CDM: Assignment 4

Due on Wednesday June 24

Section I

Answer all questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Suppose that in an undirected graph, all vertices have even degrees. We can always oriented the edges in such a way that every vertex has the same in-degree and out-degree.

   (a) Give a combinatorial proof.

   (b) Give a linear algebraic proof.

   Hint. *For the algebraic part, use the matrix decomposition theorem we discussed in class.*

2. Given any undirected graph, we can always orient the edges in such a way that in every vertex, the difference between its in-degree and out-degree is at most 1.

   (a) Give a combinatorial proof.

   (b) Give a linear algebraic proof.

3. Let $n \leq 2k$ and let $A_1, A_2, \cdots, A_m$ be a k-uniform subsets of $[n]$ such that $A_i \cup A_j \neq [n]$ for all i, j. Show that $m \leq (1 - k/n)\binom{n}{k}$.

4. Here is an application of min-flow problem. An airline has p flight legs that it wishes to service by the fewest possible planes. To do so, it must determine the most efficient way to combine these legs into flight schedules. The starting time for flight i is a_i and the finishing time is b_i. The plane requires r_{ij} hours to return from the point of destination of flight i to the point of origin of flight j. Transform the above scenario into a flow problem and solve it (assuming that you already know there is a poly-time min-flow algorithm).

Section II

Answer one of the following two questions. Make sure to explain your solutions completely to receive full credit.

1. Use Min-flow-max-cut theorem to prove Dilworth's theorem.

2. Prove the following: given a digraph $G = (V, E)$, where $b(V) = 0$, $l \in (\mathbb{R} \cup \{0\})^{|E|}$ and $u \in \mathbb{R}^{|E|}$ with $l \leq u$, there exists $x \in \mathbb{R}^{|E|}$ such that (1) $f_x(v) = b_v$, $\forall v \in V$, and (2) $l_e \leq x_e \leq u_e$, $\forall e \in E$ if and only if given any subgraph $A \subseteq V$, the following holds

$$u(\delta(\overline{A})) \geq b(A) + l(\delta(A)).$$