Bipartite Matching

The problem

Bipartite graph $G = (V_1 \cup V_2, E)$

All edges are of the form (u, v) where $u \in V_1$ and $v \in V_2$

Matching = set of vertex-disjoint edges in G

If M is a matching, each node (vertex) occurs at most once as an endpoint of an edge in M

Goal: find a matching of maximum cardinality, i.e., the largest possible amount of edges

Applications

Examples:

- scheduling, where each machine can process a subset of the jobs, and we want to maximize the number of jobs done
- video processing: matching moving objects based on a sequence of snapshots
- . . .

Finding a matching

We look for a matching of maximum cardinality, i.e., the largest possible amount of edges

How can this be solved?

...Formulate as a maximum flow problem!

Bipartite matching as a maximum flow problem

- add a source s and a sink t
- add edges from s to every node in V_1
- add edges from every node in V_2 to t
- all edges have capacity 1

How large is this network? It has $n+2$ nodes and $m' = m+n$ edges

We have $m \geq n/2$ since every vertex in V has at least one incident edge

Thus, $m \leq m' = m+n \leq 3m$, so $m' = \Theta(m)$.

Lemma 1. *Integer-valued flows in G' correspond to matchings in G*

Proof. Matching $M \longrightarrow$ flow f: if $(u,v) \in M$, set $f(s,u) = f(u,v) = f(v,t) = 1$.

For all other edges $(u,v) \in E'$, set $f(u,v) = 0$.

Each edge $(u,v) \in M$ corresponds to one unit of flow in G' that traverses the path $s \longrightarrow u \longrightarrow v \longrightarrow t$.

Flow from s to t is equal to $|M|$

Flow $f \rightarrow$ matching M: define

$$M = \{(u,v) : u \in V_1, v \in V_2, f(u,v) > 0\}$$

Why is this a matching? Flow conservation: if some amount of flow enters some node u from s, it must leave.

Integrality: if a unit of flow enters u, it leaves along exactly one edge (u,v). At most one edge leaving u has positive flow.

By symmetry, same holds for a node $v \in V_2$.

Why is $|M| = |f|$? We have $f(s,u) = 1$ for every matched $u \in L$, and $f(u,v) = 0$ for every edge $(u,v) \in E - M$. Thus net flow across cut $\{(u,v) : u \in V_1, v \in V_2\}$ is $|M|$ $(= |f|)$. □

Completing the proof

Even if a flow is integer-valued, it could be that for some edges, $f(u,v)$ is not integer (only the sum over each cut is).

Applying Ford-Fulkerson or Dinitz returns a flow which is integral on all edges.

Therefore, this flow must correspond to a maximum matching using the method on the previous slide

Minimum Cost Bipartite Perfect Matching

A transportation problem in a bipartite graph

$G = (V_1 \cup V_2, E \subseteq V_1 \times V_2)$ with

supply$(v) = 1$ for $v \in V_1$,

supply$(v) = -1$ for $v \in V_2$.

An integral flow defines a matching

Each edge has a cost

We want to find a perfect matching of minimum cost

Perfect matching: all vertices are matched

The stable marriage problem

n guys, n girls

Everyone has a preference list of length n

How to match everyone?

Theorem: there exists a matching such that no couple wants to elope

Applications

- Origin: graduating medical students and hospital jobs
  - students rank hospitals in order of preference
  - hospitals rank students
- high-school students to universities?

Possible algorithm

Start with any matching

If there exists a non-matched couple which prefers being together over their current partners, switch them

Repeat. . .

May not terminate!

A better algorithm: Gale-Shapley

Ultra-short summary: “men propose, women accept”

As long as some man is unmatched:

- ☐ propose to first woman on his preference list that he has not yet proposed to
- ☐ if woman is unmatched, she accepts
- ☐ if woman is matched, she compares him to her current fiance; if she can improve, she breaks off her engagement and accepts the new offer

Remarks

Does the algorithm terminate?

- Each woman, once proposed to, remains matched (she uses a greedy algorithm)
- Each man proposes to each woman only once
- Suppose some man and woman remain unmatched
- At some point, this man must have proposed to her
- She was unmatched, so she must have accepted: a contradiction

Remarks

Do we find a stable matching?

- Note that the women only improve over time
- Assume we have an unhappy pair, man X and woman b, who are currently matched as $X - a$ and $Y - b$.
- X must have proposed to b before a since he prefers b
- Two possibilities:
  - b rejected X because she was already better matched
  - b accepted X, but later rejected him in favor of someone else
- In both cases, b must prefer Y to X, a contradiction

Comments

Much further work has been done, e.g.

- stable roommates problem
- Random / Fair stable matchings
- Many variants of stable matching are also solvable (indifferences, groups, forbidden pairs, . . .)
- What happens if some participants lie about their preferences?
- Stable roommates with indifferences: NPcomplete