CDM: Assignment 6

Due on Wednesday July 29

Section I

Answer all questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Prove Ramsey's Theorem for an r-colored graph: For positive integers $s_1, \ldots, s_r$ such that $s_i \geq 2$, then there exists some natural number $n = R(s_1, \ldots, s_r)$ such that if we color the edges of K_n with r colors, there is always some $i \in [r]$ such that there is an s_i-clique whose edges are all colored in the i^{th} color.

2. Prove Ramsey's Theorem for a 2-colored k-set, i.e. that the number $R(k; s, t)$ exists by showing that

$$R(k; s, t) \leq R(k-1; R(k; s-1, t), R(k; s, t-1)) + 1.$$

3. Let G satisfy the conditions of Brooks' Theorem. Show that by removing at most n/d edges we can find a subgraph G' with chromatic number $\chi(G'') \leq d - 1$.

4. How many trees T are there on the set of vertices $\{1, 2, 3, 4, 5, 6, 7\}$ in which the vertices of 2 and 3 have degree 3, vertex 5 has degree 2, and hence all others have degree 1?

   Hint: Consider the second technique we used to prove Cayley's Formula.

5. Show that a graph G that has a minor isomorphic to K_5 or $K_{3,3}$ also contains a subdivision of K_5 or $K_{3,3}$.

Section II

Answer at least four questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Define, for integer $p \geq 3$,

$$S_p(n) = \sum_{k=0}^{n} \binom{pn}{k}$$

for $n \geq 0$. Use the Lagrange Inversion Formula (backwards) to show that

$$\sum_{n\geq 0} S_p(n)x^n(1+x)^{-pn-1} = \frac{1}{(1-x)(1-(p-1)x)}.$$

2. (a) Prove that

$$R(s_1,\ldots,s_r) \leq 2 + \sum_{i=1}^{r}[R(s_1,\ldots,s_{i-1},s_i-1,s_{i+1},\ldots,s_r)-1].$$

   (b) Use the formula in part (a) to calculate

$$R(\underbrace{3,3,3,\ldots,3}_{n \text{ times}}).$$

3. (a) Prove that, for any integer $t > 0$,

$$R(t,t) \leq 2^{2t-1}.$$

   Note: You cannot use the statement of part (b) to solve this problem.

   (b) Prove that

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

   where $R(s,t)$ is the smallest number n such that on the red/green-colored complete graph K_n, there exists at least one red s-clique or one green t-clique.

4. The edges of K_n are colored red and blue in such a way that each red edge is in at most one triangle. Show that there exists a subgraph K_k with $k \geq \lfloor\sqrt{2n}\rfloor$ that contains no red triangle.

5. We say an $m \times m$ matrix, P, is a *principal submatrix* of an $n \times n$ matrix, A, if P is obtained from A by removing any $n-m$ rows and the same $n-m$ columns. Let m be given. Show that if n is large enough, every $n \times n$ (0,1)-matrix has a principal submatrix of size m, in which all the elements below the diagonal are the same, and all the elements above the diagonal are the same.

6. Prove that for a tree T on n vertices,

$$\chi_T(r) = r(r-1)^{n-1}$$

   where $\chi_T(r)$ is the number of proper coloring of T using r colors.

7. Determine all pairs (d_1, d_2) of integers with $d_1, d_2 \geq 2$, so that there exists a planar graph that is regular of degree d_1 and such that all faces have degree d_2.