CDM: Definitions and Theorems from Lectures 1 & 2

The following is a list of definitions and theorems from lectures 1 and 2. The material from the powerpoint shown in lecture 1 will *not* be included in this document.

Definition 1. Let G be a graph with n nodes. A *matching* M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. In the boy-girl scenario, a matching is a set of couples. A *maximal matching* for a graph G is a matching M such that there does not exist a matching M' of G where $M \subset M'$. A *maximum matching* is a matching that contains the largest possible number of edges. A *perfect matching* in G is a matching consisting of $n/2$ edges, i.e. the largest possible matching, or the set of couples that encompasses all the boy and girls.

Theorem 1 (Hall's Marriage Theorem)**.** *Suppose we have a set of n boys, denoted B, and a set of n girls, denoted G, and there exist several possible ways to make couples. For each set B', let $PW(B')$ denote the set of possible wives for the boys in B'. Then $\forall B' \subseteq B$, $|B'| \leq |PW(B')|$ if and only if there exists a perfect matching.*

Definition 2. Let $G = (V, E)$ be a graph such that the vertex set V can be partitioned into two sets A and B. If there does not exist any edges $\{u, v\} \in E$ such that u and v are either both in A or both in B, then G is known as a *bipartite graph.*

Definition 3. A *vertex cover* in a graph G is a set of vertices that includes at least one endpoint of each edge, and a vertex cover is *minimum* if no other vertex cover has fewer vertices.

Theorem 2 (König's Theorem)**.** *Let $G = (V, E)$ be a bipartite graph. Then the number of edges in a maximum matching of G equals the number of vertices in a minimum vertex cover of G.*

Definition 4. Let M be a matching in graph G. An *alternating path* is a path in which the edges belong alternatively to M and not to M. An *augmenting path* is an alternating path that starts from and ends on unmatched vertices.

Theorem 3 (Berge's Theorem)**.** *Let $G = (V, E)$ be a graph with a matching M. Then M is a maximum matching if and only if it contains for augmenting paths.*

Definition 5. Let A be a 0-1 matrix. A set S of rows and columns is a *cover* of A if A becomes a zero matrix after all the lines in S have been deleted. Two 1's are said to be *dependent* if they are in the same row or the same column; otherwise, they are *independent.* If A is an $n \times n$ matrix with exactly n independent 1's, A is a *permutation matrix.*