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Colorings of graphs and Ramsey's theorem

We shall first look at a few so-called coloring problems for graphs.

A *proper coloring* of a graph G is a function from the vertices to a set C of 'colors' (e.g. $C = \{1, 2, 3, 4\}$) such that the ends of every edge have distinct colors. (So a graph with a loop will admit no proper colorings.) If $|C| = k$, we say that G is *k-colored*.

The *chromatic number* $\chi(G)$ of a graph G is the minimal number of colors for which a proper coloring exists.

If $\chi(G) = 2$ (or $\chi(G) = 1$, which is the case when and only when G has no edges), then G is called *bipartite*. A graph with no odd polygons (equivalently, no closed paths of odd length) is bipartite as the reader should verify.

The famous 'Four Color Theorem' (K. Appel and W. Haken, 1977) states that if G is planar, then $\chi(G) \leq 4$.

Clearly $\chi(K_n) = n$. If k is odd then $\chi(P_k) = 3$. In the following theorem, we show that, with the exception of these examples, the chromatic number is at most equal to the maximum degree (R. L. Brooks, 1941).

Brooks' Theorem

Theorem 3.1. Let $d \geq 3$ and let G be a graph in which all vertices have degree $\leq d$ and such that K_{d+1} is not a subgraph of G . Then $\chi(G) \leq d$.

PROOF 1: As is the case in many theorems in combinatorial analysis, one can prove the theorem by assuming that it is not true, then considering a *minimal* counterexample (in this case a graph with the minimal number of vertices) and arriving at a contradiction. We shall use the technique of recoloring: it is possible to change

the colors of certain vertices to go from one proper coloring to another. For example, let S be a subset of the set C of colors. On any connected component of the subgraph induced by the vertices with colors from S , we arbitrarily permute the colors (without changing those of the vertices with colors in $C \setminus S$). Clearly we again have a proper coloring.

So let G be a counterexample with the minimum number of vertices. Let $x \in G$ and let $\Gamma(x) = \{x_1, \dots, x_l\}$, $l \leq d$. Since G is a minimal counterexample, the graph H , obtained by deleting x and the edges incident with x , has a d -coloring, say with colors $1, 2, \dots, d$. If one of these colors is not used in the coloring of $\Gamma(x)$, then we can assign this color to x and obtain a d -coloring of G . It follows that $l = d$ and every d -coloring of H must use all the colors on the set $\Gamma(x)$. Let us assume that x_i has color i for $i = 1, 2, \dots, d$.

Now consider x_i and x_j and the induced subgraph H_{ij} of H with colors i and j . If x_i and x_j were in different connected components of H_{ij} , then we could interchange the colors in one of these components, after which x_i and x_j would have the same color, which is impossible. So x_i and x_j are in the same component (say C_{ij}) of H_{ij} . We shall now show that this component is (the graph of) a *simple path* (with alternating colors i and j) from x_i to x_j . If two neighbors of x_i in H had color j , then the neighbors of x_i in H would have at most $d - 2$ different colors. Then we could recolor x_i and that is impossible. Suppose y is the first vertex on a path from x_i to x_j in C_{ij} that has degree ≥ 3 . The neighbors of y in H have at most $d - 2$ colors, so we can recolor y to some color $\notin \{i, j\}$ and then x_i and x_j are no longer connected in H_{ij} , which we know to be impossible. So such a y does not exist, proving that C_{ij} is a path.

Suppose that z is a vertex $\neq x_i$ on C_{ij} and on C_{ik} . Then z has two neighbors with color j and two with color k . Again the neighbors of z in H have at most $d - 2$ colors and z can be recolored to some color $\notin \{i, j, k\}$, again a contradiction. Hence $C_{ij} \cap C_{ik} = \{x_i\}$.

Our assumption that $K_{d+1} \not\subseteq G$ shows that there are two vertices in $\Gamma(x)$, say x_1 and x_2 , that are not connected by an edge. We have the situation of Fig. 3.1. The vertex a is the neighbor of x_1 with color 2 on C_{12} .

We recolor H by interchanging the colors 1 and 3 on the subgraph C_{13} . For the new coloring, we have new paths that we call C'_{ij} . Clearly $a \in C'_{23}$ (since x_1 now has color 3). However, on C_{12} no point except x_1 has changed color, so $a \in C'_{12}$. Hence $C'_{12} \cap C'_{23} \neq \{x_2\}$, contradicting what we proved above. The contradiction shows that our assumption that a minimal counterexample exists is false. \square

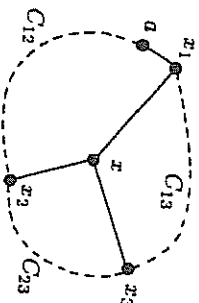


Figure 3.1

In preparation for a second proof of Brook's Theorem, the reader should do the following problem without applying the theorem.

Problem 3A. Fix an integer $d \geq 3$. Let H be a simple graph with all degrees $\leq d$ which cannot be d -colored and which is minimal (with the fewest vertices) subject to these properties. (We claim H is complete on $d+1$ vertices, but we don't know that yet.) (i) Show that H is nonseparable (this means that every graph obtained from H by deleting a vertex is connected). (ii) Then show that if the vertex set $V(H)$ is partitioned into sets X and Y with $|Y| \geq 3$, then there are at least three vertices $a, b, c \in Y$ each of which is adjacent to at least one vertex in X .