CDM: Definitions and Theorems from Lectures 17-19

The following is a list of definitions, theorems, and notation from lectures 17-19.

Definition 1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two exponential families with disjoint picture sets P_1 and P_2. We define the exponential family $\mathcal{F}$ such that for $n \geq 1$, then $\mathcal{D}_n = \mathcal{D}'_n \cup \mathcal{D}''_n$ where $\mathcal{D}_n$, $\mathcal{D}'_n$, and $\mathcal{D}''_n$ are the decks of weight n for families $\mathcal{F}$, $\mathcal{F}_1$, and $\mathcal{F}_2$, respectively. Then we call $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ the *merger* of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Lemma 2 (The Fundamental Lemma of Labeled Counting)**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two exponential families with disjoint picture sets, and let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ be their merger. If $\mathcal{H}_1(x,y)$, $\mathcal{H}_2(x,y)$, and $\mathcal{H}(x,y)$ are the respective two variable hand enumerators of these families, then*

$$\mathcal{H}(x,y) = \mathcal{H}_1(x,y)\mathcal{H}_2(x,y).$$

Claim 3. *Let $\mathcal{F}$ be an exponential family such that for some fixed $r \in \mathbb{N}$, the deck $\mathcal{D}_n$ is empty if $n \neq r$, and $\mathcal{D}_r$ is a set of d_r cards, where $d_r \geq 1$. Then*

$$\mathcal{H}(x,y) = \exp\left\{\frac{yd_r x^r}{r!}\right\}$$

where $\mathcal{H}(x,y)$ is the two variable hand enumerator for $\mathcal{F}$.

Theorem 4 (The Exponential Formula for Two Variable Hand Enumeration)**.** *Let $\mathcal{F}$ be an exponential family with deck and hand enumerators $\mathcal{D}(x)$ and $\mathcal{H}(x,y)$, respectively. Then*

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)},$$

and the number of hands of weight n with k cards is

$$h(n,k) = \left[\frac{x^n}{n!}\right]\left\{\frac{\mathcal{D}(x)^k}{k!}\right\}.$$

Corollary 5 (The Generalized Exponential Formula)**.** *Let $\mathcal{F}$ be an exponential family with deck enumerator $\mathcal{D}(x)$, and let $\mathcal{H}(x) = \sum_{n\geq 0} \frac{h_n}{n!}x^n$ where h_n denotes the number of hands of weight n. Then*

$$\mathcal{H}(x) = e^{\mathcal{D}(x)}.$$

Definition 6. A *tree* is a connected graph without cycles. A *rooted tree* that has a distinguished vertex called a *root.* A *forest* is a graph such that each connected component is a tree. A *rooted forest* is a graph such that each connected component is a rooted tree.

Theorem 7 (Cayley's Formula)**.** *For $n \geq 1$, there are exactly n^{n-2} labeled trees on n vertices.*

Theorem 8 (The Lagrange Inversion Formula)**.** *Let* $f(u) = \sum_{n\geq 0} f_n u^n$ *and* $g(u) = \sum_{n\geq 0} g_n u^n$ *such that* $g(0) = 1$*. Then there exists a unique* $u = u(t) = \sum_{n\geq 0} u_n t^n$ *such that*

$$u = t \cdot g(u).$$

Furthermore,

$$[t^n]\left\{f(u(t))\right\} = \frac{1}{n}\left[u^{n-1}\right]\left\{f'(u)g(u)^n\right\}.$$

Theorem 9 (Mantel's Theorem)**.** *If we are given a graph* $G = (V, E)$ *such that* $|V| = 2n$ *and* $|E| = n^2 + 1$*, then* G *contains a triangle.*

Definition 10. A k-*clique* is a graph on k vertices such that every two vertices share an edge.

Theorem 11 (Turán's Theorem)**.** *Let* $G = (V, E)$ *be a graph such that* $|V| = n$ *and* G *does not contain a* $(k+1)$*-clique. Then*

$$|E| \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2}.$$

Ramsey's Theorem for Graphs

Definition 12. Let $s_1, \ldots, s_r$ be positive integers such that $s_i \geq 2$, for all $i \in [r]$. Then the *Ramsey Number* $R(s_1, \ldots, s_r)$ is the smallest integer n such that if we color the edges of K_n with r colors, there is always some $i \in [r]$ such that there is an s_i-clique whose edges are all colored in the i^{th} color.

Claim 13. $R(3, 3) = 6$.

Claim 14. $R(3, 4) = R(4, 3) = 9$.

Claim 15. $R(4, 4) = 18$.

Theorem 16 (Ramsey's Theorem for 2-Colored Graphs)**.** *For any* $s, t \in \mathbb{N}$*, there exists a natural number* $n = R(s, t)$ *such that if we color the edges of* K_n *red and green, there is always either a red* s*-clique or a green* t*-clique.*