CDM: Assignment 2

Due on Wednesday May 27

Section I

Answer all questions in this section. Make sure to explain your solutions completely to receive full credit.

1. Build an example to show that if we remove a maximal chain from a poset, it is possible that the width of the poset remains the same.
2. Let $0 < a_1 < a_2 < \cdots < a_{sr+1}$ integers. Prove that we can select either $s + 1$ of them, no one of which divides any other, or $r + 1$ of them, each dividing the following one.

   Hint. *Use Dilworth's Theorem.*
3. Use Max-Flow-Min-Cut theorem to prove König's Theorem.
4. Use Dilworth's theorem to prove König's Theorem.
5. Let $X = \{1, \ldots, n\}$ and let A, B be distinct subsets of X such that $A \subset B$. How many maximal chains of subsets of X contain both A and B?

Section II

Pick three questions in this section to answer. Make sure to explain your solutions completely to receive full credit.

1. (a) Prove that the maximum deficiency sets are closed under union and intersection, that is, if B' and B'' are both maximum deficiency sets, then so there $B' \cap B''$ and $B' \cup B''$.

   (b) Let M be a maximum matching and direct all edges in M from B to G while all other edges from G to B. Recall that we define a *reachable set* R as the set of vertices that can be reached from the set of boys exposed by M. Prove that R happens to be the "core" of the maximum deficiency sets, i.e., it is the smallest maximum deficiency set.
2. Consider the following game. The board is a graph $G = (V, E)$. The players alternately choose vertices in V, producing a sequence $u_1, v_1, u_2, v_2, \ldots$ of vertices (so $u_1, u_2, \ldots$ are the moves of player I and $v_1, v_2, \ldots$ are the moves of Player II). The players must obey the following rules:

   (i) No vertex is chosen more than once (so $u_1, v_1, \ldots$ is a sequence of distinct vertices),

(ii) Each vertex chosen must be connected by an edge to the last vertex chosen.

Note that $u_1, v_1, u_2, v_2, \ldots$ is a path in G. The first player who cannot make a move is the loser.

(a) Show that if G has a perfect matching then the second player has a winning strategy.

(b) Show that if G does not have a perfect matching then the first player has a winning strategy.

3. Let $\mathcal{F}$ be a collection of subsets of $\{1, \ldots, n\}$. We say that $\mathcal{F}$ is intersecting if every pair of sets in $\mathcal{F}$ intersect; to be precise, the collection $\mathcal{F}$ is intersecting if

$$A, B \in \mathcal{F} \Rightarrow A \cap B \neq \emptyset.$$

(a) Prove that if $\mathcal{F}$ is an intersecting collection of subsets of $\{1, \ldots, n\}$ then $|\mathcal{F}| \leq 2^{n-1}$.

(b) Give an example of an intersecting collection $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ such that $|\mathcal{F}| = 2^{n-1}$.

4. Let $t < n/2$ and let $\mathcal{F}$ be a collection of subsets of $\{1, \ldots, n\}$. Suppose that:

(i) For all sets $A \in \mathcal{F}$, then $|A| \leq t$, and

(ii) $\mathcal{F}$ is an antichain.

Let $\mathcal{F}_t$ denote the collection of all the t-element subsets of $\{1, \ldots, n\}$ that contain at least one member of $\mathcal{F}$. Prove that $|\mathcal{F}| \leq |\mathcal{F}_t|$.

5. Let (X_1, Y_1) and (X_2, Y_2) be minimum cuts in a transportation network. Show that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is also a minimum cut.