

Automated theorem proving by resolution in non-classical logics

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Overview

- Motivation
- Resolution-based theorem proving in non-classical logics
 1. Finitely-valued logics (superposition; ordered chaining)
 2. Infinitely-valued logics
 3. Modal logics and generalizations
 - logics based on distributive lattices with operators
 - applications: description logics, terminological reasoning
 4. Beyond modal logics: deciding uniform word problems
- Conclusions and future work

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- Huge variety of non-classical logics
- Huge variety of methods for automated theorem proving in these logics
 - sequent calculi, semantic tableaux, various extensions of resolution

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This talk: Embedding into first-order logic + resolution

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3. modal logics and generalizations
4. beyond modal logics

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1. finitely-valued logics (first-order)
 - superposition and ordered chaining
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Many-valued logics

Definition: $\mathcal{L} = (X, \text{Op}, \text{Pred}, \Sigma, \text{Quant})$, A : set of truth values

Interpretations: $\mathcal{I} = (D, I, d)$

– interpretation of terms: as in classical logic

– $P \in \text{Pred}$ $I(P) : D^{a(P)} \rightarrow A$

– $\sigma \in \Sigma$ $I(\sigma) = \sigma_A : A^{a(\sigma)} \rightarrow A$

– $Q \in \text{Quant}$ $I(Q) = Q_A : \mathcal{P}(A) \setminus \emptyset \rightarrow A$

$\mathcal{I}(Qx\phi(x)) = Q_A(\{\mathcal{I}(\phi)(d) \mid d \in D\})$

Examples

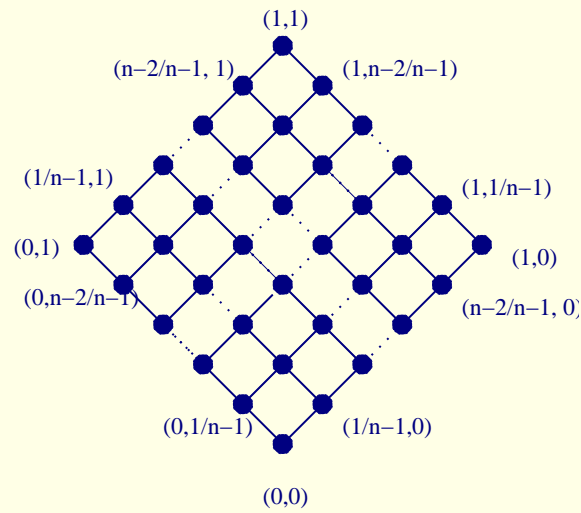
- 3-valued logics $A = \{0, \frac{1}{2}, 1\}$
 - Łukasiewicz $\frac{1}{2}$: possible
 - Bochvar $\frac{1}{2}$: meaningless
 - Kleene $\frac{1}{2}$: undefined

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- n -valued logics $A = \{0, \frac{1}{n-1}, \dots, \frac{n-1}{n-1}\}$
 - \mathfrak{L}_n Łukasiewicz logic
 - G_n Gödel logic
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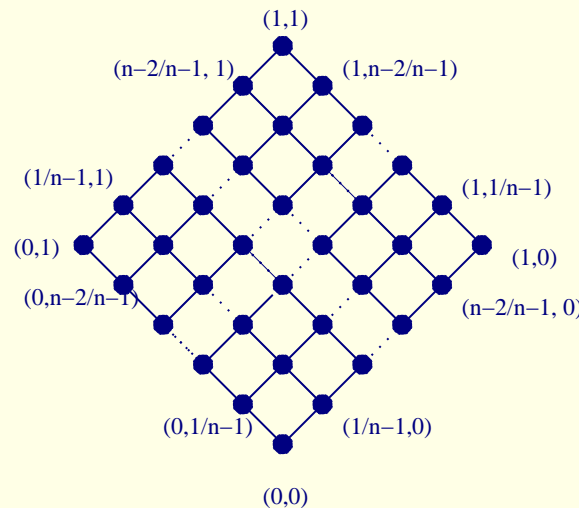
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- Other examples SH_n -logics [Iturrioz, Orłowska 1996]



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- many (propositional) logics are characterized by one single algebra: the **Lindenbaum algebra** (usually difficult to effectively describe)

Examples: fuzzy logics

Logic	Truth values	Connectives	*
\mathbf{L}_n	$\{0, \frac{1}{n-1} \dots \frac{n-1}{n-1}\}$	$x \circ y = \max(0, x+y-1)$ \rightarrow : the right residuum of \circ	prop. 1 st order
\mathbf{L}_∞	$[0, 1]$		prop. 1 st order
\mathbf{G}_n	$\{0, \frac{1}{n-1} \dots \frac{n-1}{n-1}\}$	$x \circ y = \min(x, y)$ \rightarrow : the right residuum of \circ	prop. 1 st order
\mathbf{G}_∞	$[0, 1]$		prop. 1 st order
$\mathbf{\Pi}_\infty$	$[0, 1]$	$x \circ y = x \cdot y$ \rightarrow : the right residuum of \circ	prop. 1 st order

* in all cases above for the 1st order version of the logic the quantifiers are:
 $\forall = \inf$; $\exists = \sup$

Examples: fuzzy logics

Logic	Truth values	Connectives	*	Complexity (validity)
\mathbf{L}_n	$\{0, \frac{1}{n-1} \dots \frac{n-1}{n-1}\}$	$x \circ y = \max(0, x+y-1)$ \rightarrow : the right residuum of \circ	prop. 1 st order	co-NP
\mathbf{L}_∞	$[0, 1]$		prop. 1 st order	co-NP Π_2 -complete
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\mathbf{G}_∞	$[0, 1]$		prop. 1 st order	co-NP Σ_1 -complete
$\mathbf{\Pi}_\infty$	$[0, 1]$	$x \circ y = x \cdot y$ \rightarrow : the right residuum of \circ	prop. 1 st order	co-NP Π_2 -hard

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Automated theorem proving (finite-valued logics)

- translation to clause normal form (signed literals)

$$\begin{array}{l} \text{[Baaz and Fermüller 1995]} \quad \phi \quad \mapsto \quad \bigwedge (\bigvee L^v) \\ \phi \text{ valid} \quad \text{iff} \quad \bigwedge (\bigvee L^v) \text{ unsatisfiable} \end{array}$$

- many-valued resolution rules

Automated theorem proving (finite-valued logics)

- translation to clause normal form (signed literals)

[Baaz and Fermüller 1995] $\phi \mapsto \bigwedge (\bigvee L^v)$

ϕ valid iff $\bigwedge (\bigvee L^v)$ unsatisfiable

$\phi^{\mathcal{I}} \in D \quad \bigwedge_{a \in D} \bigvee_{b \neq a} \phi = b \mapsto \bigwedge (\bigvee L^v)$

- many-valued resolution rules

Automated theorem proving (finite-valued logics)

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[Baaz and Fermüller 1995] $\phi \mapsto \bigwedge (\bigvee L^v)$

- many-valued resolution rules

$$\frac{C \vee L^v \quad D \vee L^u}{C \vee D}$$

provided that $u \neq v$

Summarizing

Truth values	MV-ATP	Signed lit.
A arbitrary	MV-resolution [Baaz, Fermüller'95]	L^v
(A, \leq) poset	annotated resolution [Kifer, Lozinskii'92] [Lu, Murray, Rosenthal'98]	$\uparrow v:L$ $\sim \uparrow v:L$
(A, \leq) total order $(v_1 < \dots < v_n)$	regular resolution [Hähnle'94, '96]	$\uparrow v_i:L$ $\downarrow v_i:L$

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(A, \leq) poset	annotated resolution [Kifer, Lozinskii'92] [Lu, Murray, Rosenthal'98]	$\uparrow v:L$ $\sim \uparrow v:L$	$C \vee \uparrow v_1:L \quad D \vee \sim \uparrow v_2:L$ $C \vee D$ provided that $v_1 \geq v_2$
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Summarizing

Truth values	MV-ATP	Signed lit.	$C \vee L=v \quad D \vee L=u$
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(A, \leq) poset	annotated resolution [Kifer, Lozinskii'92] [Lu, Murray, Rosenthal'98]	$\uparrow v:L$ $\sim \uparrow v:L$	$C \vee L \geq v_1 \quad D \vee L \not\geq v_2$ $C \vee D$ provided that $v_1 \geq v_2$
(A, \leq) total order $(v_1 < \dots < v_n)$	regular resolution [Hähnle'94, '96]	$\uparrow v_i:L$ $\downarrow v_i:L$	

A simple translation to classical logic

Truth values	MV-ATP	Signed lit.	Classical lit.	Theory ax.
A arbitrary	MV-resolution [Baaz, Fermüller'95]	L^v	$L = v$	Eq Φ_A, Fin
(A, \leq) poset	annotated resolution [Kifer, Lozinskii'92] [Lu, Murray, Rosenthal'98]	$\uparrow v:L$ $\sim \uparrow v:L$	$v \leq L$ $v \not\leq L$	Trans \leq Φ_A, Fin (Sup, Min)
(A, \leq) total order $(v_1 < \dots < v_n)$	regular resolution [Hähnle'94, '96]	$\uparrow v_i:L$ $\downarrow v_i:L$	$v_i \leq L$ $L < v_{i+1}$ or $v_{i+1} \not\leq L$	Trans \leq Tot \leq Φ_A, Min

$A \quad + \quad \Phi \quad \mapsto \quad \Phi^c$ classical clauses
 Φ mv-unsat. $\Leftrightarrow \Phi^c$ classically unsat.

Automated theorem proving (finite-valued logics)

[Ganzinger & VS 2000]: Specialize **superposition** and **ordered chaining**
(calculi that encode inferences with **congruence** resp. **transitivity**)
to many-valued logics

Advantages

- direct encoding
- reconstruct known completeness results
- but** much more restricted calculi
 - ordering, selection
 - **simplification/elimination of redundancies**
- allows use of efficient implementations (SPASS, Saturate)

Comments

Limitations

- The method relies on a suitable translation to clause form.
- May not be applicable:
 - for infinitely-valued logics
 - when the semantics is given in terms of a class of algebras.

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2. infinitely-valued logics
3. modal logics and generalizations
4. beyond modal logics

Infinitely-valued logics

- Propositional Łukasiewicz and Gödel logics [Hähnle 1994, 1996]
 - CNF translation; reduction to mixed integer programming

Infinitely-valued logics

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- **Łukasiewicz logics**
 - propositional: McNaughton's theorem [Aguzzoli, Ciabattoni 2000]
reduction to finitely-valued Łukasiewicz logics
but number of truth values exponential in size of formula
 - first-order: resolution-like calculus [Mundici, Olivetti 1998]

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 - first-order: resolution-like calculus [Mundici, Olivetti 1998]
- **Gödel logics + projections** (prenex) [Baaz, Fermüller, Ciabattoni 2001]
 - CNF translation + ordered chaining for dense total orderings
 - process might not terminate (the fragment is undecidable)

Overview

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3. modal logics and generalizations

restrict to propositional logics

4. beyond modal logics

uniform word problems in various classes of

distributive lattices (or Boolean algebras) with operators

Modal logics and generalizations

Algebraic models for propositional non-classical logics

Distributive lattices, lattices, semilattices with operators

$$\Box(x \wedge y) = \Box(x) \wedge \Box(y)$$

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$$

$$\Diamond(x \vee y) = \Diamond(x) \vee \Diamond(y)$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$$

Description logics:

operators with numeric values

$$\text{maxcost}, \text{mincost} : \mathcal{P}(X) \rightarrow \mathbb{N},$$

$$\text{maxcost}(U \cup V) = \max\{\text{maxcost}(U), \text{maxcost}(V)\}$$

$$\text{mincost}(U \cup V) = \min\{\text{mincost}(U), \text{mincost}(V)\}$$

Galois connections:

$$f : L_1 \rightarrow L_2, \quad g : L_2 \rightarrow L_1$$

$$y \leq f(x) \quad \text{iff} \quad x \leq g(y)$$

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Many-sorted DLO

$$\mathbf{L} = (\{L_s\}_{s \in S}, \{f_L\}_{f \in \Sigma}) \quad a(f) = s_1 \dots s_n \rightarrow s$$

$$f_L : L_{s_1} \times \dots \times L_{s_n} \rightarrow L_s \text{ join homomorphism}$$

Examples

- B Boolean algebra;

$$S = (\mathbf{b}, \mathbf{b}_d)$$

- L lattice;

$$S = (\mathbf{l}, \mathbf{l}_d)$$

- L lattice; C_n ;

$$S = (\mathbf{l}, \mathbf{l}_d, \mathbf{n}, \mathbf{n}_d)$$

- $L_1 \xrightleftharpoons[g]{f} L_2$ G.c.;

$$S = \{\mathbf{l}_1, \mathbf{l}_{1_d}, \mathbf{l}_2, \mathbf{l}_{2_d}\}$$

Operator	Join homomorphism of type
$\diamond : B \rightarrow B$	$\mathbf{b} \rightarrow \mathbf{b}$
$\square : B \rightarrow B$	$\mathbf{b}_d \rightarrow \mathbf{b}_d$
$\rightarrow : L \times L \rightarrow L$	$\mathbf{l}, \mathbf{l}_d \rightarrow \mathbf{l}_d$
$\text{maxcost} : L \rightarrow C_n$	$\mathbf{l} \rightarrow \mathbf{n}$
$\text{mincost} : L \rightarrow C_n$	$\mathbf{l} \rightarrow \mathbf{n}_d$
$f : L_1 \rightarrow L_2$	$\mathbf{l}_1 \rightarrow \mathbf{l}_{2_d}$
$g : L_2 \rightarrow L_1$	$\mathbf{l}_2 \rightarrow \mathbf{l}_{1_d}$

Interesting problems

Word Problems: $\mathcal{V} \models s = s'$

- $\mathcal{V} \models \phi = 1$

validity in modal, intuitionistic logic

- $\mathcal{V} \models \phi \geq e$

validity in relevant/substructural logics

- $\mathcal{V} \models C \leq C'$

concept subsumption (description logics)

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concept subsumption with respect to terminologies (description logics)

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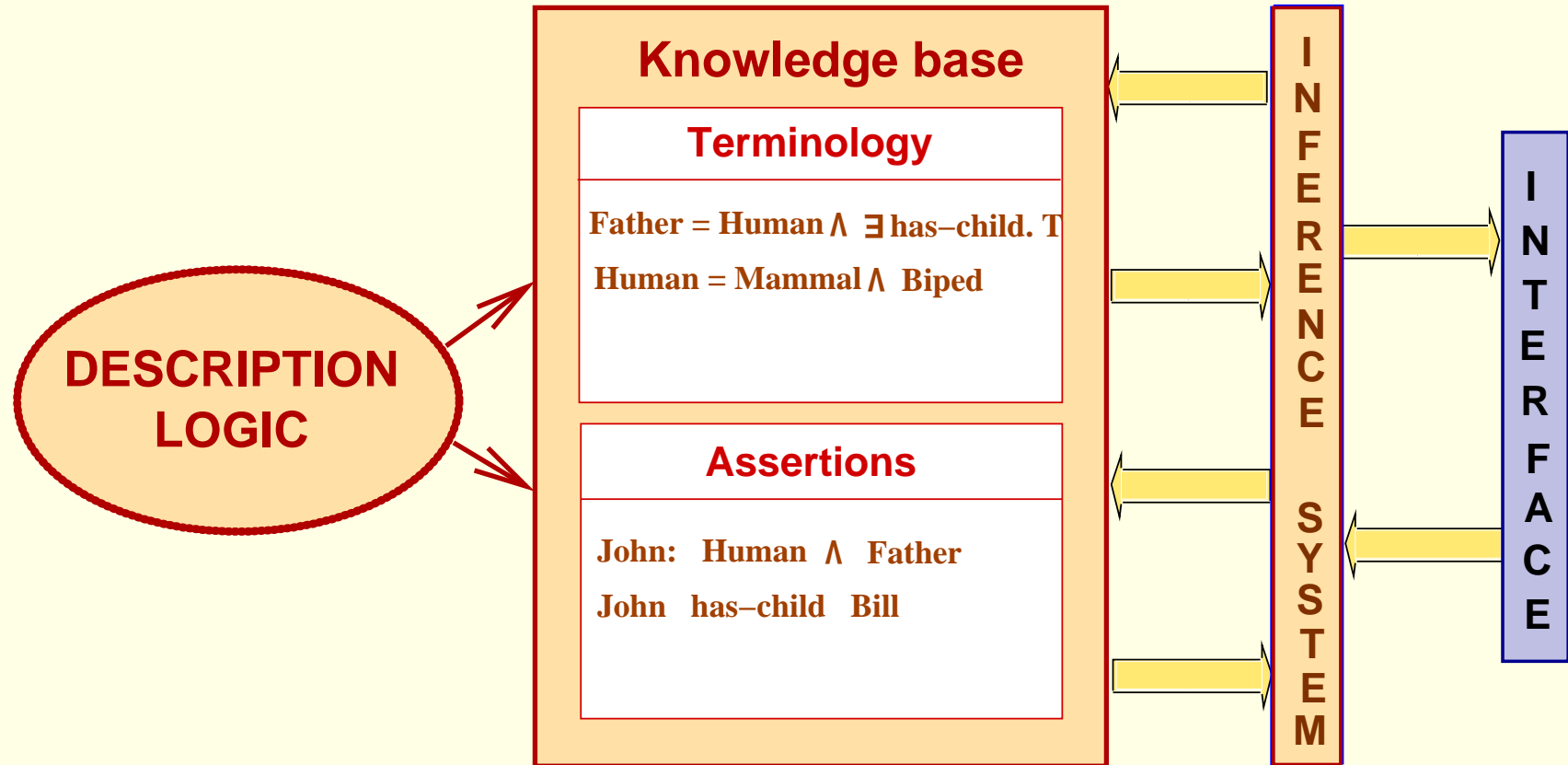
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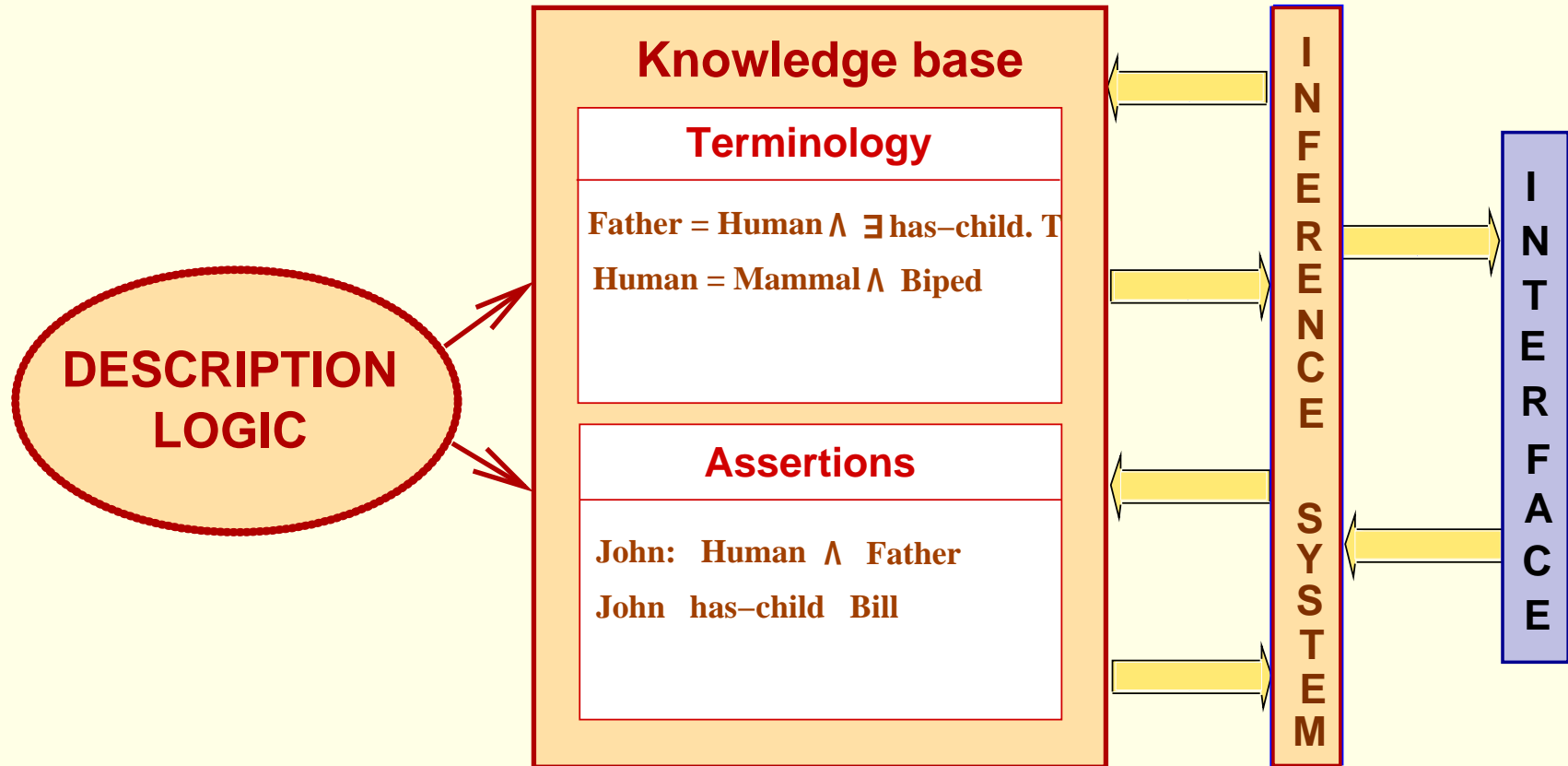
Unification problems: $\mathcal{V} \models \exists \bar{y} \bigwedge_{i=1}^n s_i = s'_i$

- $\mathcal{V} \models \exists \bar{C} \bigwedge_{i=1}^n C_i = C'_i$ unification of concept terms

Description logics



Description logics



concepts

e.g. Mammal, Biped, Human, Father

roles

e.g. has-child

terminology:

definitions of concepts (TBox)

Assertions:

statements about individuals (ABox)

Problems:

- subsumption
 - $\text{Father} \subseteq_{\mathcal{T}} \text{Human}$
 - $\text{Father} \subseteq_{\mathcal{T}} \text{Biped}$
- consistency

Description logics

Description logics are characterized by a set of constructors that allow to build complex **concepts** and **roles** from atomic ones.

- **concepts** correspond to classes / interpreted as sets of objects
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Constructors

logical: $C \wedge D$, $C \vee D$, $\neg C$ restrictions: $\exists R.C$, $\forall R.C$

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TBoxes

- concept definitions: $A = C$
- axioms: $C \sqsubseteq D$

ABoxes

- concept assertions: $a : C$
- role assertions: $(a_1, a_2) : R$

TBox subsumption and u.w.p.

- Subsumption: $- C \subseteq D$
w.r.t. TBox \mathcal{T} : $- C \subseteq_{\mathcal{T}} D$

- Translation to algebraic form:

– concept expressions:

$$\begin{array}{llll} \overline{\neg C} & = & \neg \overline{C} & \overline{\exists R.C} & = & f_{\exists R}(\overline{C}) \\ \overline{C_1 * C_2} & = & \overline{C_1} * \overline{C_2} & \overline{\forall R.C} & = & f_{\forall R}(\overline{C}) \end{array}$$

- TBoxes and subsumption problems

$$\begin{array}{ll} \mathcal{T} = \{C_1 = D(C_1), \dots, C_n = D(C_n)\} & \mapsto \quad \overline{\mathcal{T}} = \bigwedge_{i=1}^n C_i = \overline{D(C_i)} \\ C \subseteq_{\mathcal{T}} D & \mapsto \quad \bigwedge_{i=1}^n C_i = \overline{D(C_i)} \rightarrow C \leq D \end{array}$$

TBox subsumption and u.w.p.

BAO_{N_R}	$(B, \vee, \wedge, \neg, 0, 1, \{f_{\exists R}, f_{\forall R}\}_{R \in N_R})$	$f_{\forall R}(x) = \neg f_{\exists R}(\neg x)$ $f_{\exists R}$ join hemimorphism $f_{\forall R}$ meet hemimorphism
$\text{DLO}_{N_R}^{\forall}$	$(L, \vee, \wedge, 0, 1, \{f_{\forall R}\}_{R \in N_R})$	$f_{\forall R}$ meet hemimorphism
$\text{DLO}_{N_R}^{\exists}$	$(L, \vee, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$	$f_{\exists R}$ join hemimorphism
$\text{SLO}_{N_R}^{\forall}$	$(L, \wedge, 1, \{f_{\forall R}\}_{R \in N_R})$	$f_{\forall R}$ meet hemimorphism
$\text{SLO}_{N_R}^{\exists}$	$(L, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$	$f_{\exists R}$ monotone, and $f_{\exists R}(0) = 0$

Theorem: Let Cons be the set of allowed concept constructors;
 C, C' concept expressions, $\mathcal{T} = \{C_i = D(C_i) \mid i = 1, n\}$ a TBox

(1) $\text{Cons} = \{\vee, \wedge, \neg, 0, 1, \{\exists R, \forall R\}_{R \in N_R}\}$:

$$C \subseteq_{\mathcal{T}} C' \quad \text{iff} \quad \text{BAO}_{N_R} \models \bigwedge_{i=1}^n C_i = \overline{D(C_i)} \rightarrow C \leq C'$$

(2) $\text{Cons} = \{\vee, \wedge, 0, 1, \{\forall R\}_{R \in N_R}\}$:

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(3) $\text{Cons} = \{\vee, \wedge, 0, 1, \{\exists R\}_{R \in N_R}\}$:

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(4) $\text{Cons} = \{\wedge, 1, \{\forall R\}_{R \in N_R}\}$:

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\mathcal{ALC}

ExpTime

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\mathcal{FL}_0

PSpace

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\mathcal{EL}

PTime

Interesting problems

Word Problems: $\mathcal{V} \models s = s'$

- $\mathcal{V} \models \phi = 1$ validity in modal, intuitionistic logic
- $\mathcal{V} \models \phi \geq e$ validity in relevant/substructural logics
- $\mathcal{V} \models C \leq C'$ concept subsumption (description logics)

Uniform word problems: $\mathcal{V} \models \bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s'$

- $\mathcal{V} \models \bigwedge_{i=1}^n A_i = D_i \rightarrow C \leq C'$ concept subsumption with respect to terminologies (description logics)

Unification problems: $\mathcal{V} \models \exists \bar{y} \bigwedge_{i=1}^n s_i = s'_i$

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Automated theorem proving

Modal logic



DLO (HAO, BAO)

uniform word problems

- equational reasoning modulo ACI: difficult

- alternative method [VS 1999–2003]

- representation as lattices of sets

- embedding into FOL + resolution

applications to various non-classical logics

Automated theorem proving

Modal logic



DLO (HAO, BAO)

validity/satisfiability [Ohlbach,Hustadt,Schmidt]

- relational translation + (hyper)resolution

uniform word problems

- equational reasoning modulo ACI: difficult

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applications to various non-classical logics

Similar ideas applicable to a wide class of algebraic structures, many of which occur in a natural way in applications.

Advantages: - link between algebraic / relational semantics
- decision procedures with optimal complexity

Representation theorems

Boolean algebras (with operators)

- Stone (1936)
- Jónsson and Tarski (1951)

Distributive lattices (with operators)

- Priestley (1970)
- Goldblatt (1989)
- VS (2000 – 2002)

Lattices (with operators)

- Urquhart (1978)
- Allwein and Dunn (1993)
- Dunn, Hartonas (1997)

General Idea:

- $A \mapsto D(A)$ topological space
with additional structure

- $A \cong \text{Closed}(D(A)) \hookrightarrow \mathcal{P}(D(A))$

closed wrt: topological structure

order structure

...

- operators \mapsto relations on $D(A)$

Representation theorems

Boolean algebras (with operators)

- Stone (1936)
- Jónsson and Tarski (1951)

Distributive lattices (with operators)

- Priestley (1970)
- Goldblatt (1986),
- VS (2000 – 2002)

Lattices (with operators)

- Urquhart (1978)
- Allwein and Dunn (1993)
- Dunn, Hartonas (1997)

General Idea:

- $A \mapsto D(A)$ topological space
with additional structure

$$D(A) = (\mathcal{F}_p(A), \subseteq, \tau, \{R_f\}_{f \in \Sigma})$$

- $A \cong \text{Closed}(D(A)) \hookrightarrow \mathcal{O}(D(A))$

closed wrt: topological structure
order structure

...

- operators \mapsto relations on $D(A)$
 $f: n\text{-ary, j.h.} \mapsto R_f : n + 1\text{-ary}$
 $R_f(F_1 \dots F_n, F)$ iff $f(F_1 \dots F_n) \subseteq F$

Uniform word problems

$$DLO_{\Sigma}^S \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{\mathcal{O}} \end{array} RT_{\Sigma}^S$$

U|

U|

$$\mathcal{V} \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{\mathcal{O}} \end{array} \mathcal{K}$$

$$A \hookrightarrow \mathcal{O}(D(A))$$

Uniform word problems

$$\mathcal{V} \models \bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s'$$

$$DLO_{\Sigma}^S \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{\mathcal{O}} \end{array} RT_{\Sigma}^S$$

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Uniform word problems

$$\mathcal{V} \models \bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s'$$

iff

$$\mathcal{O}(X) \models \bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s' \quad \forall X \in \mathcal{K}$$

$$DLO_{\Sigma}^S \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{\mathcal{O}} \end{array} RT_{\Sigma}^S$$

UI UI

$$\mathcal{V} \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{\mathcal{O}} \end{array} \mathcal{K}$$

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encoding

terms:	sets:
sets	unary predicates
$t \mapsto P_t$	$x \in P_t \mapsto P_t(x)$

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$t \mapsto P_t$	$x \in P_t \mapsto P_t(x)$

- $\vee \mapsto$ set union
- $\wedge \mapsto$ set intersection
- $f \mapsto R_f$ f j.h. of type
 $\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon,$
 $\varepsilon_i, \varepsilon \in \{-1, +1\}$
- negation of ϕ :
- sort information; description of domain

Uniform word problems

$$\mathcal{O}(X) \models \bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s' \quad \forall X \in \mathcal{K}$$

encoding

terms:	sets:
sets	unary predicates
$t \mapsto P_t$	$x \in P_t \mapsto P_t(x)$

- $\vee \mapsto$ set union

$$P_{t_1 \vee t_2}(x) \leftrightarrow P_{t_1}(x) \text{ or } P_{t_2}(x)$$

- $\wedge \mapsto$ set intersection

$$P_{t_1 \wedge t_2}(x) \leftrightarrow P_{t_1}(x) \text{ and } P_{t_2}(x)$$

- $f \mapsto R_f$ f j.h. of type

$$P_{f(t_1, \dots, t_n)}^\varepsilon(x) \leftrightarrow R_f^{-1}(P_{t_1}^{\varepsilon_1}, \dots, P_{t_n}^{\varepsilon_n})(x)$$

$$\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon,$$

$$\varepsilon_i, \varepsilon \in \{-1, +1\}$$

- negation of ϕ :

$$P_{s_1}(x) \leftrightarrow P_{s'_1}(x)$$

...

$$P_{s_n}(x) \leftrightarrow P_{s'_n}(x)$$

$$P_s(x) \not\leftrightarrow P_{s'}(x)$$

- sort information; description of domain (Dom_γ)

Properties of the domain

\mathcal{V}	\mathcal{K}	(Dom)	Logic
BAO$_{\Sigma}$ $(B, \vee, \wedge, 0, 1, \neg, \{f\}_{f \in \Sigma})$	R$_{\Sigma}$ $(X, \{R\}_{R \in \Sigma})$		modal
HAO$_{\Sigma}$ $(L, \vee, \wedge, \Rightarrow, 0, 1, \{f\}_{f \in \Sigma})$	Rp$_{\Sigma}$ $(X, \leq, \{R\}_{R \in \Sigma})$	P_e monotone R monotone/antitone	intuitionistic modal
RDO $(L, \vee, \wedge, 0, 1, \circ, \rightarrow)$ $x \circ y \leq z$ iff $x \leq y \rightarrow z$	RSp (X, \leq, R_o)	P_e, R_o monotone R_{\rightarrow} antitone $R_{\rightarrow}(x, y, z) \leftrightarrow R_o(z, x, y)$	substructural
DLO$_{\Sigma}$ $(L, \vee, \wedge, 0, 1, \{f\}_{f \in \Sigma})$	Rp$_{\Sigma}$ $(X, \leq, \{R\}_{R \in \Sigma})$	P_e monotone R monotone/antitone	positive

\leq preorder

Decidability results

Ordered resolution with selection decides u.w.p. of:

- $DLO_{\Sigma}^S, RDLO_{\Sigma}^S$;
- $DLO_{\Sigma}^A = \{(L, A, \{f_{LA}\}_{f \in \Sigma_{LA}}) \mid L \in DLO, f_{LA} : L^n \rightarrow A\}$ A finite DLO
& extension with residuation rules
- BAO_{Σ}^S ;
- $BAO_{\Sigma}^A = \{(L, A, \{f_{LA}\}_{f \in \Sigma_{LA}}) \mid L \in BAO, f_{LA} : L^n \rightarrow A\}$ A finite DLO
- H (the class of Heyting algebras)

Ordered resolution: [R]DLO $_{\Sigma}$

Theorem [R]DLO $_{\Sigma} \models \phi_1 \leq \phi_2$ iff the following conjunction unsatisfiable:

(Dom)

$$x \leq x$$

$$x \leq y, y \leq z \rightarrow x \leq z$$

$$x \leq_{\varepsilon} y, R_f(x_1, \dots, x_n, x) \rightarrow R_f(x_1, \dots, x_n, y) \text{ if } f \in \Sigma_{\bar{\varepsilon} \rightarrow \varepsilon}$$

$$[R_f(x_1, \dots, x_i, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)]$$

(Her)

$$x \leq y, P_e(x) \rightarrow P_e(y)$$

(Ren)(0, 1)

$$\neg P_0(x) \quad P_1(x)$$

$$(\wedge) P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)$$

$$(\vee) P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)$$

n

$$(\Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}) P_{f(e_1, \dots, e_n)}(x) \leftrightarrow (\exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1 \dots x_n, x)))^{\varepsilon}$$

$i = 1$

(N)

$$\exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$$

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Theorem [R]DLO $_{\Sigma} \models \phi_1 \leq \phi_2$ iff the following conjunction unsatisfiable:
(Dom)

$$[R_f(x_1, \dots, x_i, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)]$$

(Her)

$$\begin{aligned} \text{(Ren)(0, 1)} \quad & \neg P_0(x) \quad P_1(x) \\ & (\wedge) P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x) \\ & (\vee) P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x) \end{aligned}$$

$$\begin{aligned} & n \\ \text{(\Sigma}_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon})} \quad & P_{f(e_1, \dots, e_n)}(x) \leftrightarrow (\exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1 \dots x_n, x)))^{\varepsilon} \end{aligned}$$

$$\text{(N)} \quad \exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$$

Ordered resolution: $[R]DLO_{\Sigma}^S$

Theorem $[R]DLO_{\Sigma} \models \phi_1 \leq \phi_2$ iff the following conjunction unsatisfiable:

(Dom)

sort information
can be kept implicit

$$[R_f(x_1, \dots, x_i, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)]$$

(Her)

$$\begin{aligned} \text{(Ren)(0, 1)} \quad & \neg P_0(x) \quad P_1(x) \\ & (\wedge) P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x) \\ & (\vee) P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x) \end{aligned}$$

$$\text{(\Sigma}_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}) \quad P_{f(e_1, \dots, e_n)}(x) \leftrightarrow (\exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1 \dots x_n, x)))^{\varepsilon}$$

$$\text{(N)} \quad \exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$$

Ordered resolution: $[R]DLO_{\Sigma}^A$

Theorem $[R]DLO_{\Sigma}^A \models \phi_1 \leq \phi_2$ iff the following conjunction unsatisfiable:

(Dom)

sort information
can be kept implicit

$$R_f(x_1, \dots, x_n, a) \rightarrow R_f(x_1, \dots, x_n, b) \quad \text{for all } a \leq b \in D(A)$$

$$[R_f(x_1, \dots, x_i, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)]$$

(Her)

$$\begin{aligned} \text{(Ren)(0, 1)} \quad & \neg P_0(x) \quad P_1(x) \\ (\wedge) \quad & P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x) \\ (\vee) \quad & P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x) \end{aligned}$$

$$\begin{aligned} (\Sigma_{\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon}) \quad & P_{f(e_1, \dots, e_n)}(x) \leftrightarrow (\exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1 \dots x_n, x)))^{\varepsilon} \\ & i = 1 \end{aligned}$$

$$\text{(N)} \quad \exists c \in X : P_{\phi_1}(c) \wedge \neg P_{\phi_2}(c)$$

Ordered chaining: HAO_Σ

Theorem $\text{HAO}_\Sigma \models \phi = 1$ iff the following conjunction unsatisfiable:

(Dom)

$$x \leq x$$

$$x \leq y, y \leq z \rightarrow x \leq z$$

$$x \leq_\epsilon y, R_f(x_1, \dots, x_n, x) \rightarrow R_f(x_1, \dots, x_n, y) \text{ if } f \in \Sigma_{\bar{\epsilon} \rightarrow \epsilon}$$

(Her)

$$x \leq y, P_e(x) \rightarrow P_e(y)$$

(Ren)(0, 1) $\neg P_0(x) \quad P_1(x)$

$$\left\{ \begin{array}{l} (\wedge) P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x) \\ (\vee) P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x) \end{array} \right.$$

$$(\Sigma_{\epsilon_1 \dots \epsilon_n \rightarrow \epsilon}) P_{f(e_1, \dots, e_n)}(x) \leftrightarrow (\exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\epsilon_i} \wedge R_f(x_1 \dots x_n, x)))^\epsilon$$

$$(\rightarrow) P_{e_1 \rightarrow e_2}(x) \leftrightarrow \forall y (x \leq y \wedge P_{e_1}(y) \rightarrow P_{e_2}(y))$$

(N)

$$\exists c \in X : \neg P_\phi(c)$$

Automated Theorem Proving

The same ideas can be used for giving resolution-based decision procedures for uniform word problems, i.e. for deciding:

$$(R)DLO_{\Sigma}^S \models \forall \bar{x} \left(\bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s' \right)$$

$$(R)DLO_{\Sigma}^A \models \forall \bar{x} \left(\bigwedge_{i=1}^n s_i = s'_i \rightarrow s = s' \right)$$

Class of algebras	Complexity of resolution decision procedure
[R]DLO _Σ , [R]DLO _Σ ^S	EXPTIME
BAO _Σ , BAO _Σ ^S	EXPTIME
DLO _Σ ^A , RDLO _Σ ^A	EXPTIME
HA	DEXPTIME (ordered chaining with selection)

Conclusions

- Resolution-based theorem proving in non-classical logic
 1. Finitely-valued logics (superposition; ordered chaining)
 2. Infinitely-valued logics
 3. Modal logics and generalizations
 - logics based on distributive lattices with operators
 - applications: knowledge representation, terminological reasoning
 4. Beyond modal logics: uniform word problems
- Advantages
 - use existing powerful theorem provers for first-order logic
 - no sophisticated encodings
 - obtain decision procedures with optimal complexity.