Expander Decomposition and Hierarchies: Exercise 2

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1 Online multicut

In the online multicut problem, you are given a graph $G = (V, E)$ and an online sequence of requests $R = (s_1, t_1), (s_2, t_2), \dots (s_{|R|}, t_{|R|}) \in V \times V$. After seeing each request (s_i, t_i), you must choose a set D_i of edges to be deleted from the current graph G_i so that s_i and t_i are not connected in $G_{i+1} = G_i - D_i$. You must choose D_i before seeing the nest request. Your online algorithm is c-competitive if your total cut size is always at most the size of the (offline) optimal solution (which depends on the whole sequence R). Design a polylog(n)-competitive algorithm.

Hint: reduce to the online set cover problem, which admits polylog(n)-competitive algorithms.

2 Correctness of the BSE hierarchy construction

Recall the dynamic expander decomposition subroutine.Let $G = (V, E)$ be a fixed graph and ϕ be a parameter. Let $D \subseteq E$ be a set of edges. Let A be a node weighting. Both D and A undergo *incremental* updates. The subroutine $\text{DynED}(G, \phi, A, D)$ maintains an *incremental* edge set $C \supseteq D$ such that, at any time,

- A is ϕ-expanding in $G - C$, and
- $|C| - |D| \le \phi |A| \log(n)$.

In Lecture 3, we describe the following algorithm for constructing a BSE hierarchy of a graph $G = (V, E)$:

- Initialize:
  - $C_0 \leftarrow E$, $C_i \leftarrow \emptyset$ for all $i \ge 1$
  - $\phi \le \frac{1}{16 \log n}$ and $\ell := \frac{\log m}{\log(1/4\phi \log n)} + 1$
- For $0 \le i \le \ell$, maintain until there is no update

$$C_{i+1} \leftarrow \text{DynED}(G, \phi, A_i := \deg_{C_i}, D_i := C_{i+2})$$

- Return $E_i = C_i - C_{i+1}$ for all $i \leq \ell$

Our goal is to show that $C_{\ell+1} = \emptyset$, which proves the correctness of the whole algorithms (See the slides).

Lemma. *For each i, $|C_i| \leq (2q)^i m$ where $q = 2\phi \log(n) \leq 1/8$. In particular, $C_{\ell+1} = \emptyset$.*

In this exercise, we will guide you to prove this lemma.

1. For $0 \leq i \leq \ell$, prove that
$$|C_{i+1}| - |C_{i+2}| \leq q|C_i|.$$
2. Prove by induction that
$$|C_{i+1}| \leq 2^i q^{i+1} m + 2|C_{i+2}|.$$
3. Prove by induction again that
$$|C_i| \leq (2q)^i m.$$

3 Flow shortcuts

We say that a flow F G has hopbound h if every flow path in F contains at most h edges. Let $G = (V, E)$ be a graph.

Definition 3.1. An edge set E' is a *flow shortcut of G with κ congestion and h hopbound* if, for any $\deg_G$-respecting demand D,

- If D is routable in G, then D is routable in $G' := G \cup E'$ with hopbound h.
- If D is routable in G', then D is routable in G with congestion κ.

We call G' a *shortcut graph*. We allow E' to contain vertices not in G.

Flow shortcuts with low congestion and hopbound are desirable. They allow us to assume that the target flow has small hopbound, and finding flow with small hopbound is usually easier. There are many recent flow algorithms based on this. This concept is closely related to Cohen's hopset which has the same purpose for distance-based problems, instead of congestion-based problems.

Exercise. We will construct flow shortcuts from expanded hierarchies.

1. Let $E_0, \ldots, E_\ell$ be a ϕ-boundary-separator-expanding hierarchy of G. For each i and each level-i cluster H in $G - E_{>i}$, let E_H be a star with a Steiner root vertex r_H and leaf set $V(H)$. For each $v \in V(H)$, the set the capacity $\text{cap}(r_H, v) = \deg_{E_{\geq i}}(v)$. Set $E' \leftarrow \bigcup_H E_H$. Prove that E' is a flow shortcut of G with congestion $\kappa = O(\ell/\phi)$ and hopbound $h = O(\ell)$. Conclude that every graph admits a flow shortcut with congestion $\log^2(n)$ and hopbound $O(\log n)$.

2. Let $E_0, \dots, E_\ell$ be a ϕ-separator-expanding hierarchy of G. For each i and each level-i cluster H in $G - E_{>i}$, let E_H be a star with a Steiner root vertex r_H and leaf set $V(H)$. For each $v \in V(H)$, the set the capacity $\mathrm{cap}(r_H, v) = \deg_{E_i}(v)$. Set $E' \leftarrow \bigcup_H E_H$. Prove that E' is a flow shortcut of G with congestion $\kappa = O(\ell/\phi)$ and hopbound $h = O(2^\ell)$.

3. Read Problem 4. Prove the same thing when $E_0, \dots, E_\ell$ form a weak ϕ-separator-expanding hierarchy.

4. *Optional:* A flow shortcut can be defined for vertex-capacitated graphs too. Let us call that a vertex-flow shortcut. By (2) and generalizing the definition ϕ-separator-expanding hierarchy to vertex-capacitated graphs, conclude that every graph admits a vertex-flow shortcut with congestion and hopbound $(\log n)^{O(\sqrt{\log n})}$.

It is open if every graph admits a vertex-flow shortcut with congestion and hopbound $\mathrm{poly}\log n$. I would be very interested to see this.

4 A bottom-up construction of (weak) separator-expanding hierarchies

Definition 4.1. Let H be a graph. We say that demand D is *constrained by components of* H if, for each (u, v), $D(u, v) > 0$ implies that u, v are in the same connected component of H. We also say that D is H-component-constrained.

That is, we can have only set a demand in D between connected pairs of vertices of H.

Definition 4.2. We say that a node weighting A is H-component-constrained ϕ-expanding in G if every H-component-constrained demand routable in G with congestion $1/\phi$.

Let $H \subseteq G$. Observe that if A is ϕ-expanding in H, then A is H-component-constrained ϕ-expanding in G. That is, the former is a stronger condition. For intuition, the latter only requires ability to route between vertices in the same components of H, but we could use edges in the supergraph $G \supseteq H$ to route.

Definition 4.3. We say that a partition $E_0, \dots, E_\ell$ of E forms a *weak ϕ-separator-expanding hierarchy* if, for each i, E_i is $(G - E_{>i})$-component-constrained ϕ-expanding in G.

Observe that a ϕ-separator-expanding hierarchy is also a weak ϕ-separator-expanding hierarchy. There is an extremely simple algorithm for building a weak ϕ-separator-expanding hierarchy by making $O(\log n)$ calls to expander decomposition:

- $C_0 \leftarrow E$ and $\phi \le \frac{1}{2\log n}$
- While $C_i \neq 0$,
  - $C_{i+1} \leftarrow \phi$-expander decomposition of $A_i := \deg_{C_i}$.
  - $i \leftarrow i + 1$
- Return $E_i = C_i \setminus C_{>i}$ for each i.

Exercise. Prove that $\{E_0, \ldots, E_\ell\}$ forms a weak ϕ-separator-expanding hierarchy with $\ell \leq \frac{\log m}{\log(1/\phi \log n)}$ levels.