Combinations of logical theories

An introduction

Viorica Sofronie-Stokkermans

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Combinations of decision procedures
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Overview

- **Theories**

- **Extensions and combinations of theories**

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- **Theories**
  - decidability for specific fragments

- **Extensions and combinations of theories**
  - decidability and modularity for specific fragments

    positive answers and limitations

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- **Theories** (syntactic vs. semantic view)
  - decidability for specific fragments

- **Extensions and combinations of theories**
  - decidability and modularity results

    positive answers and limitations

Logical theories

Theories have a syntactic and a semantic aspect.

Syntactic view

first-order theory: given by a set $\mathcal{F}$ of (closed) first-order Σ-formulae.

the models of $\mathcal{F}$: $\quad \mathsf{Mod}(\mathcal{F}) = \{\mathcal{A} \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$

Semantic view

given a class $\mathcal{M}$ of Σ-algebras

the first-order theory of $\mathcal{M}$: $\mathsf{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$

Theories

$\mathcal{F}$ set of (closed) first-order formulae

$$\mathsf{Mod}(\mathcal{F}) = \{A \in \Sigma\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{F}\}$$

$\mathcal{M}$ class of Σ-algebras

$$\mathsf{Th}(\mathcal{M}) = \{G \in F_\Sigma(X) \text{ closed} \mid \mathcal{M} \models G\}$$

$\mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$
represents exactly the set of consequences of $\mathcal{F}$

Theories

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$\text{Th}(\text{Mod}(\mathcal{F}))$ the set of formulae true in all models of $\mathcal{F}$
represents exactly the set of consequences of $\mathcal{F}$

Note: $\mathcal{F} \subseteq \text{Th}(\text{Mod}(\mathcal{F}))$ (typically strict)

$\mathcal{M} \subseteq \text{Mod}(\text{Th}(\mathcal{M}))$ (typically strict)

Examples

1. Groups

Let $\Sigma = (\{e/0, */2, i/1\}, \emptyset)$

Let $\mathcal{F}$ consist of all (universally quantified) group axioms:

$$\begin{array}{ll} \forall x, y, z & x * (y * z) \approx (x * y) * z \\ \forall x & x * i(x) \approx e \quad \wedge \quad i(x) * x \approx e \\ \forall x & x * e \approx x \quad \wedge \quad e * x \approx x \end{array}$$

Every group $\mathcal{G} = (G, e_G, *_G, i_G)$ is a model of $\mathcal{F}$

Mod($\mathcal{F}$) is the class of all groups

$\mathcal{F} \subset \mathrm{Th}(\mathrm{Mod}(\mathcal{F}))$

Examples

2. Presburger arithmetic

Let $\Sigma_{\mathsf{Pres}} = (\{0/0, s/1, +/2\}, \{\leq /2\})$

Let $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +, \leq)$ the standard interpretation of integers.

Th($\mathbb{Z}_+$) is called Presburger arithmetic

3. Uninterpreted function symbols

Let $\Sigma = (\Omega, \Pi)$ be arbitrary

Let $\mathcal{M} = \Sigma$-alg be the class of all Σ-structures

The theory of uninterpreted function symbols is Th(Σ-alg) the family of all first-order formulae which are true in all Σ-algebras.

Examples

4. Lists

Let $\Sigma = (\{\mathsf{car}/1, \mathsf{cdr}/1, \mathsf{cons}/2\}, \emptyset)$

Let $\mathcal{F}$ be the following set of list axioms:

$$\begin{array}{rcl} \mathsf{car}(\mathsf{cons}(x, y)) & \approx & x \\ \mathsf{cdr}(\mathsf{cons}(x, y)) & \approx & y \\ \mathsf{cons}(\mathsf{car}(x), \mathsf{cdr}(x)) & \approx & x \end{array}$$

$\mathsf{Mod}(\mathcal{F})$ class of all models of $\mathcal{F}$

$\mathsf{Th}_{\mathsf{Lists}} = \mathsf{Th}(\mathsf{Mod}(\mathcal{F}))$ theory of lists (axiomatized by $\mathcal{F}$)

Decidable theories

Let $\Sigma = (\Omega, \Pi)$ be a signature.

Let $\mathcal{M}$ be a class of Σ-algebras, and $\mathsf{Th}(\mathcal{M})$ the theory of $\mathcal{M}$.

$\mathcal{T} = \mathsf{Th}(\mathcal{M})$ is decidable

iff

there is an algorithm which, for every closed first-order formula ϕ, can decide (after a finite number of steps) whether ϕ is in $\mathcal{T}$ or not.

Decidable theories

Let $\Sigma = (\Omega, \Pi)$ be a signature.

Let $\mathcal{F}$ be a class of (closed) first-order formulae.

Let $\mathcal{T} = Th(Mod(\mathcal{F}))$ the theory axiomatized by $\mathcal{F}$.

The theory $\mathcal{T}$ is decidable

iff

there is an algorithm which, for every closed first-order formula ϕ, can decide (in finite time) whether ϕ is in $\mathcal{T}$ or not.

iff

there is an algorithm which, for every closed first-order formula ϕ, can decide (in finite time) whether $\mathcal{F} \models \phi$ or not.

Examples

Decidable theories

Th($\mathbb{Z}_+$) (Presburger arithmetic) dec.in 3EXPTIME [Presburger'29]

Th($\mathbb{R}$, $\{+/2, */2\}$, $\{\leq /2\}$) dec. in 2EXPTIME [Tarski'30]

Undecidable theories

Th(($\mathbb{Z}$, $\{+, *\}$, $\{\leq\}$))

Th(Σ-alg)

Examples

Decidable theories

$\text{Th}(\mathbb{Z}_+)$ (Presburger arithmetic) dec.in 3EXPTIME [Presburger'29]

$\text{Th}(\mathbb{R}, \{+/2, */2\}, \{\leq /2\})$ dec. in 2EXPTIME [Tarski'30]

Undecidable theories

$\text{Th}((\mathbb{Z}, \{+, *\}, \{\leq\}))$

Th(Σ-alg) (decision procedures are known for subfragments

e.g. for universal formulae or clauses)

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Problems

The $\mathcal{T}$-validity problem

Let $\mathcal{T}$ be a first-order theory in signature Σ

Let $\mathcal{L}$ be a class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Problems

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Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Common restrictions on $\mathcal{L}$

	$\Pi = \emptyset$	$\{\phi \in \mathcal{L} \mid \mathcal{T} \models \phi\}$
$\mathcal{L}=\{\forall x A(x) \mid A \text{ atomic}\}$	word problem	
$\mathcal{L}=\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	uniform word problem	$\mathrm{Th}_{\forall \mathrm{Horn}}$
$\mathcal{L}=\{\forall x C(x) \mid C(x) \text{ clause}\}$	clausal validity problem	$\mathrm{Th}_{\forall,\mathrm{cl}}$
$\mathcal{L}=\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	universal validity problem	$\mathrm{Th}_{\forall}$

Problems

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Common restrictions on $\mathcal{L}$

	$\Pi = \emptyset$	$\{\phi \in \mathcal{L} \mid \mathcal{T} \models \phi\}$
$\mathcal{L}=\{\exists x A_1 \wedge \ldots \wedge A_n \mid A_i \text{ atomic}\}$	unification problem	$\mathrm{Th}_{\exists}$
$\mathcal{L}=\{\forall x \exists x A_1 \wedge \ldots \wedge A_n \mid A_i \text{ atomic}\}$	unification with constants	$\mathrm{Th}_{\forall\exists}$

$\vdots$

Problems

The $\mathcal{T}$-validity problem

Let $\mathcal{T}$ be a first-order theory in signature Σ
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Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Examples of decidable $\mathcal{T}$-validity problems for subfragments $\mathcal{L}$:

1. Presburger arithmetic $\mathcal{T} = \mathcal{L} = \text{Th}(\mathbb{Z}_+)$
2. Universal theory of UIF $\mathcal{T} = \text{Th}(\Sigma\text{-alg})$
   $\mathcal{L}$: all universally quantified clauses (formulae)
3. Universal theory of lists $\mathcal{T} = \text{Th}_{\text{Lists}}$
   $\mathcal{L}$: all universally quantified clauses (formulae)

$\mathcal{T}$-validity

$\mathcal{T}$ first-order theory in signature Σ; $\mathcal{L}$ class of (closed) Σ-formulae

Common restrictions on $\mathcal{L}$

	$\Pi = \emptyset$	$\{\phi \in \mathcal{L} \mid \mathcal{T} \models \phi\}$
$\mathcal{L}=\{\forall x(A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	uniform word problem	$\mathrm{Th}_{\forall_{\mathrm{Horn}}}$
$\mathcal{L}=\{\forall x C(x) \mid C(x) \text{ clause}\}$	clausal validity problem	$\mathrm{Th}_{\forall,\mathrm{cl}}$
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Remarks:

- $\mathcal{T}$-validity for $\mathrm{Th}_{\forall,\mathrm{cl}}$ decidable iff $\mathcal{T}$-validity for $\mathrm{Th}_{\forall}$ decidable
- For convex theories:
  $\mathcal{T}$-validity for $\mathrm{Th}_{\forall,\mathrm{Horn}}$ decidable iff $\mathcal{T}$-validity for $\mathrm{Th}_{\forall,\mathrm{cl}}$ decidable.

Convex theories

Definition A first-order Σ-theory $\mathcal{T}$ is Σ_0-convex ($\Sigma_0 \subseteq \Sigma$) iff

whenever $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{m} B_j$, where $A_1, \ldots, A_n$ are Σ-atoms,

and $B_1, \ldots, B_m$ are Σ_0-atoms

there exists $k \in \{1, \ldots, m\}$ such that $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow B_k$,

Examples of convex theories

1. Let $\mathbb{Q}$ be the theory of rational numbers with linear arithmetic.

   Then $\mathbb{Q}$ is convex with respect to equality atoms, i.e.

   if $\mathbb{Q} \models \bigwedge_{i} A_i \rightarrow \bigvee_{j} t_i \approx t_i'$ then $\mathbb{Q} \models \bigwedge_{i=1}^{n} A_i \rightarrow t_k \approx t_k'$ for some k

Convex theories

Definition A first-order Σ-theory $\mathcal{T}$ is Σ_0-convex $(\Sigma_0 \subseteq \Sigma)$ iff

whenever $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{m} B_j$, where $A_1, \ldots, A_n$ are Σ-atoms,

and $B_1, \ldots, B_m$ are Σ_0-atoms

there exists $k \in \{1, \ldots, m\}$ such that $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow B_k$,

Examples of convex theories

2. Let $\mathbb{Q}$ be the theory of rational numbers with linear arithmetic.

   $\mathbb{Q}$ is not convex with respect to inequality atoms:

   $\mathbb{Q} \models x \leq y \vee y \leq x$ but $\mathbb{Q} \not\models x \leq y$ and $\mathbb{Q} \not\models y \leq x$.

Convex theories

Definition A first-order Σ-theory $\mathcal{T}$ is Σ_0-convex ($\Sigma_0 \subseteq \Sigma$) iff

whenever $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{m} B_j$, where $A_1, \ldots, A_n$ are Σ-atoms,
and $B_1, \ldots, B_m$ are Σ_0-atoms

there exists $k \in \{1, \ldots, m\}$ such that $\mathcal{T} \models \bigwedge_{i=1}^{n} A_i \rightarrow B_k$,

Examples of convex theories

3. Theories axiomatized by sets of Horn clauses are convex

(Any theory $\mathcal{T}$ such that Mod($\mathcal{T}$) closed under products is convex).

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T}$-validity: Let $\mathcal{T}$ be a first-order theory in signature Σ
Let $\mathcal{L}$ be a class of (closed) Σ-formulae

Given ϕ in $\mathcal{L}$, is it the case that $\mathcal{T} \models \phi$?

Remark: $\mathcal{T} \models \phi$ iff $\mathcal{T} \cup \neg\phi$ unsatisfiable

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

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Remark: $\mathcal{T} \models \phi$ iff $\mathcal{T} \cup \neg\phi$ unsatisfiable

Every $\mathcal{T}$-validity problem has a dual $\mathcal{T}$-satisfiability problem:

$\mathcal{T}$-satisfiability: Let $\mathcal{T}$ be a first-order theory in signature Σ
Let $\mathcal{L}$ be a class of (closed) Σ-formulae

$$\neg\mathcal{L} = \{\neg\phi \mid \phi \in \mathcal{L}\}$$

Given ψ in $\neg\mathcal{L}$, is it the case that $\mathcal{T} \cup \psi$ is satisfiable?

$\mathcal{T}$-satisfiability

$\mathcal{T}$-satisfiability: Let $\mathcal{T}$ be a first-order theory in signature Σ

Let $\mathcal{L}$ be a class of (closed) Σ-formulae

$$\neg\mathcal{L} = \{\neg\phi \mid \phi \in \mathcal{L}\}$$

Given ψ in $\neg\mathcal{L}$, is it the case that $\mathcal{T} \cup \psi$ is satisfiable?

Common restrictions on $\mathcal{L}$ / $\neg\mathcal{L}$

$\mathcal{L}$	$\neg\mathcal{L}$
$\{\forall x A(x) \mid A \text{ atomic}\}$	$\{\exists x \neg A(x) \mid A \text{ atomic}\}$
$\{\forall x (A_1 \wedge \ldots \wedge A_n \rightarrow B) \mid A_i, B \text{ atomic}\}$	$\{\exists x (A_1 \wedge \ldots \wedge A_n \wedge \neg B) \mid A_i, B \text{ atomic}\}$
$\{\forall x \bigvee L_i \mid L_i \text{ literals}\}$	$\{\exists x \bigwedge L'_i \mid L'_i \text{ literals}\}$
$\{\forall x \phi(x) \mid \phi(x) \text{ unquantified}\}$	$\{ \exists x\ \phi'(x) \mid \phi'(x) \text{ unquantified}\}$

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T} \models \forall x A(x)$	iff	$\mathcal{T} \cup \exists x \neg A(x)$ unsatisfiable
$\mathcal{T} \models \forall x (A_1 \wedge \cdots \wedge A_n \rightarrow B)$	iff	$\mathcal{T} \cup \exists x (A_1 \wedge \cdots \wedge A_n \wedge \neg B)$ unsatisfiable
$\mathcal{T} \models \forall x (\bigvee_{i=1}^n A_i \vee \bigvee_{j=1}^m \neg B_j)$	iff	$\mathcal{T} \cup \exists x (\neg A_1 \wedge \cdots \wedge \neg A_n \wedge B_1 \wedge \cdots \wedge B_m)$ unsatisfiable

$\mathcal{T}$-validity vs. $\mathcal{T}$-satisfiability

$\mathcal{T} \models \forall x A(x)$	iff	$\mathcal{T} \cup \exists x \neg A(x)$ unsatisfiable
$\mathcal{T} \models \forall x (A_1 \wedge \cdots \wedge A_n \rightarrow B)$	iff	$\mathcal{T} \cup \exists x (A_1 \wedge \cdots \wedge A_n \wedge \neg B)$ unsatisfiable
$\mathcal{T} \models \forall x (\bigvee_{i=1}^{n} A_i \vee \bigvee_{j=1}^{m} \neg B_j)$	iff	$\mathcal{T} \cup \exists x (\neg A_1 \wedge \cdots \wedge \neg A_n \wedge B_1 \wedge \cdots \wedge B_m)$ unsatisfiable

$\mathcal{T}$-satisfiability vs. Constraint Solving

The field of Constraint Solving also deals with satisfiability problems

But be careful:

- in Constraint Solving one is interested if a formula is satisfiable in a given, fixed model of $\mathcal{T}$.
- in $\mathcal{T}$-satisfiability one is interested if a formula is satisfiable in any model of $\mathcal{T}$ at all.

Overview

- **Theories** (syntactic vs. semantic view)
  - decidability for specific fragments

- **Combinations of theories**
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    positive answers and limitations

Problems

The combined validity problem

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ-formulae

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$

Let $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ be a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$

Given ϕ in $\mathcal{L}_1 \bigoplus \mathcal{L}_2$, is it the case that $\mathcal{T}_1 \bigoplus \mathcal{T}_2 \models \phi$?

Problems

The combined decidability problem I

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ-formulae

such that the $\mathcal{T}_i$-validity problem for $\mathcal{L}_i$ is decidable

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$

Let $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ be a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$

Is the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$-validity problem for $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ decidable?

Problems

The combined decidability problem II

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ-formulae
- P_i decision procedure for $\mathcal{T}_i$-validity for $\mathcal{L}_i$

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$
Let $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ be a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$

Can we combine P_1 and P_2 modularily into a decision procedure for the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$-validity problem for $\mathcal{L}_1 \bigoplus \mathcal{L}_2$?

The combined decidability problem I

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ-formulae

such that the $\mathcal{T}_i$-validity problem for $\mathcal{L}_i$ is decidable

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$

Let $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ be a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$

Is the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$-validity problem for $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ decidable?

In general: No (restrictions needed for affirmative answer)

Main issue: How are $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ and $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ defined?

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Combinations of theories and models

Forgetting symbols

Let $\Sigma = (\Omega, \Pi)$ and $\Sigma' = (\Omega', \Pi')$

Suppose $\Sigma \subseteq \Sigma'$, that is, $\Omega \subseteq \Omega'$ and $\Pi \subseteq \Pi'$

For $\mathcal{A} \in \Sigma'$-alg, we denote by $\mathcal{A}_{|\Sigma}$ the Σ-structure for which:

$$
\begin{array}{rcll}
U_{\mathcal{A}_{|\Sigma}} & = & U_{\mathcal{A}} & \\
f_{\mathcal{A}_{|\Sigma}} & = & f_{\mathcal{A}} & \text{for } f \in \Omega \\
P_{\mathcal{A}_{|\Sigma}} & = & P_{\mathcal{A}} & \text{for } f \in \Omega
\end{array}
$$

(ignore functions and predicates associated with symbols in $\Sigma' \backslash \Sigma$)

$\mathcal{A}_{|\Sigma}$ is called the restriction of $\mathcal{A}$ to Σ.

One possibility of combining theories

Example:

$\Sigma' = (\{+/2, */2, 1/0\}, \{\leq /2, \mathsf{even}/1, \mathsf{odd}/1\})$

$\Sigma = (\{+/2, 1/0\}, \{\leq /2\}) \subseteq \Sigma'$

$\mathcal{N} = (\mathbb{N}, +, *, 1, \leq, \mathsf{even}, \mathsf{odd})$

$\mathcal{N}_{|\Sigma} = (\mathbb{N}, +, 1, \leq)$

Combinations of theories and models

Restrictions for classes of models

Let $\mathcal{M}$ be a class of Σ'-algebras

$\mathcal{M}_{|\Sigma} = \{\mathcal{A}_{|\Sigma} \mid \mathcal{A} \in \mathcal{M}\}$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

$\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2\}$

where $\Sigma_1 \cup \Sigma_2 = (\Omega_1, \Pi_1) \cup (\Omega_2, \Pi_2) = (\Omega_1 \cup \Omega_2, \Pi_1 \cup \Pi_2)$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

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Semantic view: Let $\mathcal{M}_i = \mathsf{Mod}(\mathcal{T}_i), i = 1, 2$

$\mathcal{M}_1 + \mathcal{M}_2 = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i \text{ for } i = 1, 2\}$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

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$$
\begin{aligned}
\mathcal{A} \in \mathrm{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) \quad & \text{iff} \quad \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2 \\
& \text{iff} \quad \mathcal{A}_{|\Sigma_i} \models G, \text{ for all } G \text{ in } \mathcal{T}_i, i = 1, 2 \\
& \text{iff} \quad \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i, i = 1, 2 \\
& \text{iff} \quad \mathcal{A} \in \mathcal{M}_1 + \mathcal{M}_2
\end{aligned}
$$

One possibility of combining theories

Syntactic view: $\mathcal{T}_1 + \mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq F_{\Sigma_1 \cup \Sigma_2}(X)$

$\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2) = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A} \models G, \text{ for all } G \text{ in } \mathcal{T}_1 \cup \mathcal{T}_2\}$

Semantic view: Let $\mathcal{M}_i = \mathsf{Mod}(\mathcal{T}_i), i = 1, 2$

$\mathcal{M}_1 + \mathcal{M}_2 = \{\mathcal{A} \in (\Sigma_1 \cup \Sigma_2)\text{-alg} \mid \mathcal{A}_{|\Sigma_i} \in \mathcal{M}_i \text{ for } i = 1, 2\}$

Remark: $\mathcal{A} \in \mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2)$ iff ($\mathcal{A}_{|\Sigma_1} \in \mathsf{Mod}(\mathcal{T}_1)$ and $\mathcal{A}_{|\Sigma_2} \in \mathsf{Mod}(\mathcal{T}_2)$)

Consequence: $\mathsf{Th}(\mathsf{Mod}(\mathcal{T}_1 \cup \mathcal{T}_2)) = \mathsf{Th}(\mathcal{M}_1 + \mathcal{M}_2)$

Example

1. Presburger arithmetic + theory of free function symbols

$\mathrm{Th}(\mathbb{Z}_+) \cup \mathit{Free}_\Sigma$

Models: $(A, 0, s, +, \{f_A\}_{f \in \Omega}, \leq, \{P_A\}_{P \in Pi})$

where $(A, 0, s, +, \leq) \in \mathrm{Mod}(\mathrm{Th}(\mathbb{Z}_+)$.

2. Presburger arithmetic + theory of lists

$\mathrm{Th}(\mathbb{Z}_+) \cup \mathrm{Th}_{\mathrm{Lists}}$

Models: $(A, 0, s, +, \mathrm{car}_A, \mathrm{cdr}_A, \mathrm{cons}_A, \leq)$

where $(A, 0, s, +, \leq) \in \mathrm{Mod}(\mathrm{Th}(\mathbb{Z}_+))$.

$(A, \mathrm{car}_A, \mathrm{cdr}_A, \mathrm{cons}_A) \in \mathrm{Mod}(\mathrm{Th}_{\mathrm{Lists}})$

Example

3. The theory of reals + the theory of a monotone function f

$\mathrm{Th}(\mathbb{R}) \cup \mathrm{Mon(f)}$ $\qquad$ $\mathrm{Mon(f)}: \forall x, y (x \leq y \rightarrow f(x) \leq f(y))$

Models: $(A, +, *, f_A, \{\leq\})$, where

where $(A, +, *, \leq) \in \mathrm{Mod}(\mathrm{Th}(\mathbb{R}))$.

$(A, f_A, \leq) \models \mathrm{Mon}(f)$, i.e. $f_A : A \rightarrow A$ monotone.

Note:

The signatures of the two theories share the $\leq$ predicate symbol

Combinations of theories

Definition. A theory is consistent if it has at least one model.

Question: Is the union of two consistent theories always consistent?

Answer: No. (Not even when the two theories have disjoint signatures)

Example: $\Sigma_1 = (\Omega_1, \emptyset), \quad \Sigma_2 = (\{c/0.d/0\}, \emptyset), \quad c, d \not\in \Omega_1$

$\mathcal{T}_1 = \{\exists x, y, z(x \not\approx y \ \wedge \ x \not\approx z \ \wedge \ y \not\approx z)\}$

$\mathcal{T}_2 = \{\forall x(x = c \ \vee \ x = d)\}$

$\mathcal{A} \in \mathrm{Mod}(\mathcal{T}_1) \quad \text{iff} \quad |A| \geq 3.$

$\mathcal{B} \in \mathrm{Mod}(\mathcal{T}_2) \quad \text{iff} \quad |A| \leq 2.$

Overview

- Theories (syntactic vs. semantic view)
  - decidability for specific fragments

- Combinations of theories
  - decidability and modularity results

    positive answers and limitations

The combined decidability problem I

For $i = 1, 2$

- let $\mathcal{T}_i$ be a first-order theory in signature Σ_i
- let $\mathcal{L}_i$ be a class of (closed) Σ-formulae

such that the $\mathcal{T}_i$-validity problem for $\mathcal{L}_i$ is decidable

Let $\mathcal{T}_1 \bigoplus \mathcal{T}_2$ be a combination of $\mathcal{T}_1$ and $\mathcal{T}_2$

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Is the $\mathcal{T}_1 \bigoplus \mathcal{T}_2$-validity problem for $\mathcal{L}_1 \bigoplus \mathcal{L}_2$ decidable?

In general: No (restrictions needed for affirmative answer)

Undecidable combination of decidable theories

Word problem for a theory $\mathcal{T}$: Decide if $\mathcal{T} \models \forall x(s \approx t)$

$\mathcal{A}$: theory of associativity

$\mathcal{G}$ finite set of ground equations
(presentation for semigroup
with undecidable word problem)

$\uparrow$

($\exists$ finitely-presented semigroup with
undecidable word problem [Matijasevic'67])

Word problem: decidable for $\mathcal{A}, \mathcal{G}$; undecidable for $\mathcal{A} \cup \mathcal{G}$

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Simpler instances: combinations of theories over disjoint signatures, theories sharing constructors, compatibility with shared theory ...

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- Nelson/Oppen procedure for reasoning in combinations of theories

    combinations of theories over disjoint signatures

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  for specific types of theories and problems

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- Local theory extensions

  consider only extensions of a theory

  (results can be also used for some combinations of theories over non-disjoint signatures)

Other possibilities of combination

- Many-sorted combinations

Example: Flat acyclic lists of integer elements

Two sorts: elements, lists

$(E, L, +, \leq, =, \text{car}, \text{cdr}, \text{cons})$

$(E, +, \leq, =) \in \text{Mod}(\text{Th}(\mathbb{Z}_+))$

car : $L \rightarrow E$

cdr : $L \rightarrow L$

cons : $E \times L \rightarrow L$

satisfy list axioms + acyclicity:

$x \neq t(x)$ if t built only using cons

Extension with a length function.

Overview

- SAT checking modulo a theory

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- Applications