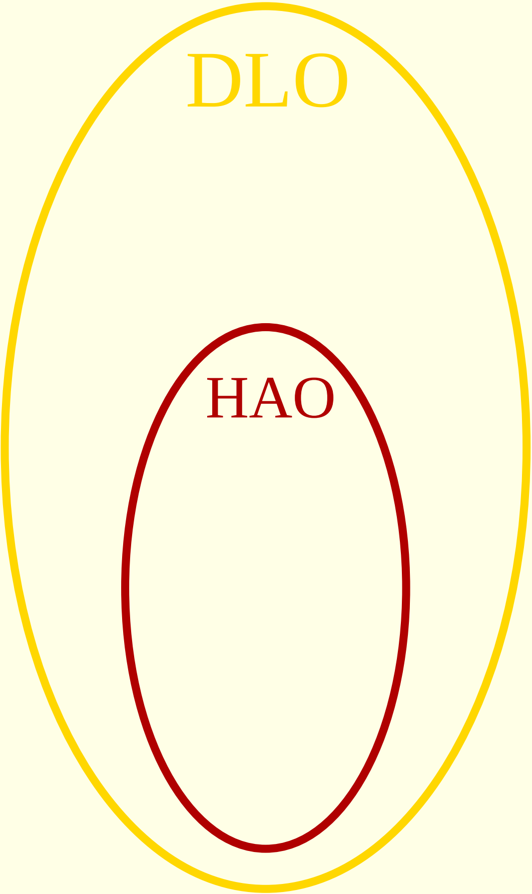

- positive logics [Dunn'95]

- (modal) intuitionistic logic
- Gödel logics [Gödel'30]
- SH_n, SHK_n logics [Iturrioz'82]
- Post logics and generalizations

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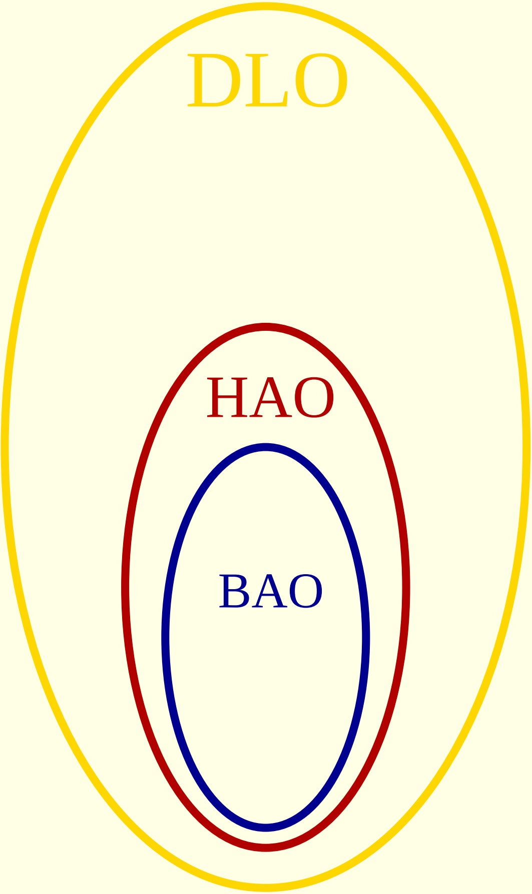

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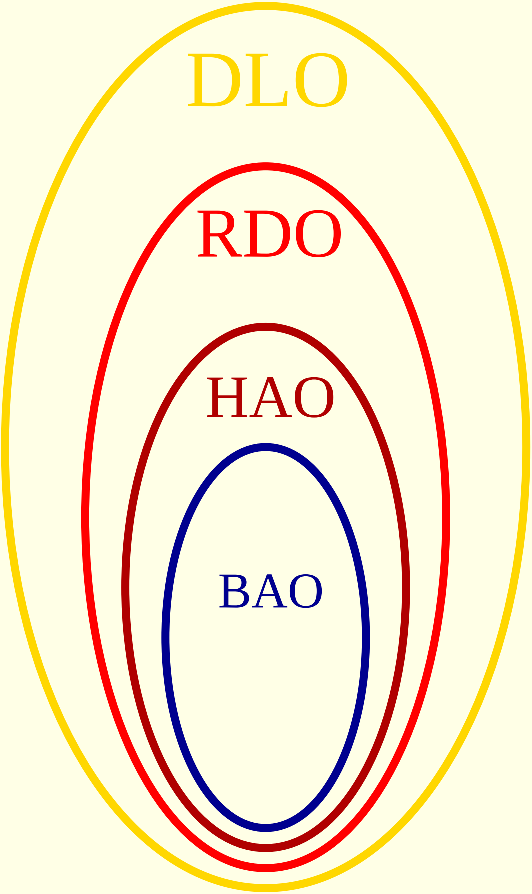

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- fuzzy logics

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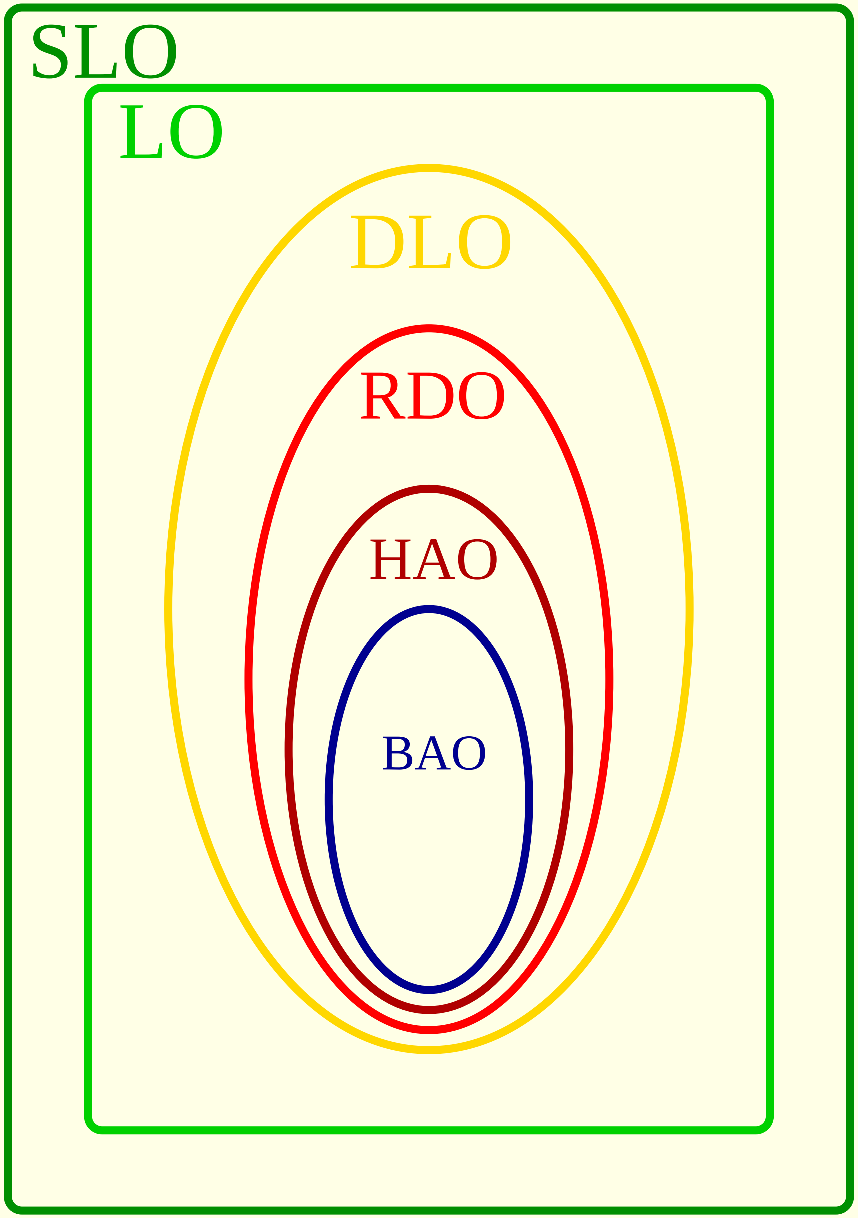

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- BCC and related logics
- Lambek calculus; linear logic ...

Motivation. Semantics

Algebraic models

(A, D)

$$\begin{tikzcd}\mathsf{Var} \arrow[r, "f"] \arrow[d] & A \\ \mathsf{Fma(Var)} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Motivation. Semantics

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Kripke-style models

$(W, \{R_W\}_{R \in \mathsf{Rel}})$ $m : \mathsf{Var} \to \mathcal{P}(X)$

meaning function

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Relational models

algebras of relations

Motivation. Decidability results

Logical calculi

- Gentzen-style calculi
- natural deduction
- hypersequent calculi [Avron 1991]

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Automated theorem proving

- embedding into FOL + resolution
- tableau methods
- natural deduction; labelled deductive systems

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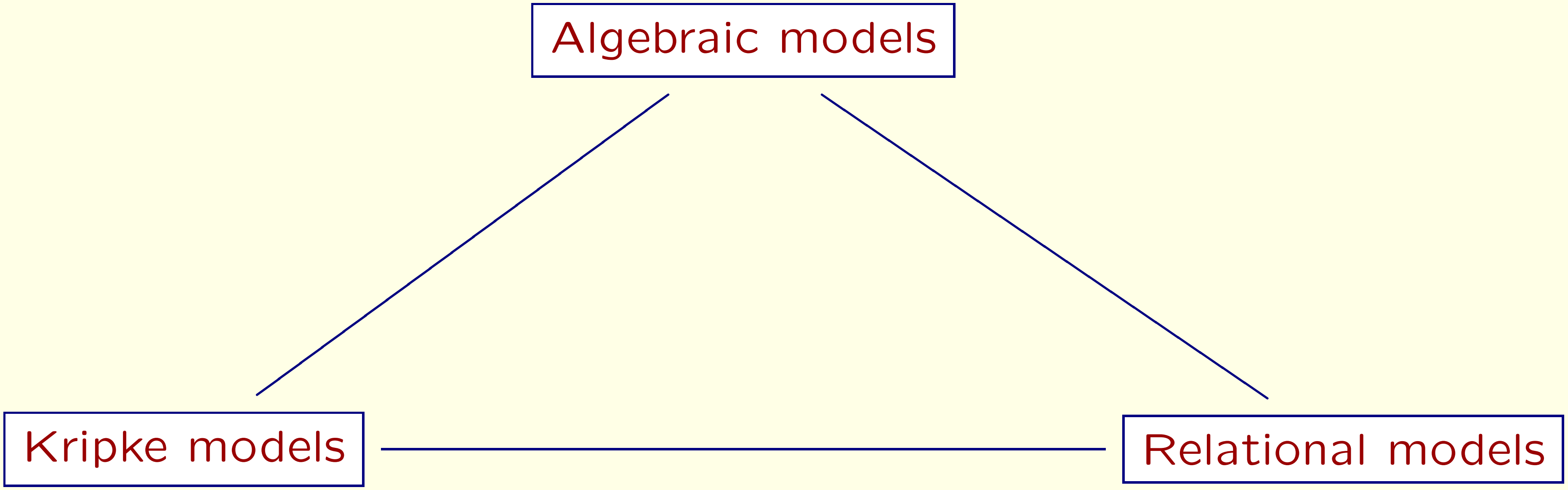
Connections between classes of models



Algebraic models

Kripke models

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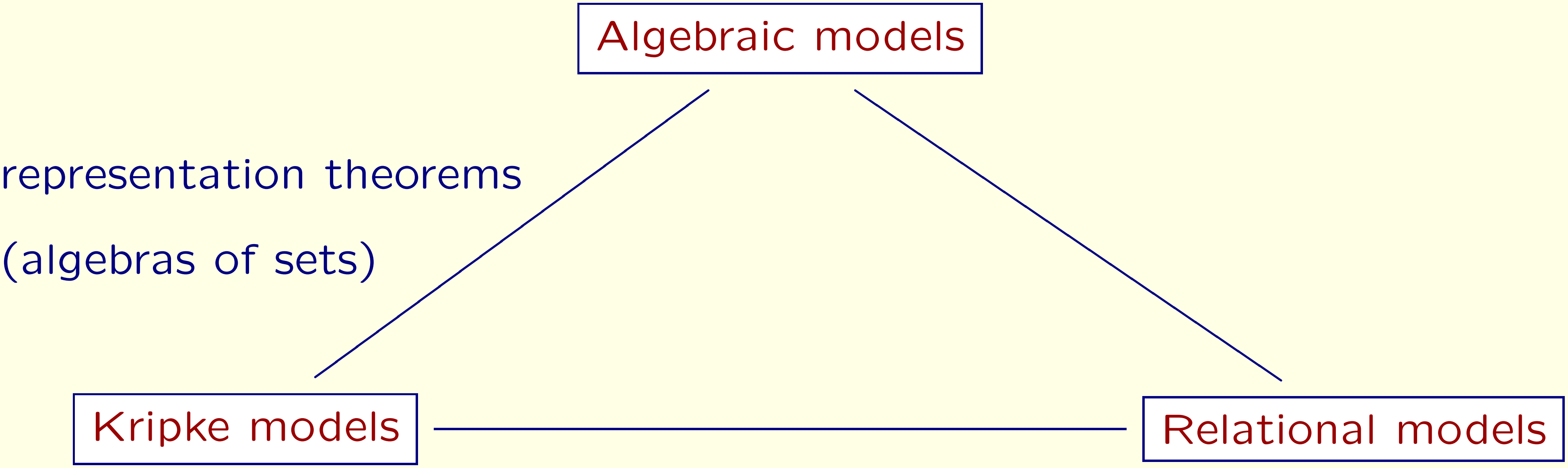


Algebraic models

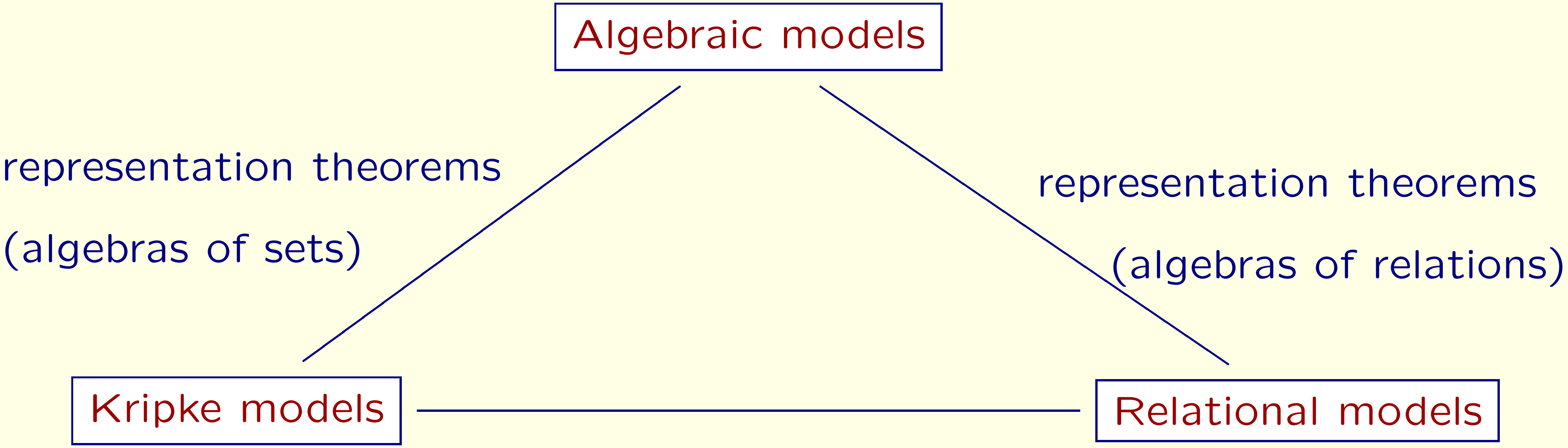
representation theorems

(algebras of sets)

Kripke models

Relational models

Connections between classes of models



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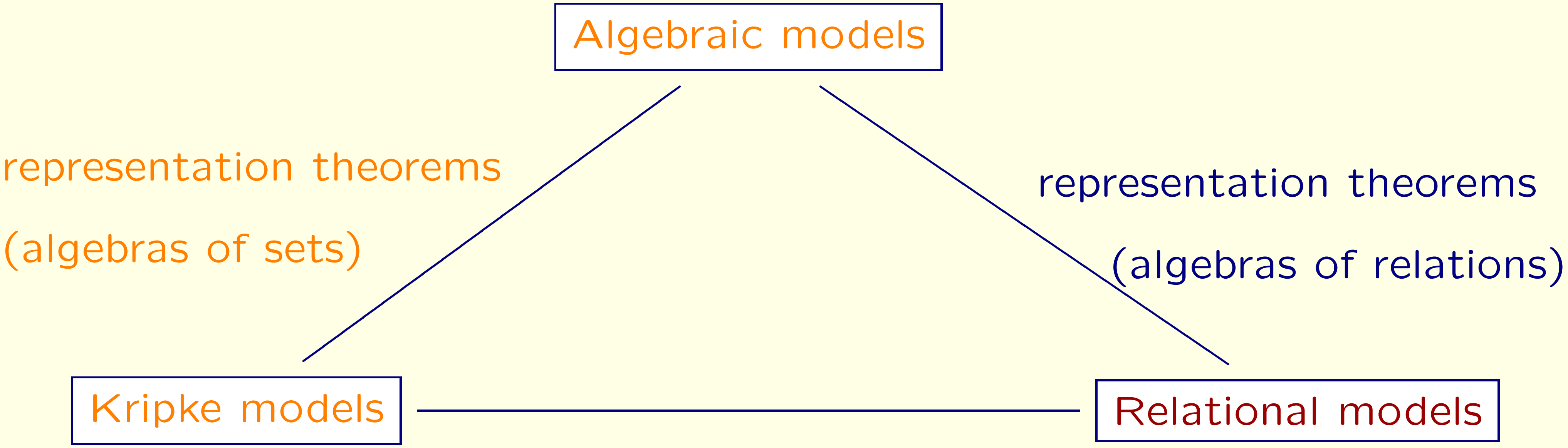

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Relational models

Algebraic and Kripke-style semantics

Algebraic models — Kripke-style models

Algebraic and Kripke-style semantics

Algebraic models ———— Kripke-style models

(C)

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathbb{D}"] & \mathcal{R} \arrow[l, shift left, "\mathbb{E}"] \end{tikzcd}$$

(i) $\mathbb{E}(K) \subseteq \mathcal{P}(K)$ algebra of subsets of K

(ii) $i : A \to \mathbb{E}(\mathbb{D}(A))$ injective homomorphism

Algebraic and Kripke-style semantics

Algebraic models —— Kripke-style models

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Algebraic and Kripke-style semantics

$$\begin{tikzcd} \boxed{\text{Algebraic models}} \arrow[r, dash] & \boxed{\text{Kripke-style models}} \end{tikzcd}$$

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Kripke-style models (K, m) $K \in \mathcal{R}; m : \mathsf{Var} \rightarrow \mathbb{E}(K) \subseteq \mathcal{P}(K)$

Algebraic and Kripke-style semantics

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Kripke-style models (K, m) $K \in \mathcal{R}; m : \mathsf{Var} \to \mathbb{E}(K) \subseteq \mathcal{P}(K)$

$(K, m) \overset{r}{\models}_x \phi$ iff $x \in \overline{m}(\phi)$

Algebraic and Kripke-style semantics

Algebraic models —— Kripke-style models

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$$(K, m) \overset{r}{\models}_x \phi \text{ iff } x \in \overline{m}(\phi)$$

$$K \overset{r}{\models} \phi \text{ iff } \mathbb{E}(K) \overset{a}{\models} \phi = 1.$$

Algebraic and Kripke-style semantics

Algebraic models —— Kripke-style models

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$\overset{r}{\models}$

Theorem If $\mathcal{A}, \mathcal{R}$ satisfy (C)(i,ii) then $\mathcal{A} \overset{a}{\models} \phi$ iff $\mathcal{R} \overset{r}{\models} \phi$.

Algebraic and relational semantics

Algebraic models —— Relational models

(C)
$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathbb{D}"] & \mathcal{R} \arrow[l, shift left, "\mathbb{E}"] \end{tikzcd}$$

(i) $\mathbb{E}(K)$ algebra of relations

(ii) $i : A \to \mathbb{E}(\mathbb{D}(A))$ injective homomorphism

Relational models (K, f) $K \in \mathcal{R}; f : \mathsf{Var} \to \mathbb{E}(K)$

$$\overset{a}{\models}$$

Theorem If $\mathcal{A}, \mathcal{R}$ satisfy (C)(i,ii) then $\mathcal{A} \overset{a}{\models} \phi$ iff $\mathcal{R} \overset{a}{\models} \phi$.

Representation theorems

Stone 1940: Bool

$B \overset{\sim}{\rightarrow} \mathsf{Clopen}(\mathcal{F}_m(B), \tau)$

$\eta_B(x) = \{F \in \mathcal{F}_m(L) \mid x \in F\}$

Priestley 1972: D_{01}

$L \overset{\sim}{\rightarrow} \mathsf{ClopenOF}(\mathcal{F}_p(L), \subseteq, \tau)$

$\eta_L(x) = \{F \in \mathcal{F}_p(L) \mid x \in F\}$

Representation theorems

Natural Dualities: $\mathcal{V} = ISP(P)$

$A \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{Rel}}(D(A), \underline{P})$

$\underline{P}$ 'alter-ego' of P

$D(A) = \mathsf{Hom}_{\mathcal{V}}(A, P)$

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Stone 1940: $\mathsf{Bool} = ISP(B_2)$

$B \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{St}}(D(B), \underline{B_2})$

$\eta_B(x)(h) = h(x)$

Priestley 1972: D_{01}

$L \overset{\sim}{\to} \mathsf{ClopenOF}(\mathcal{F}_p(L), \subseteq, \tau)$

$\eta_L(x) = \{F \in \mathcal{F}_p(L) \mid x \in F\}$

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Stone 1940: $\mathsf{Bool} = ISP(B_2)$

$B \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{St}}(D(B), \underline{B_2})$

$\eta_B(x)(h) = h(x)$

Priestley 1972: $D_{01} = ISP(L_2)$

$L \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{Pr}}(D(L), \underline{L_2})$

$\eta_L(x)(h) = h(x)$

Representation theorems

Natural Dualities: $\mathcal{V} = ISP(P)$

$A \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{Rel}}(D(A), \underline{P})$

$\underline{P}$ 'alter-ego' of P

$D(A) = \mathsf{Hom}_{\mathcal{V}}(A, P)$

Stone 1940: $\mathsf{Bool} = ISP(B_2)$

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$\eta_B(x)(h) = h(x)$

Priestley 1972: $D_{01} = ISP(L_2)$

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Semilattices: $SL = ISP(S_2)$

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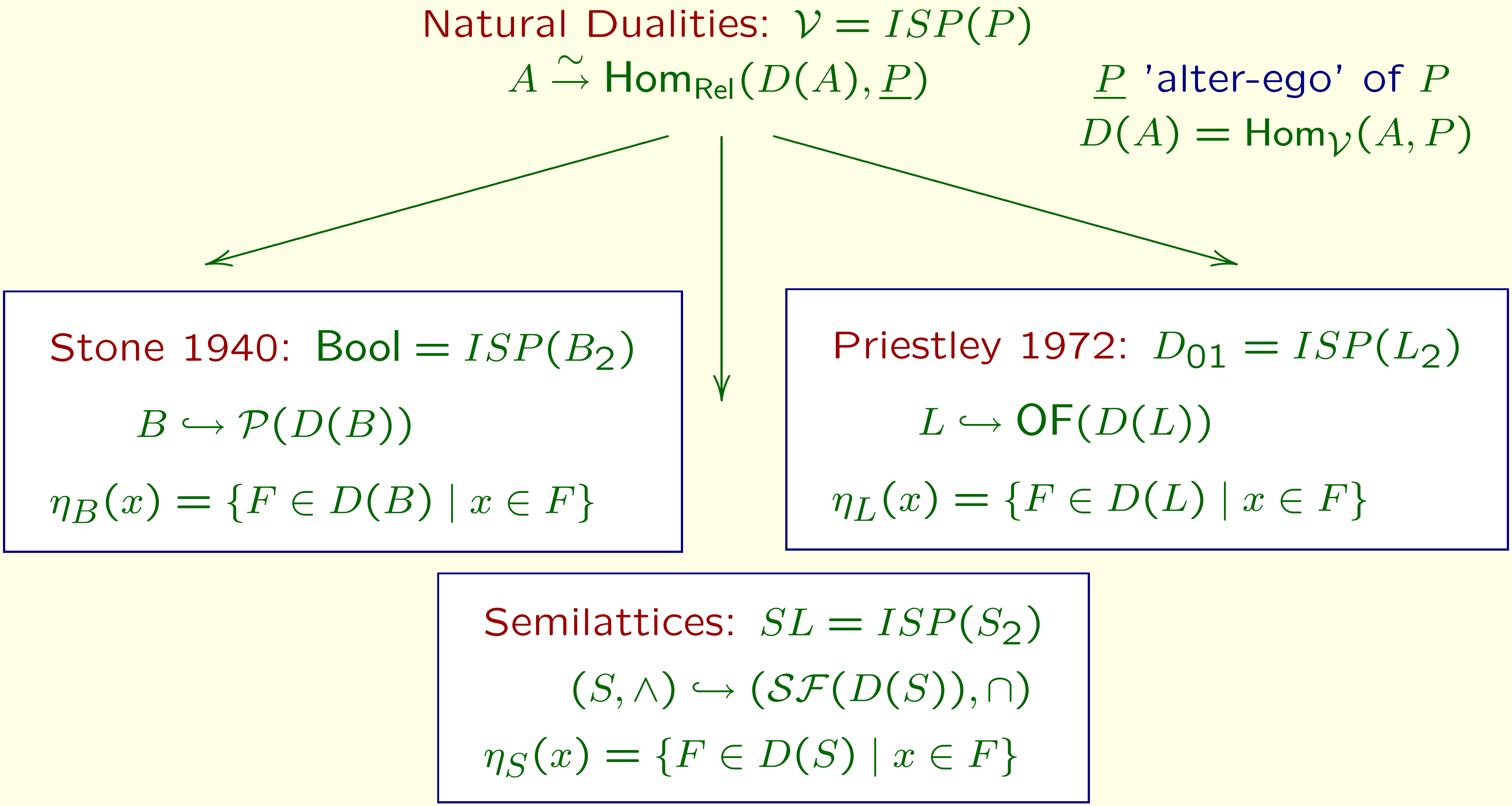
Representation theorems



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Semilattices: $SL = ISP(S_2)$

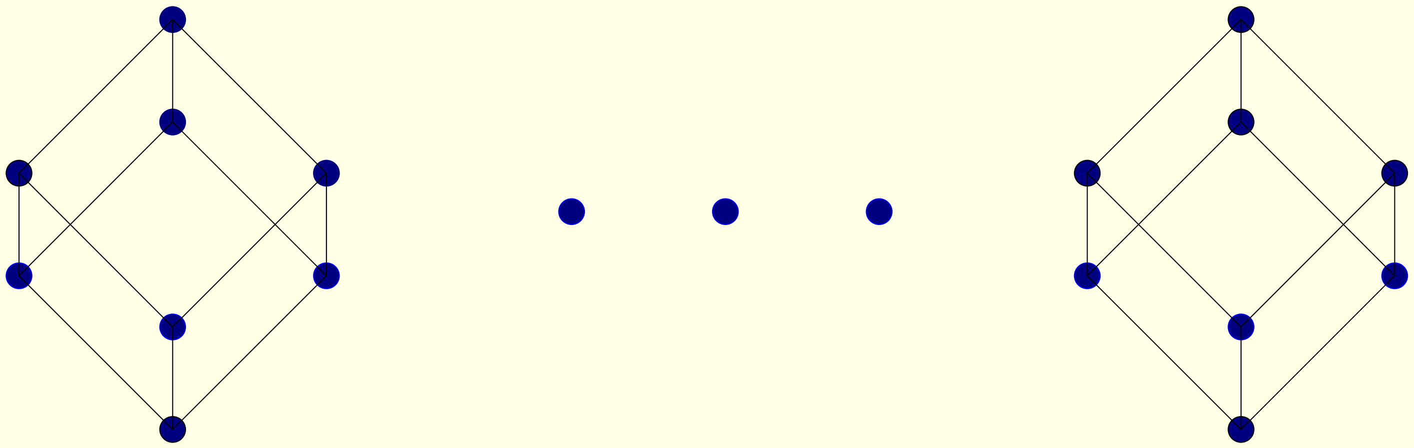
$(S, \wedge) \hookrightarrow (\mathcal{SF}(D(S)), \cap)$

$\eta_S(x) = \{F \in D(S) \mid x \in F\}$

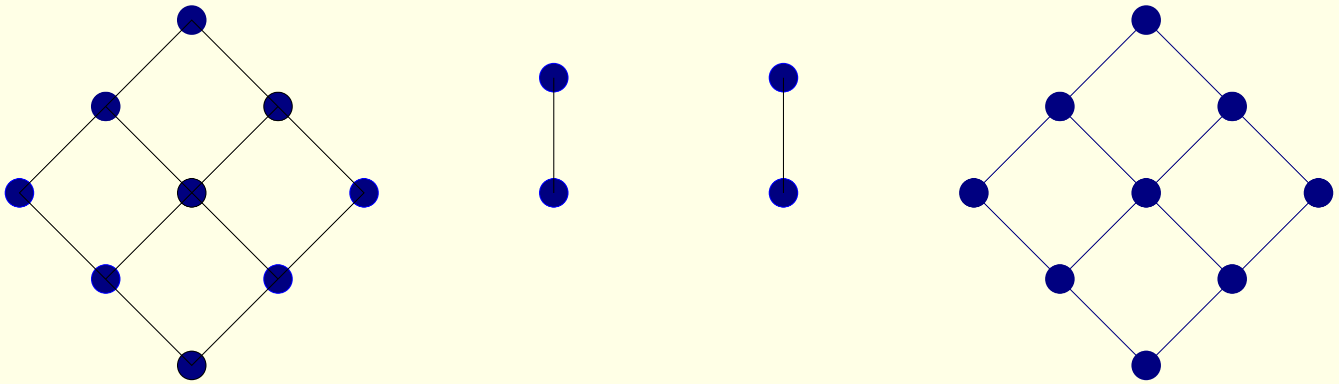
Lattices: $\eta_L : (L, \wedge, \vee) \hookrightarrow (\mathcal{SF}(D(L)), \cap, \vee)$

$\eta_L(x) := \{F \in D(L) \mid x \in F\}$

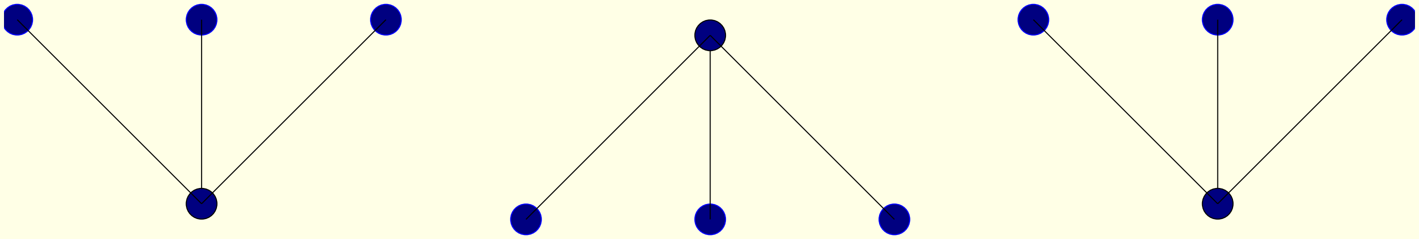
Example 1. Boolean algebras



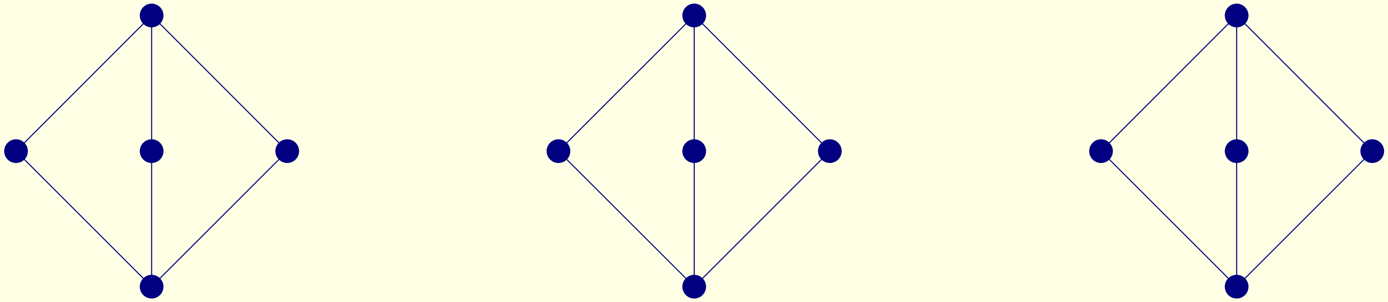
Example 2. Distributive lattices



Example 3. Semilattices



Example 4. Lattices



Other representation theorems

Boolean algebras with operators

- Jónsson and Tarski (1951)

Other representation theorems

Boolean algebras with operators

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Distributive lattices with operators

- Goldblatt (1986), VS (2000)

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General Idea:

- $A \mapsto D(A)$ topological space with additional structure
- $A \cong$ ClosedSubsets of $D(A)$

closed wrt: topological structure
order structure
...

- operators $\mapsto$ relations on $D(A)$

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"Gaggles", "tonoids" Dunn (1990, 1993)

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$f \in \Sigma_{\varepsilon_1 \dots \varepsilon_n \to \varepsilon}$: $\boxed{f_A : A^{\varepsilon_1} \times \cdots \times A^{\varepsilon_n} \to A^{\varepsilon} \text{ join-hemimorphism}}$

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$\mathbb{D}(A)$ $\quad R_f(F_1, \dots, F_n, F)$ iff $f(F_1^{\varepsilon 1}, \dots, F_n^{\varepsilon n}) \subseteq F^{\varepsilon}$

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Example $\quad x \circ y \leq z$ iff $x \leq y \to z$

$\circ$ has type $+1, +1 \to +1$ $\quad R_\circ(F_1, F_2, F_3)$ iff $F_1 \circ F_2 \subseteq F_3$

$\to$ has type $+1, -1 \to -1$ $\quad R_\to(F_1, F_2, F_3)$ iff $F_1 \to F_2^c \subseteq F_3^c$

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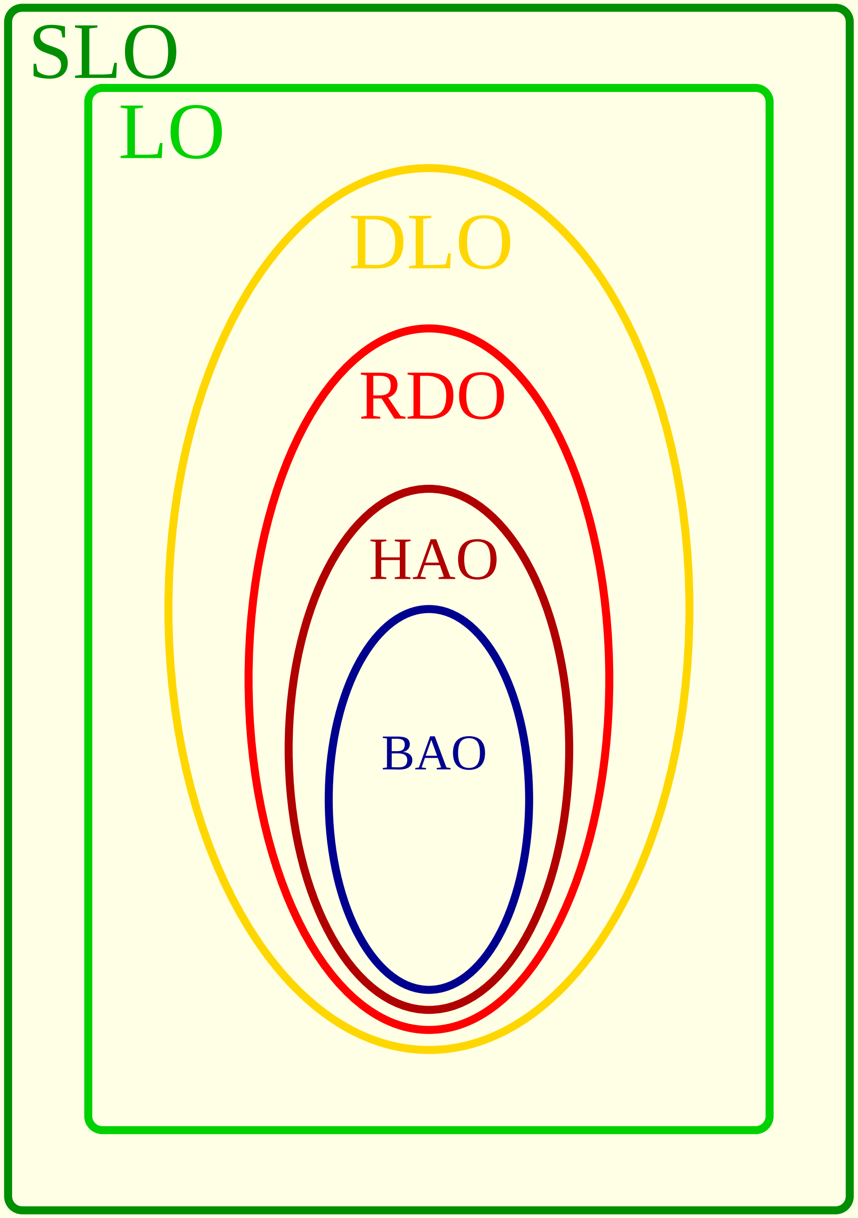
Algebraic and Kripke-style semantics



(C)

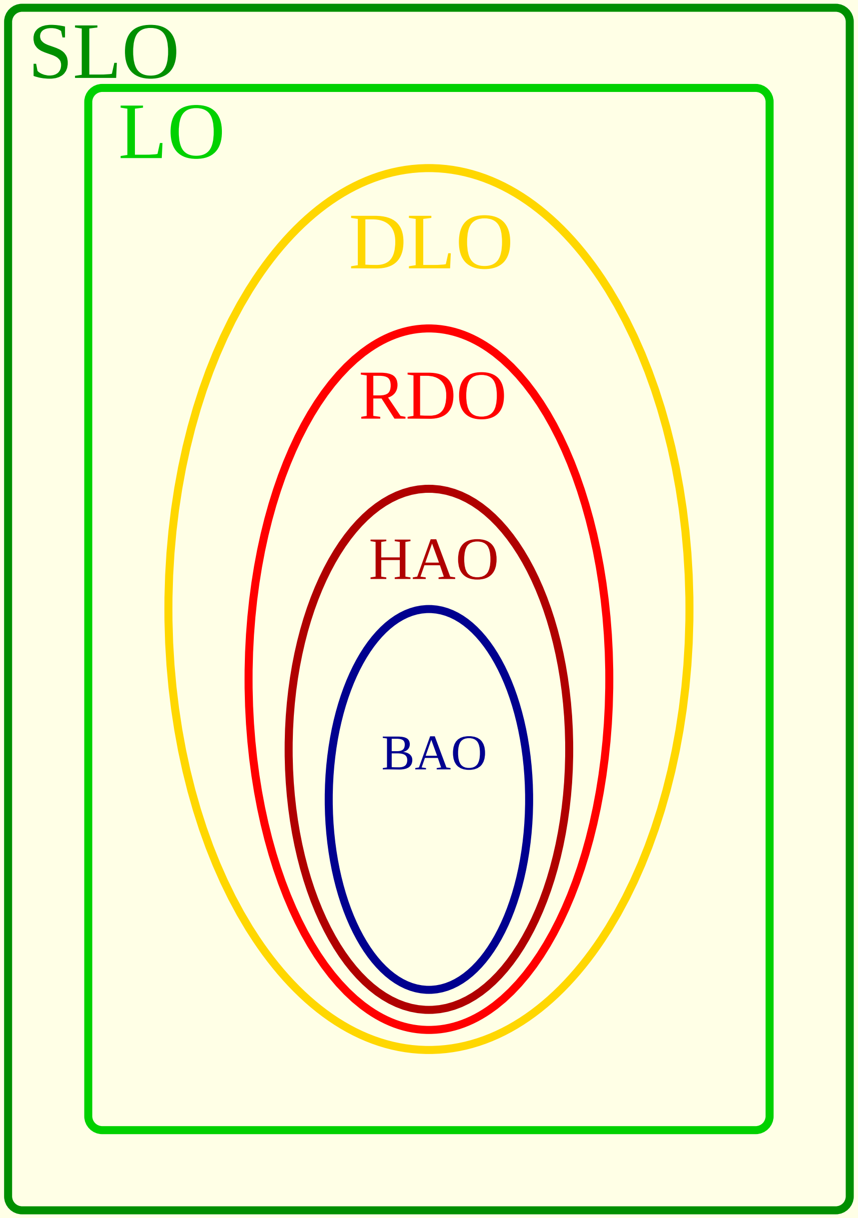
$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "\mathbb{D}"] & \mathcal{R} \arrow[l, shift left, "\mathbb{E}"] \end{tikzcd}$$

(i) $\mathbb{E}(K) \subseteq \mathcal{P}(K)$

algebra of subsets of K

(ii) $i : A \hookrightarrow \mathbb{E}(\mathbb{D}(A))$

Algebraic and Kripke-style semantics


SLO

LO

DLO

RDO

HAO

BAO

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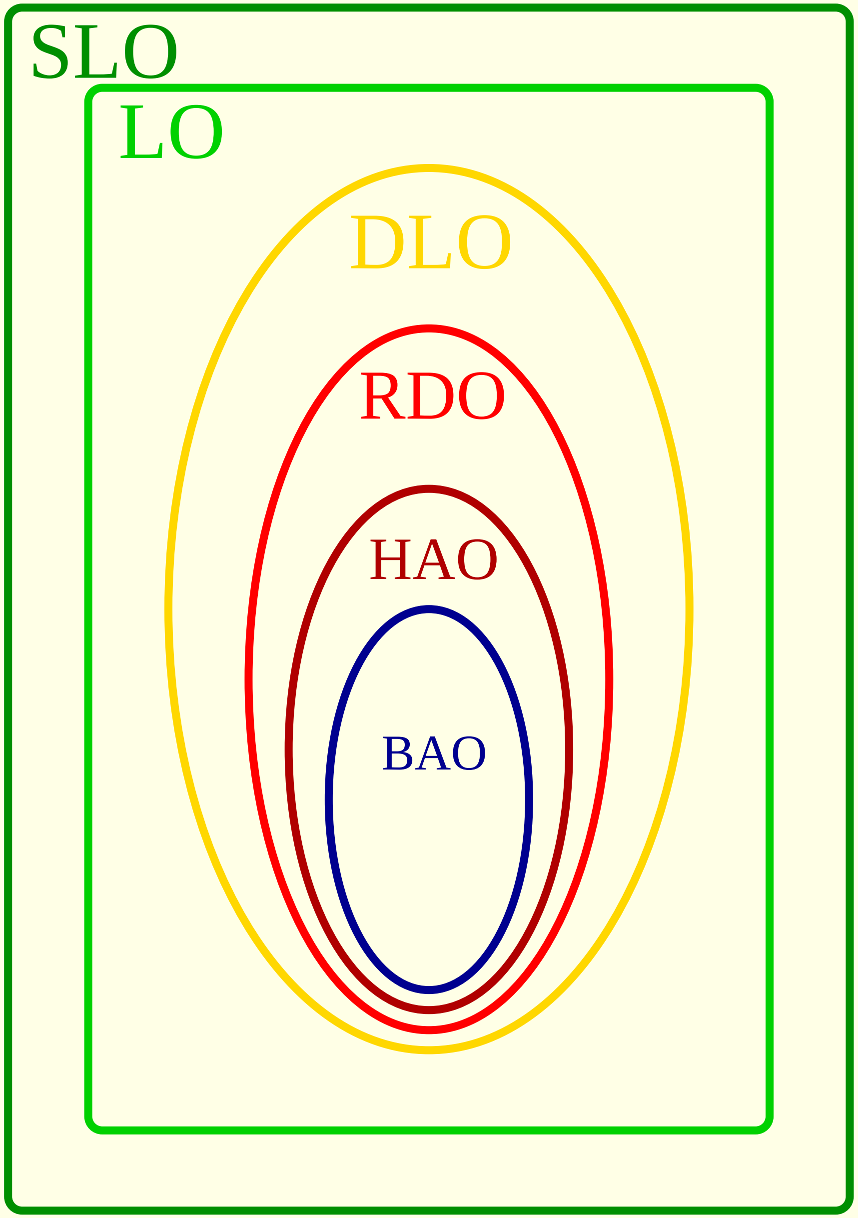

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DLO	Priestley representation
	$\eta_A : A \to \mathsf{OF}(D(A))$
SLO, LO	Representation for (semi)lattices
	$\eta_A : A \to \mathcal{SF}(D(A))$

Logic	Algebraic models	Kripke-style models	meaning functions
Positive	DLO$_{\Sigma}$ $(L, \vee, \wedge, 0, 1, \{f\}_{f \in \Sigma})$	Rp$_{\Sigma}$ $(X, \leq, \{R\}_{R \in \Sigma})$	m : Var $\to$ OF(X)
Post-style	HAO$_{\Sigma}$ $(L, \vee, \wedge, \Rightarrow, 0, 1, \{f\}_{f \in \Sigma})$	Rp$_{\Sigma}$ $(X, \leq, \{R\}_{R \in \Sigma})$	m : Var $\to$ OF(X)
	BAO$_{\Sigma}$ $(B, \vee, \wedge, 0, 1, \neg, \{f\}_{f \in \Sigma})$	BAO$_{\Sigma}$ $(X, \{R\}_{R \in \Sigma})$	m : Var $\to \mathcal{P}(X)$
Łukasiewicz-style	RDO $(L, \vee, \wedge, 0, 1, \circ, \to)$	RSp $(X, \leq, R_{\circ})$	m : Var $\to$ OF(X)
	RSO, RLO $(S, \wedge, 0, 1, \circ, \to)$ $(S, \vee, \wedge, 0, 1, \circ, \to)$	RSO, RLO $(X, \wedge, R_{\circ})$ $(X, \wedge, R_{\circ})$	m : Var $\to \mathcal{SF}(X)$

Overview

- Motivation
- Connection between different classes of models
- Representation theorems
- Examples
- Decidability results
- Automated theorem proving
- Conclusions

Class	u.w.p.	References
Lattices	PTIME	Skolem (1920), Burris (1995)
ResLatMon	decidable	Blok, Van Alten (1999)
ResLatIntMon	decidable	Blok, Van Alten (1999)
$\mathsf{BCK}_{\rightarrow}$	decidable	Blok, Van Alten (1999)
Modular Lattices	undecidable	Freese (1980), Herrmann (1983)
D_{01}	co-NP complete	Bloniarz et al.(1987)
DLO_{Σ}, RDO_{Σ}	EXPTIME	VS (1999, 2001)
$\mathsf{DLSgr}_{\vee,\mathsf{d}}$	decidable	Andreka
subclasses	undecidable	Urquhart (1995)
Heyting Algebras	DEXP	VS (1999)
$\mathsf{HASgr}_{\vee,\mathsf{d}}$	undecidable	Kurucz, Nemeti et al. (1993)
Boolean Algebras	co-NP complete	Cook (1971)
ResBoolMon	undecidable	Kurucz, Nemeti et al. (1993)
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Decidability results

Semantics

- Algebraic semantics

finite model property

(uniform) word problem decidable

Decidability results

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• Algebraic semantics	finite model property (uniform) word problem decidable
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- ○ embedding into FOL + ATP in first-order logic
- ○ tableau methods
- ○ natural deduction; labelled deductive systems

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- ∘ embedding into FOL + ATP in first-order logic
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Decidability results

Semantics

- Algebraic semantics
  - finite model property
  - (uniform) word problem decidable
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  - finite model property
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Resolution-based methods

Advantages

Resolution-based methods

Advantages

- direct encoding
- restricted (hence efficient) calculi
  - ordering, selection
  - simplification/elimination of redundancies
- allow use of efficient implementations

  (SPASS, Saturate)
- in many cases better than equational reasoning

  AC operators $\mapsto$ logical operations

Automated Theorem Proving: DLO$_\Sigma$

Theorem DLO$_\Sigma \models \phi_1 \leq \phi_2$ iff

the following conjunction is unsatisfiable:

(Dom) $x \leq x; \quad x \leq y, y \leq z \rightarrow x \leq z$

$R_f(x_1, \dots, x_n, x), x \bowtie_\epsilon y \Rightarrow R_f(x_1, \dots, x_n, y)$ if $f \in \Sigma_{\overline{\varepsilon} \rightarrow \varepsilon}$

(Her) $x \leq y, P_e(x) \Rightarrow P_e(y)$

(Ren)(0, 1) $\neg P_0(x) \qquad P_1(x)$

($\wedge$) $P_{e_1 \wedge e_2}(x) \Leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)$

($\vee$) $P_{e_1 \vee e_2}(x) \Leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)$

(Σ) $P_{f(e_1, \dots, e_n)}(x) \Leftrightarrow (\exists x_1, \dots \exists x_n \qquad f \in \Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}$

$(P_{e_1}(x_1)^{\varepsilon_1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon_n} \wedge R_f(x_1, \dots, x_n, x)))^{\varepsilon}$

(N) $\exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$

Automated Theorem Proving: DLO_Σ

Theorem $\mathsf{DLO}_\Sigma \models \phi_1 \leq \phi_2$ iff

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(Her)

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$(P_{e_1}(x_1)^{\varepsilon 1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon n} \wedge R_f(x_1, \dots, x_n, x)))^{\varepsilon}$

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Automated Theorem Proving: HAO$_\Sigma$

Theorem HAO$_\Sigma$ $\models \phi = 1$ iff

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$R_f(x_1, \dots, x_n, x), x \bowtie_\epsilon y \Rightarrow R_f(x_1, \dots, x_n, y)$ if $f \in \Sigma_{\overline{\varepsilon} \rightarrow \varepsilon}$

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$(P_{e_1}(x_1)^{\varepsilon_1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon_n} \wedge R_f(x_1, \dots, x_n, x)))^{\varepsilon}$

($\rightarrow$) $P_{e_1 \rightarrow e_2}(x) \Leftrightarrow \forall y (y \geq x \wedge P_{e_1}(y) \Rightarrow P_{e_2}(y))$

(N) $\exists c \in X : \neg P_\phi(c)$

Automated Theorem Proving

Class of algebras	Complexity (refinements of resolution)
DLO_Σ	EXPTIME
RDO_Σ	EXPTIME
BAO_Σ	EXPTIME
HA	DEXPTIME
HAO_Σ	?
$\mathsf{RSO}_\Sigma, \mathsf{RLO}_\Sigma$	?

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Automated theorem proving

- what presentation is better?

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  - logical calculus/semantics
  - what semantics: algebraic, Kripke or relational?

Questions

Automated theorem proving

- what presentation is better?
  - logical calculus/semantics
  - what semantics: algebraic, Kripke or relational?
- which methods for ATP are better?
  - resolution
  - tableaux
  - natural deduction
  - ...