Representation theorem
and the semantics of
(semi)lattice based logic

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Overview

- Motivation
- Connection between different classes of mo
- Representation theorems
- Examples
- Decidability results
- Automated theorem proving
- Conclusions

Motivation

Logical consequence

provability relation — logical co

$\vdash$ — $\longrightarrow$

Residuation condition

$p, q \vdash r$ if and only if $p \vdash q -$

Motivation. Premise combin

Structural rules

$$\frac{\Gamma, \Delta \vdash A}{\Gamma, Y, \Delta \vdash A}$$

(Weakening)

$$\frac{\Gamma, \Delta \vdash A}{\Delta, \Gamma \vdash A}$$

(Exchange)

$$\frac{\Gamma, X}{\Gamma, X}$$

(Co

Examples

- Relevant logic — weakening may not hold
- Linear logic — weakening, contraction do
- Lambek calculus — contraction, exchange do

Motivation. Premise combin

Logical consequence

provability relation	logical cor
$\vdash$	$\longrightarrow$
$\leq$	$\longrightarrow$

Residuation condition

$\phi, \psi \vdash \gamma$ if and only if $\phi \vdash \psi$ -

$[\phi] \circ [\psi] \leq [\gamma]$ $[\phi] \leq [\psi]$

Motivation. Premise combin

Structural rules

$$\frac{\Gamma, \phi, \Delta \vdash A}{\Gamma, \psi, \phi, \Delta \vdash A}$$

(Weakening)

$[\psi] \circ [\phi] \leq [\phi]$

$$\frac{\Gamma, \phi, \psi, \Delta \vdash A}{\Gamma, \psi, \phi, \Delta \vdash A}$$

(Exchange)

$[\phi] \circ [\psi] \leq [\psi] \circ [\phi]$

$$\frac{\Gamma, \phi}{\Gamma, \phi}$$

(Co

[ϕ

$$\frac{(\phi_1, \phi_2), \phi_3 \vdash A}{\phi_1, (\phi_2, \phi_3) \vdash A}$$

(Regrouping)

associativity of $\circ$

$$\frac{\Gamma \vdash A \quad \Delta, A, \Delta' \vdash B}{\Delta, \Gamma, \Delta' \vdash B}$$

(Cut)

$\leq$ partial order; $\circ$ monoto

Definitions

$(M, \leq)$ poset; $\circ, \rightarrow$: $M^2 \rightarrow M$

$\rightarrow$ is the left residuation associated with $\circ$ if $a \circ b \leq c$ iff

$\rightarrow$ is the right residuation associated with $\circ$ if $b \circ a \leq c$ iff

$(M, \leq, \circ, \rightarrow, 1)$ left residuated monoid if

- $(M, \circ, 1)$ monoid; $\circ$ monotone in all arguments
- $\rightarrow$ left residuation associated with $\circ$

Comm

$x \circ y =$

Integra

BCC-a

$(M, \vee, \wedge, \circ, \rightarrow)$ left residuated lattice if

- $(M, \vee, \wedge)$ lattice; $\circ$ join-hemimorphism in both argu
- $\rightarrow$ left residuation associated with $\circ$.

Examples

Positive logics [Goldblatt 1974, Dunn 1995] Binary logi

- no implication in the language $\phi \vdash \psi$
- algebraic models: lattices with operators

Logics based on Heyting algebras Post-style

- algebraic models: Heyting algebras with operators

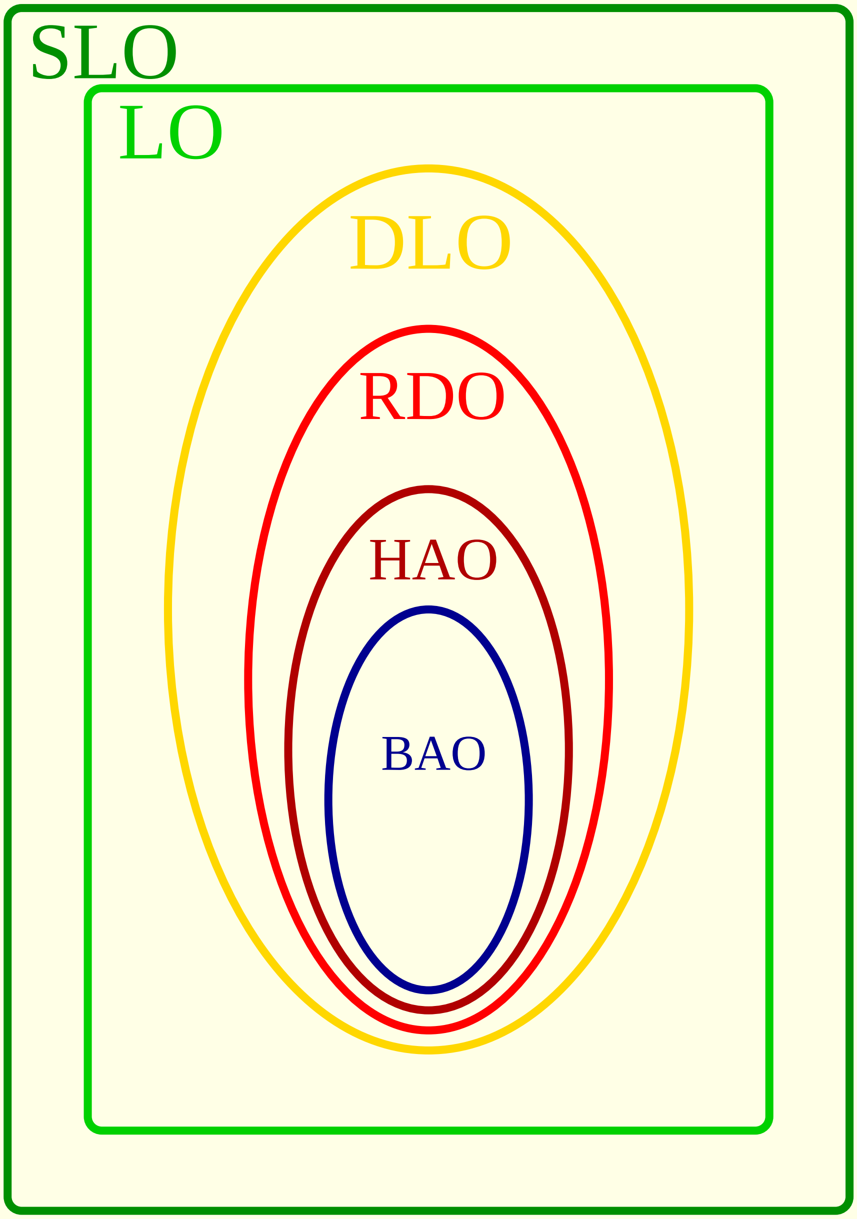
$p \wedge q \leq r$ iff

Logics based on residuated (semi)lattices Łukasiewic

- algebraic models: residuated (semi)lattices with opera

$p \circ q \leq r$ iff

Examples



- positive logics [Dunn
- (modal) intuitionistic
- Gödel logics [Gödel 19
- SH_n, SHK_n logics [Itu
- Post logics and gener
- modal logic, dynamic
- relevant logic RL [Urq
- fuzzy logics

  Gödel, Łukasiew
- BCC and related logic
- Lambek calculus; line

Motivation. Semantics

Algebraic models

(A, D)

Var −

Fma(Var

Kripke-style models

$(W, \{R_W\}_{R \in \mathsf{Rel}})$

m : Va

meaning

Relational models

algebras of relations

Motivation. Decidability res

Logical calculi

- Gentzen-style calculi
- natural deduction
- hypersequent calculi [Avron 1991]

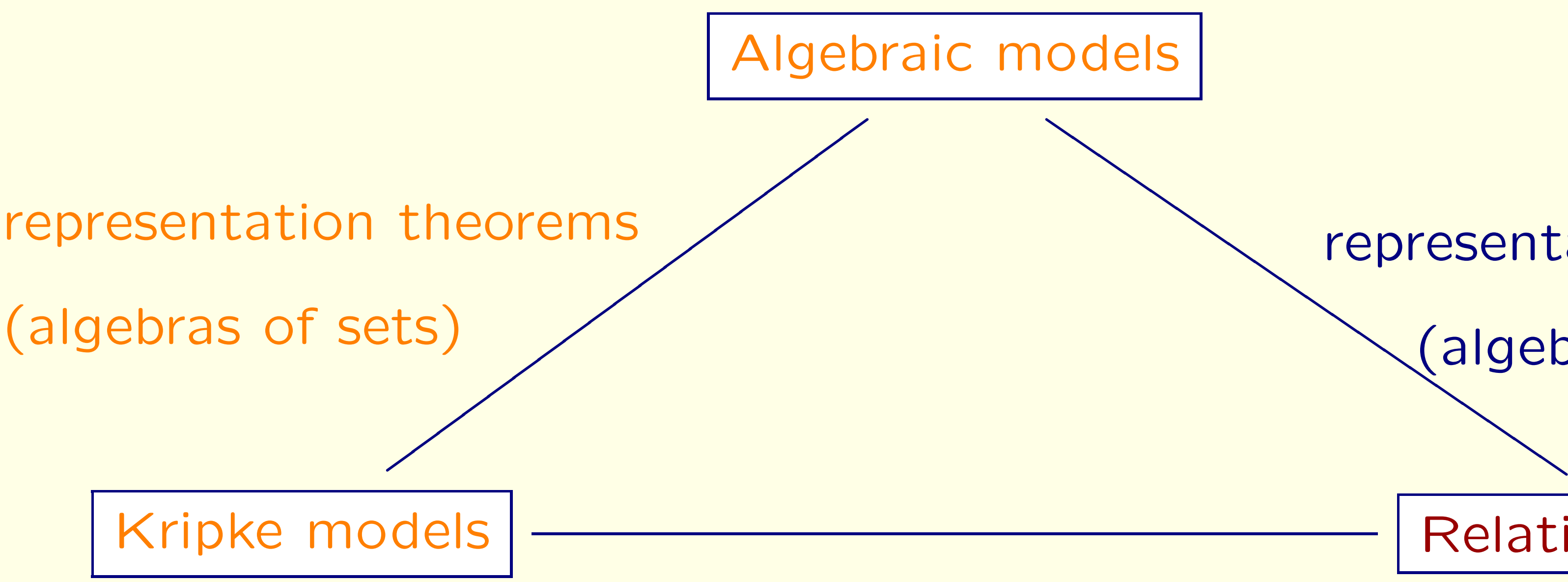
Semantics

- Algebraic semantics
- Kripke-style semantics
- Relational semantics

Automated theorem proving

- embedding into FOL + resolution
- tableau methods
- natural deduction; labelled deductive systems

Connections between classes o



Algebraic and Kripke-style sen

Algebraic models —— Kripke-style mod

(C) $\mathcal{A} \underset{\mathbb{E}}{\overset{\mathbb{D}}{\rightleftarrows}} \mathcal{R}$

(i) $\mathbb{E}(K) \subseteq \mathcal{P}(K)$ algebra of subsets of K

(ii) $i : A \to \mathbb{E}(\mathbb{D}(A))$ injective homomorphism

Kripke-style models (K, m) $K \in \mathcal{R}; m :$ Var $\to \mathbb{E}($

$\overset{r}{\models}$

Theorem If $\mathcal{A}, \mathcal{R}$ satisfy (C)(i,ii) then $\mathcal{A} \overset{a}{\models} \phi$ if

Algebraic and relational sem

Algebraic models ——— Relational model

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(i) $\mathbb{E}(K)$ algebra of relations

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Relational models (K, f) $K \in \mathcal{R}$; f : Var $\to \mathbb{E}(K$

$\overset{a}{\models}$

Theorem If $\mathcal{A}, \mathcal{R}$ satisfy (C)(i,ii) then $\mathcal{A} \overset{a}{\models} \phi$ if

Representation theorem

Natural Dualities: $\mathcal{V} = ISP(P)$

$A \overset{\sim}{\to} \mathsf{Hom}_{\mathsf{Rel}}(D(A), \underline{P})$

$\underline{P}$ 'al

$D(A)$

Stone 1940: $\mathsf{Bool} = ISP(B_2)$

$B \hookrightarrow \mathcal{P}(D(B))$

$\eta_B(x) = \{F \in D(B) \mid x \in F\}$

Priestley 1972: D

$L \hookrightarrow \mathsf{OF}(D(L$

$\eta_L(x) = \{F \in D(L$

Semilattices: $SL = ISP(S_2)$

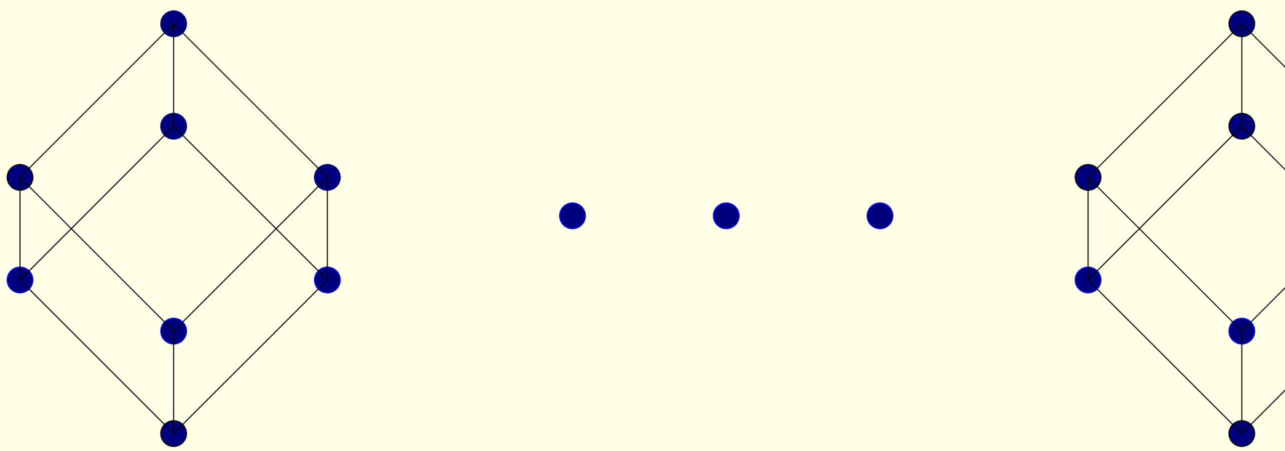
$(S, \wedge) \hookrightarrow (\mathcal{SF}(D(S)), \cap)$

$\eta_S(x) = \{F \in D(S) \mid x \in F\}$

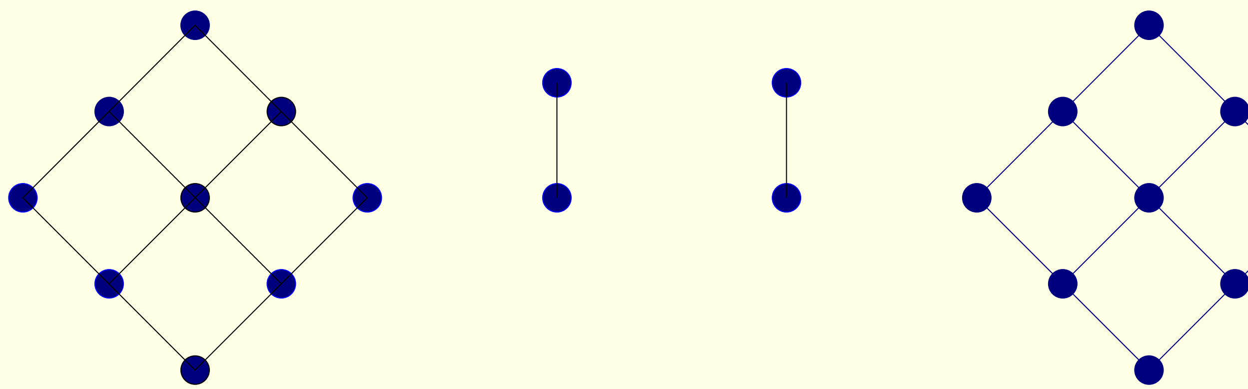
Lattices: $\eta_L : (L, \wedge, \vee) \hookrightarrow (\mathcal{SF}(D(L)), \cap, \vee)$

$\eta_L(x) := \{F \in D(L) \mid x \in F\}$

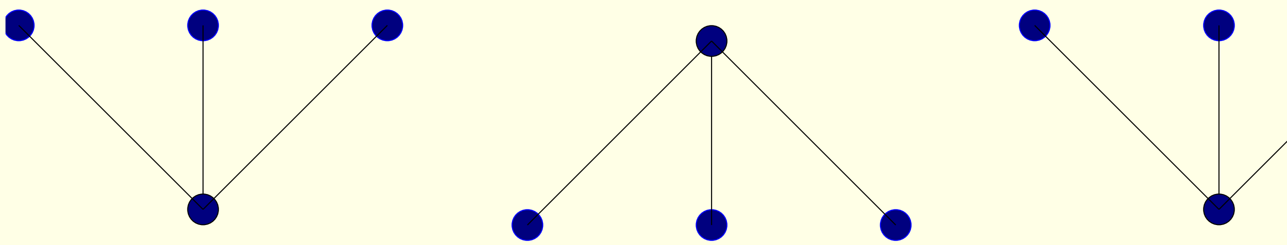
Example 1. Boolean algeb



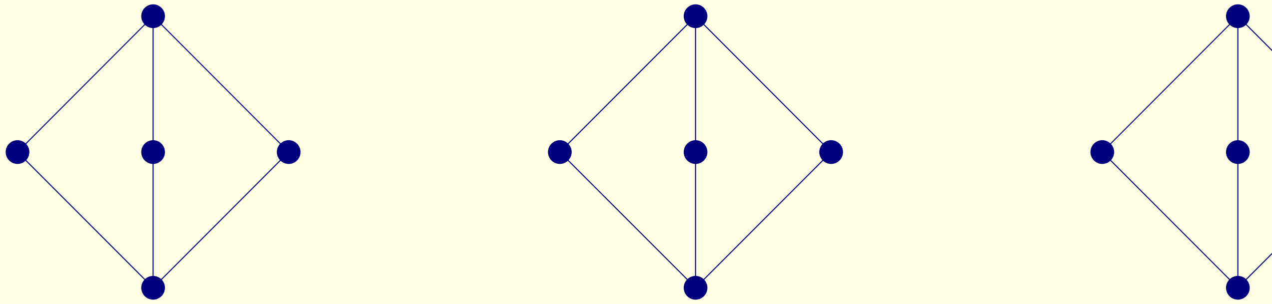
Example 2. Distributive lat



Example 3. Semilattice



Example 4. Lattices



Other representation theor

Boolean algebras with operators

- Jónsson and Tarski (1951)

Distributive lattices with operators

- Goldblatt (1986), VS (2000)

Lattices (with operators)

- Urquhart (1978)
- Allwein and Dunn (1993)
- Dunn and Hartonas (1997)
- Hartonas (1997)

"Gaggles", "tonoids" Dunn (1990, 1993)

General Idea:

- $A \mapsto D(A)$ topologi

with additi

- $A \cong$ ClosedSubsets

closed wrt: topolo

order

...

- operators $\mapsto$ relatio

Representation theorem

$f \in \Sigma_{\varepsilon_1 \dots \varepsilon n \to \varepsilon}$: $f_A : A^{\varepsilon 1} \times \cdots \times A^{\varepsilon n} \to A^{\varepsilon}$ join-hemimorphis

$\mathsf{DLO}_\Sigma \underset{\mathbb{E}}{\overset{\mathbb{D}}{\rightleftarrows}} \mathsf{Rp}_\Sigma$

$\mathsf{SLO}_\Sigma \underset{\mathbb{E}}{\overset{\mathbb{D}}{\rightleftarrows}} \mathsf{SLSp}_\Sigma$

$\mathbb{D}(A)$ $R_f(F_1, \dots, F_n, F)$ iff $f(F_1^{\varepsilon 1}, \dots, F_n^{\varepsilon n}) \subseteq F^{\varepsilon}$

$\mathbb{E}(X)$ $f_R(U_1, \dots, U_n) = (R^{-1}(U_1^{\varepsilon 1}, \dots, U_n^{\varepsilon n}))$

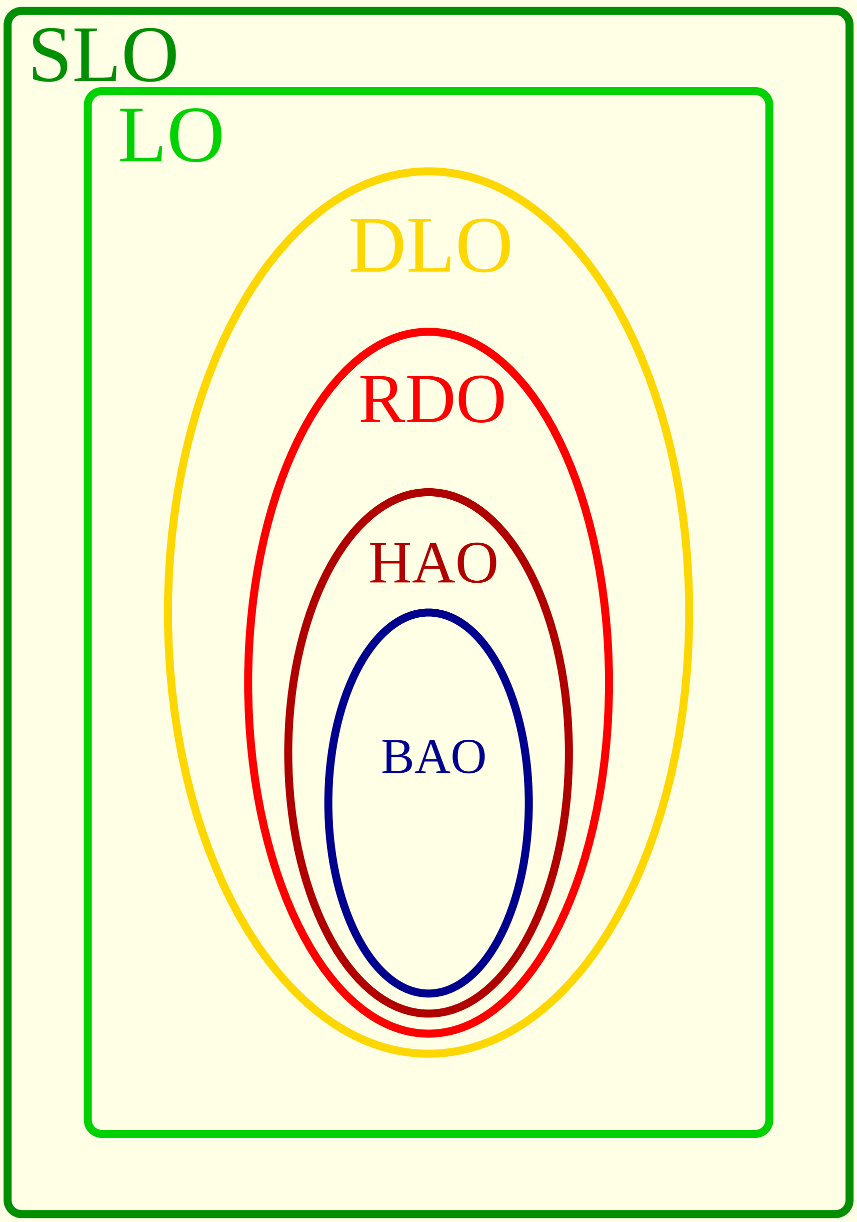
Example $x \circ y \leq z$ iff $x \leq y \to z$

$\circ$ has type $+1, +1 \to +1$ $R_\circ(F_1, F_2, F_3)$ iff $F_1 \circ F_2$

$\to$ has type $+1, -1 \to -1$ $R_\to(F_1, F_2, F_3)$ iff $F_1 \to F$

$R_\to(F_1, F_2, F_3)$ iff $R_\circ(F_3,$

Algebraic and Kripke-style ser



(C) $\mathcal{A} \underset{\mathbb{E}}{\overset{\mathbb{D}}{\rightleftarrows}}$

(i) $\mathbb{E}(K) \subseteq \mathcal{P}(K)$

algebra of s

(ii) $i : A \hookrightarrow \mathbb{E}(\mathbb{D}($

(K, m), $m :$ Var $\rightarrow$

$(K, m) \overset{r}{\models}_x \phi$ iff x

DLO	Priestley r
	$\eta_A : A \rightarrow$ C
SLO, LO	Representa
	(semi)latti
	$\eta_A : A \rightarrow \mathcal{S}$

Logic	Algebraic models	Kripke-style models	mear
Positive	DLO_{Σ} $(L, \vee, \wedge, 0, 1, \{f\}_{f \in \Sigma})$	Rp_{Σ} $(X, \leq, \{R\}_{R \in \Sigma})$	$m :$
Post-style	HAO_{Σ} $(L, \vee, \wedge, \Rightarrow, 0, 1, \{f\}_{f \in \Sigma})$	Rp_{Σ} $(X, \leq, \{R\}_{R \in \Sigma})$	$m :$
	BAO_{Σ} $(B, \vee, \wedge, 0, 1, \neg, \{f\}_{f \in \Sigma})$	BAO_{Σ} $(X, \{R\}_{R \in \Sigma})$	$m :$
Łukasiewicz-style	RDO $(L, \vee, \wedge, 0, 1, \circ, \rightarrow)$	RSp $(X, \leq, R_{\circ})$	$m :$
	RSO, RLO $(S, \wedge, 0, 1, \circ, \rightarrow)$ $(S, \vee, \wedge, 0, 1, \circ, \rightarrow)$	RSO, RLO $(X, \wedge, R_{\circ})$ $(X, \wedge, R_{\circ})$	$m :$

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- Decidability results
- Automated theorem proving
- Conclusions

Class	u.w.p.	References
Lattices	PTIME	Skolem (1920), Burr
ResLatMon	decidable	Blok, Van Alten (19
ResLatIntMon	decidable	Blok, Van Alten (19
$BCK_{\rightarrow}$	decidable	Blok, Van Alten (19
Modular Lattices	undecidable	Freese (1980), Herr
D_{01}	co-NP complete	Bloniarz et al.(1987)
DLO_{Σ}, RDO_{Σ}	EXPTIME	VS (1999, 2001)
$DLSgr_{\vee,d}$	decidable	Andreka
subclasses	undecidable	Urquhart (1995)
Heyting Algebras	DEXP	VS (1999)
$HASgr_{\vee,d}$	undecidable	Kurucz, Nemeti et a
Boolean Algebras	co-NP complete	Cook (1971)
ResBoolMon	undecidable	Kurucz, Nemeti et a
$BoolSgr_{\vee,d}$	undecidable	Kurucz, Nemeti et a
$BoolSgr_{\vee}$	decidable	Gyuris (1992)

Decidability results

Semantics

- Algebraic semantics — finite model property
  (uniform) word problem decida
- Kripke-style semantics — finite model property
  embedding into decidable frag
  devise sound and complete de
- Relational semantics — relational proof systems

Automated theorem proving

  - embedding into FOL + ATP in first-order logic
  - tableau methods
  - natural deduction; labelled deductive systems

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Resolution-based metho

Advantages

- direct encoding
- restricted (hence efficient) calculi
  - ordering, selection
  - simplification/elimination of redundancies
- allow use of efficient implementations
  (SPASS, Saturate)
- in many cases better than equational reasoning
  AC operators $\mapsto$ logical operations

Automated Theorem Proving

Theorem $\mathsf{DLO}_\Sigma \models \phi_1 \leq \phi_2$ iff

the following conjunction is unsatisfiable:

(Dom) $x \leq x$; $x \leq y, y \leq z \rightarrow x \leq z$

$R_f(x_1, \dots, x_n, x), x \bowtie_\epsilon y \Rightarrow R_f(x_1, \dots, x_n, y)$ if

(Her) $x \leq y, P_e(x) \Rightarrow P_e(y)$

(Ren)(0, 1) $\neg P_0(x)$ $P_1(x)$

($\wedge$) $P_{e_1 \wedge e_2}(x) \Leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)$

($\vee$) $P_{e_1 \vee e_2}(x) \Leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)$

(Σ) $P_{f(e_1, \dots, e_n)}(x) \Leftrightarrow (\exists x_1, \dots \exists x_n$ $f \in$

$(P_{e_1}(x_1)^{\varepsilon 1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon n} \wedge R_f(x_1, \dots, x_n$

(N) $\exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$

Automated Theorem Proving

Theorem $\mathsf{DLO}_\Sigma \models \phi_1 \leq \phi_2$ iff

the following conjunction is unsatisfiable:

(Dom)

(Her)

(Ren)(0, 1) $\neg P_0(x) \qquad P_1(x)$

$(\wedge)$ $P_{e_1 \wedge e_2}(x) \Leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)$

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(Σ) $P_{f(e_1,\dots,e_n)}(x) \Leftrightarrow (\exists x_1, \dots \exists x_n \qquad f \in$

$(P_{e_1}(x_1)^{\varepsilon_1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon_n} \wedge R_f(x_1, \dots, x_n,$

(N) $\exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$

Automated Theorem Proving

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(Her) $x \leq y, P_e(x) \Rightarrow P_e(y)$

(Ren)(0, 1) $\neg P_0(x) \qquad P_1(x)$

($\wedge$) $P_{e_1 \wedge e_2}(x) \Leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)$

($\vee$) $P_{e_1 \vee e_2}(x) \Leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)$

(Σ) $P_{f(e_1, \dots, e_n)}(x) \Leftrightarrow (\exists x_1, \dots \exists x_n \qquad f \in$

$(P_{e_1}(x_1)^{\varepsilon 1} \wedge \dots \wedge P_{e_n}(x_n)^{\varepsilon n} \wedge R_f(x_1, \dots, x_n,$

($\to$) $P_{e_1 \to e_2}(x) \Leftrightarrow \forall y(y \geq x \wedge P_{e_1}(y) \Rightarrow P_{e_2}(y))$

(N) $\exists c \in X : \neg P_\phi(c)$

Automated Theorem Prov

Class of algebras	Complexity (refinements of resol
DLO_{Σ}	EXPTIME
RDO_{Σ}	EXPTIME
BAO_{Σ}	EXPTIME
HA	DEXPTIME
HAO_{Σ}	?
$RSO_{\Sigma}, RLO_{\Sigma}$	?

Overview

- Representation theorems
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Questions

Automated theorem proving

- what presentation is better?
  - logical calculus/semantics
  - what semantics: algebraic, Kripke o
- which methods for ATP are better?
  - resolution
  - tableaux
  - natural deduction
  - ...

D_{01}	the class of bounded distributive lattices
DLO_{Σ}	the class of bounded distributive lattices with
HA	the class of Heyting algebras
HAO_{Σ}	the class of Heyting algebras with operators i
Bool	the class of Boolean algebras
BAO_{Σ}	the class of Boolean algebras with operators
RD	the class of all residuated distributive lattices
RDO_{Σ}	the class of residuated distributive lattices wi
Lat	the class of lattices
(R)LO	lattices with operators
SL	the class of semilattices
(R)SL(O)	(residuated) semilattices (with operators)

ResLatMon	the class of residuated lattice-ordered m
ResLatIntMon	the class of residuated lattice-ordered in
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