CDM: Assignment 5

Due on Wednesday July 8

Section I

Answer all questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Let $h_0, h_1, h_2, \ldots$ be the sequence defined by $h_n = \binom{n}{3}$. Determine the generating function for this sequence.

2. Let B_n be the set of strings in $\{0,1\}^n$ which do not contain $1,1,1$ (in consecutive positions). Let $b_n = |B_n|$.

   (a) What are b_1, b_2, b_3, b_4?

   (b) Establish a recurrence for the sequence $b_0, b_1, b_2, \ldots$.

   (c) Determine the generating function for this sequence.

3. Let $a_0, a_1, a_2, \ldots$ be a sequence of integers such that $a_0 = 1$. Let $f(x)$ be the polynomial $f(x) = \sum_{n \geq 0} a_n x^n$. Prove that there exists a unique polynomial $g(x) = \sum_{n \geq 0} b_n x^n$ such that

$$f(x)g(x) = 1.$$

4. Consider a collection of n circles drawn in the plane such that:

   (i) Each pair of circles intersects in two distinct points, and

   (ii) the intersection of any three circles is empty (i.e. no point lies at the intersection of three circles).

   Let h_n be the number of regions in the plane created by the collection of intersecting circles (e.g. $h_1 = 2$ as there is a region inside the circle and a region outside the circle, $h_2 = 4$ and $h_3 = 8$).

   (a) For $n = 2, 3, \ldots$ write h_n as a function of $h_1, h_2, \ldots, h_{n-1}$ and n.

   (b) Use your answer to (a) to determine the generating function for the sequence $h_0, h_1, \ldots$.

   (c) Use your answer to (b) to give a closed form for h_n.

Section II

Answer at least three questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Let z_n denote the number of ways to color the squares of a $1 \times n$ board with the colors orange, white, blue and green in such a way that the number of squares colored orange is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $z_0, z_1, z_2, \ldots$, and then write a simple formula for z_n.

2. Determine the number of strings in $\{a, b, c, d, e, f\}^n$ (i.e. strings of length n with entries in the set $\{a, b, c, d, e, f\}^n$) that satisfy all 3 of the following conditions:

   (i) symbol a occurs an even number of times and symbol b occurs an even number of times,

   (ii) symbols c, d, e, f each occur exactly once, and

   (iii) symbols c and d appear before symbols e and f.

3. Find the exponential generating function for the number of undirected, labeled graphs on n vertices, in which every vertex is of degree 2.

4. Find the exponential generating function for the number of bipartite, vertex-labeled graphs on n vertices (*Note:* Part of this answer is a summation for which there is no easy closed formula. It is ok to leave it in summation form).

5. An involution of $[n]$ is a permutation $\sigma : [n] \mapsto [n]$ such that $\sigma^2(m) = m$ for all $m \in [n]$. Let T_n denote the number of involutions of $[n]$.

   (a) Find a recurrence that is satisfied by these numbers.

   (b) Compute $T_1, \ldots, T_6$.

   (c) Use generating functions to find a closed form for T_n.

   (d) Give a combinatorial argument for why the closed form you have found in (c) is correct.