# CDM: Definitions and Theorems from Lectures 20-24

The following is a list of definitions, theorems, and notation from lectures 20-24.

**Notation 1.** Let $R(k; s_1, \ldots, s_r)$ denote the Ramsey number for set colorings of size $k$. So for graphs, $R(s,t) = R(2; s, t)$. For simplicity, we use the notation $R_r(k; s)$ such that

$$R_r(k; s) = R(k; \underbrace{s, \ldots, s}_{r \text{ times}}).$$

**Theorem 1** (A Case of Ramsey's Theorem for Sets)**.** *Let $r, k, s$ be positive integers with $s \geq k$. Then there exists a number $n = R_r(k; s)$ where $n$ is the size of the smallest set such that if all $k$-subsets are colored with $r$ colors, then there exists an subset of size $s$ where all of its $k$-subsets are colored the same color.*

**Claim 2.** *For positice integers $r, k, s$ where $s \geq k$, we have*

$$R_{r+1}(k; s) \leq R_r(k; R(k; s, s)).$$

**Theorem 3** (Schur's Theorem)**.** *For ever $r \in \mathbb{N}$, there exists a positive integer $n = S(r)$ such that for every partition of the set $[n]$ into $r$-classes, one of the classes contains two numbers $x$ and $y$ along with their sum, $x + y$.*

**Theorem 4** (The Erdös-Szekeres Theorem for Convex Polygons)**.** *Let $m \geq 3$ be a positive integer. Then there exists a positive integer $n$ such that any set of $n$ points on the Euclidean plane, no three of which are collinear, contains $m$ points which are the vertices of a convex polygon.*

**Definition 5.** A *proper coloring* of a graph $G = (V, E)$ is a function from $V$ to a set of colors such that no two adjacent vertices are the same color. The *chromatic coloring of* $G$, denoted $\chi(G)$ is the minimum number of colors needed to properly color $G$.

**Theorem 6** (Brooks' Theorem)**.** *Let $d \geq 3$ and let $G = (V, E)$ be a graph such that $\deg(v) \leq d$ for all $v \in V$, and that $K_{d+1}$ is not a subgraph of $G$. Then $\chi(G) \leq d$.*

**Definition 7.** An *embedding of a graph $G$ on a surface $S$* is a representation (or drawing) of $G$ on $S$ such that the edges of $G$ do not cross but meet only at vertices that lie on $S$. We say $G$ is *planar* if and only if it admits an embedding on a sphere.

**Definition 8.** Let $G = (V, E)$. A *subdivision of an edge* $\{u, v\} \in E$ occurs when we add a new vetrex $w$ to $V$ such that $\{u, v\}$ is replaced by $\{u, w\}$ and $\{w, v\}$. A *subdivision of* $G$ is the result of subdividing edges in $E$.

**Definition 9.** A *minor* of a graph $G$ is a graph $G'$ obtained from $G$ by deleting and contracting edges, and possibly deleting some or all isolated vertices.

**Definition 10.** A *graph isomorphism* between two graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $\varphi$ from $V$ to $V'$ such that $\{u, v\} \in V$ if and only if $\{\varphi(u), \varphi(v)\} \in E'$.

**Theorem 11** (Kuratowski's Theorem). *A graph $G$ is planar if and only if it has no minors isomorphic to subdivisions of $K_5$ or $K_{3,3}$.*

**Definition 12.** A *face of an embedding of graph $G$ on a surface $S$* is a region bounded by edges.

**Theorem 13** (Euler's Formula). *If the finite planar graph $G = (V, E)$ is embedded on a place or a sphere, then*

$$1 - |F| + |E| - |V| + |C| = 0$$

*where $F$ is the set of faces and $C$ is the set of components of $G$.*

**Definition 14.** A graph is *simple* if it is undirected and has no multiple edges or loops.

**Proposition 15.** *Let $G = (V, E)$ be a simple planar graph. Then there exists $v \in V$ such that* $\deg(v) \leq 5$.

**Theorem 16** (The Five Color Theorem). *If $G$ is a simple planar graph, then $\chi(G) \leq 5$.*

**Definition 17.** Let $G = (V, E)$. A *spanning subgraph of $G$*, is a graph $G' = (V', E')$ such that $V' = V$ and $E' \subseteq E$. A *spanning tree of $G$*, is a spanning subgraph of $G$ that is a tree.

**Theorem 18** (Alternative Version of Cayley's Formula). *The complete graph $K_n$ has $n^{n-2}$ spanning trees.*