CDM: Assignment 1

Due on Wednesday May 13

Section I

Answer all questions in this section. Make sure to explain your solutions completely to receive full credit.

1. The dating service *Saw you in Saarland* has just been created, and so far only 5 men and 5 women have signed up. After checking out each other's profiles, all the customers came up with their own preference lists:

   Table 1: Men's Rankings

Man	1st	2nd	3rd	4th	5th
Andrew	Britney	Angela	Diane	Eva	Christina
Bob	Angela	Diane	Eva	Christina	Britney
Charles	Diane	Angela	Christina	Britney	Eva
Dave	Eva	Diane	Angela	Britney	Christina
Ed	Diane	Angela	Britney	Christina	Eva

   Table 1: Women's Rankings

Woman	1st	2nd	3rd	4th	5th
Angela	Bob	Dave	Andrew	Ed	Charles
Britney	Andrew	Bob	Dave	Charles	Ed
Christina	Andrew	Dave	Bob	Ed	Charles
Diane	Charles	Ed	Bob	Andrew	Dave
Eva	Dave	Andrew	Ed	Charles	Bob

   Find stable relationships!

2. In class, we presented König's theorem using the language of computer science: the size of the minimum vertex cover is the size of the maximum matching. Suppose, instead, we are given a 0-1 matrix. How would we express the same theorem in an alternative form so that it pertains to such a matrix?

3. Show that in a group of m girls and n boys there exist some t girls for whom husbands can be found if and only if for any subset of girls of size k, between them have at least $k + t - m$ potential husbands.

   Hint. *Invite an additional $m - t$ "very popular" boys who are potential husbands for all the girls. Show that at least t girls can find husbands in the original situation if and only if all the girls can finds husbands in the new situation. Then apply Hall's Marriage Theorem to the new situation.*

4. Let A be a 0-1 matrix of m 1's. Let s be the maximal number of 1's in a row or column of A, and suppose A has no square $r \times r$ all-1 sub-matrix. Use König's Theorem to show that we need at least $\frac{m}{sr}$ all-1 (not necessarily square) sub-matrices to cover all 1's in A.

   Hint. *There are at least m/s independent 1's, and at most r of them can be covered by one all-1 sub-matrix.*

Section II

Pick two questions in this section to answer. Make sure to explain your solutions completely to receive full credit.

1. In class, we presented and proved a simplified version of Birkhoff Theorem: given a 0-1 matrix A such that the sum of each row and each column is l, then $A = \sum_{i=1}^{l} P_i$, where P_i is a permutation matrix.

   (a) Suppose that the sum of each row and each column of A is still l. However, instead of 0-1 matrix, A's entries can be any natural number. Prove that A still can be expressed as the sum of l permutation matrices.

   (b) This time assume that the sum of each row and each column of A is 1. Suppose that A's entries can be any real positive number. Prove that A can be expressed as a convex combination of permutation matrices, i.e. that is $A = \sum_{i=1}^{t} \lambda_i P_i$, where each $\lambda_i > 0$ and $\sum_{i=1}^{t} \lambda_i = 1$. (Note that t can be large.)

2. Assume that we are given an n by n bipartite graph $G = (V, E)$.

   (a) Prove that if G is d-regular, that is, every vertex has exactly d incident edges, then we can use exactly d perfect matchings M_i, $i = 1, 2, \cdots, d$ to cover all edges: $\bigcup_{i=1}^{d} M_i = E$.

   (b) Suppose that G is d-regular, however, the graph G can be a *multi-graph*. That is, there can be more than one edge between the same pair of vertices. Again prove that we can cover E with d perfect matchings.

   (c) Suppose, instead of being d-regular, G has maximum degree d. Prove that we can use d matching (not necessarily perfect) to cover E.

3. At a certain famous university named UMC, there exist significantly more female students than their male counterparts. Noting that the university lacks enough relationships, an Indonesian professor by the name of Nhanov decides to write a dating script that creates pairs between the males and the females. Of course, interested in the results of science, Professor Nhanov tasks you with creating an algorithm for the script.

   (a) Suppose that n males and $2n$ females are paired with each other. Each male is in a relationship with exactly two females, and each female is in a relationship with one male. Give an algorithm to determine a successful pairing S between the people at this university and prove that the algorithm results in a stable pairing. You may assume that each male and female has a preference list with no ties.

(b) Now, let n males and mn females be paired together such that each male is in a relationship with m females, and each female is in a relationship with exactly one male. Determine a generalized algorithm that results in a stable pairing for any positive m.

4. Suppose we have a set of n boys, denoted B, and a set of n girls, denoted G. Each boy likes some subset of the girls (an equal amount, no preferences), and each girl likes a subset of the boys. In addition, the feelings of the members of each boy/pair is mutual, i.e. no boy like a girl without being liked back by her, and vice versa. For every subset of boys $B' \subset B$, let $PW(B')$ denote the set of potential wives for the boys of B'.

   (a) Show that if $|PW(B')| \geq |B'| - 1$ for all $B' \subseteq B$ then we can make $n - 1$ happy couples (i.e. an almost perfect matching).

   (b) Prove that if every boy likes at least $n/2$ girls, and every girl likes at least $n/2$ boys, then we have a perfect matching.