

CDM: Definitions and Theorems from Lectures 6 & 7

The following is a list of definitions and theorems from lectures 6 and 7.

Let $G = (V, E)$ be a directed graph with $s, t \in V$ where s is the source and t is the destination. For all $e \in E$, let $c_e \in \mathbb{R}^+$ denote the capacity of edge e .

Definition 1. We say f is a *flow network* of G if

1. $0 \leq f_e \leq c_e$, and
2. $\sum_{e \in \text{in}(v)} f_e = \sum_{e \in \text{out}(v)} f_e$ for each $v \in V \setminus \{s, t\}$.

The *flow volume*, denoted $|f|$, satisfies the equation

$$|f| = \sum_{e \in \text{out}(s)} f_e - \sum_{e \in \text{in}(s)} f_e.$$

Definition 2. Let $V = U \cup W$ such that $U \cap W = \emptyset$ and $s \in U, t \in W$. The *cut* K is the set of all edges $(x, y) \in E$ such that $x \in U$ and $y \in W$. The *size of cut* K satisfies the equation

$$|K| = \sum_{(x,y) \in K} c_{(x,y)}.$$

Theorem 1 (Max-Flow Min-Cut Theorem). *For graph $G = (V, E)$, the maximum volume of flow f of G is equal to the minimize size of a cut K in G .*

Definition 3. Given a flow network f , an *augmenting path* $p = (v_0, \dots, v_j)$ has the following properties:

1. $v_0 = s$,
2. $v_j = t$, and
3. Either $e = (v_i, v_{i+1}) \in E$ and $f_e < c_e$, or $e = (v_{i+1}, v_i) \in E$ and $f_e > 0$, for all $0 \leq i \leq j - 1$.

Theorem 2. *A flow network f is the maximum flow of graph G if and only if it has no augmenting paths.*

Corollary 1. *If for all $e \in E$ we have $c_e \in \mathbb{Z}^+$, then there exists a maximum flow f such that*

1. $|f| \in \mathbb{Z}^+$, and

2. $f_e \in \mathbb{Z}^+ \cup \{0\}$, $\forall e \in E$.

Theorem 3 (Menger's Theorem: Directed Graph, Edge-Disjoint Version). *Suppose we need to remove a minimum of w edges from G in order to disconnect s and t . Then there exist w edge-disjoint paths from s to t .*

Definition 4. For all $v \in V$, we let $b(v)$ denote the *demand* of vertex v . If $b(v) > 0$, then v is considered to be a *demand point*. If $b(v) < 0$, then v is considered to be a *supply point*.

Definition 5. We say that flow f is *feasible* if

1. $\sum_{e \in \text{in}(v)} f_e - \sum_{e \in \text{out}(v)} f_e = b(v)$, $\forall v \in V$, and
2. $0 \leq f_e \leq c_e$, $\forall e \in E$.

Definition 6. A *path flow* is a directed path on which the flow value of each edge of the path is the same. A *cycle flow* is a directed cycle on which the flow value of each edge of the path is the same.

Theorem 4 (Flow Decomposition Theorem). *Every flow can be decomposed into a set of path flows and cycle flows. Moreover, there are at most $|V| + |E|$ of these.*

Definition 7. A *circulation* is a flow f with the property that $|f| = 0$.

Corollary 2. *Any circulation can be decomposed into at most $|E|$ cycles.*

Definition 8. We say a set of paths in G is *vertex-disjoint* if no two paths share any vertices except s and t .

Theorem 5 (Menger's Theorem: Directed Graph, Vertex-Disjoint Version). *Suppose we need to remove a minimum of k vertices from G in order to disconnect s and t . Then there exist k vertex-disjoint paths from s to t .*