Lecture 2:
Separator-Expanding Hierarchies

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ADFOCS

Plan

Connectivity Oracles under Edge Faults [Patraşcu Thorup'07]

1. Oracle construction on expanders
2. Separator-expanding hierarchy
3. Oracle construction on general graphs

Connectivity Oracles under Edge Faults

There are 3 phases:

1. Preprocess(G) where $G = (V, E)$
2. Update(D) where $D \subset E$ has size $|D| = d$ — $\tilde{O}(d)$ time
3. Query(s, t) where $s, t \in V$

   Return if s and t are connected in $G - D$ — $\tilde{O}(1)$ time

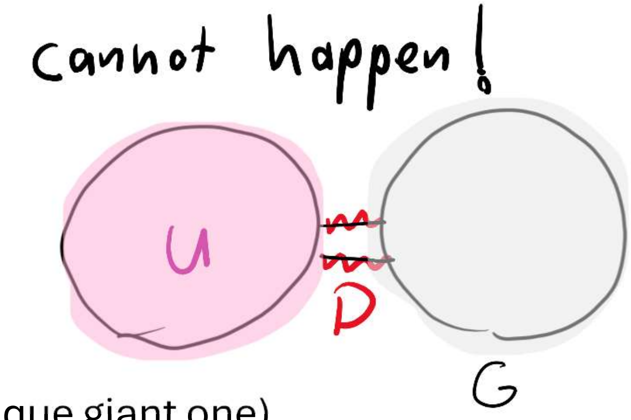
Part 1

Warm-Up:

Oracle Construction on Expanders

where (s, t) are fixed.

Robustness of Expanders



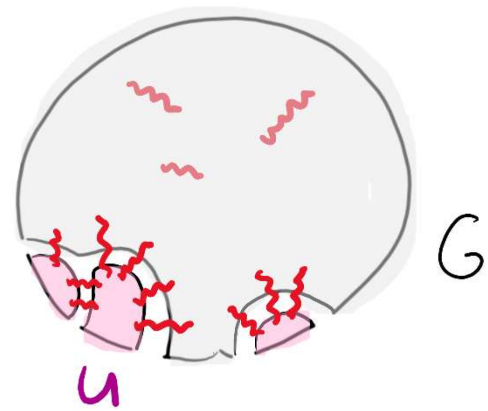
G: ϕ-expander

U: new connected component in $G - D$ (not the unique giant one)

Fact: U has *volume* $\deg_G(U) \leq d/\phi$.

Proof:

- Boundary $E_G(U, V-U)$ are all deleted.
- So $|d| \geq |E_G(U, V-U)| \geq \phi \deg_G(U)$



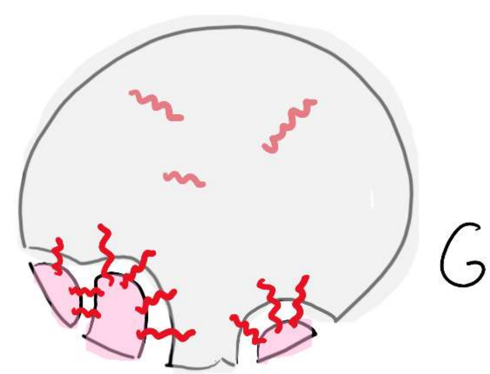
Robustness: d deletions in expanders can disconnect $\lesssim d$ volume

Oracle on ϕ-Expanders. Fixed s, t

1. DFS in $G - D$ from s. Explore $\leq 2d/\phi$ volume. Either
   - Find the isolated component U_s of s, or
   - Not done exploring. Then, set $U_s \leftarrow$ the unique giant component
2. Do the same for t.
3. Return "Yes" iff $U_s = U_t$

Correctness: clear

Update time: $O(d/\phi)$ time.



Part 2

Oracle Construction on Expanders

Connectivity Oracles under Edge Faults

There are 3 phases:

1. Preprocess(G) where $G = (V, E)$
2. Update(D) where $D \subset E$ has size $|D| = d$
3. Query(s, t) where $s, t \in V$
   Return if s and t are connected in $G - D$

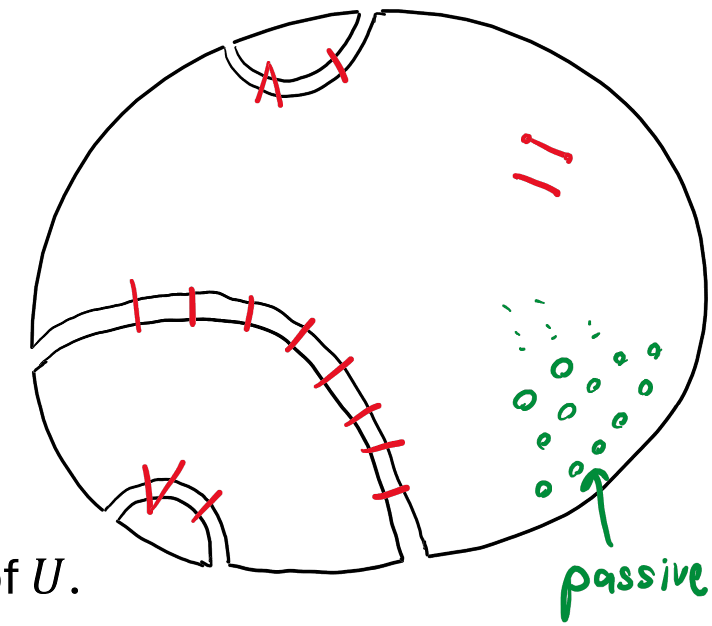
Connectivity Oracles under Edge Faults

1. Preprocess(G) where $G = (V, E)$
2. Update(D) where $D \subset E$ has size $|D| = d$

Then, get representation of connected components of $G - D$
So, can answer any connectivity queries in $G - D$ in $\tilde{O}(1)$ time.

Oracle on ϕ-Expanders

- Init passive *region* $U_v = \{v\}$ for $v \in V$
- A region U is active if
  - $\deg(U) \leq 2\mathrm{del}(U)/\phi$ $(\mathrm{del}(U) \coloneqq \deg_D U)$
  - has an unexplored neighbor in $G - D$.



- While there is an active U:
  - $U \leftarrow U \cup U_v$ where v is an unexplored neighbor of U.
  - If $\deg(U) > 2\mathrm{del}(U)/\phi$, then U becomes passive.
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- **Claim**: Components of $G - D$ = **isolated regions**, **union of passive regions**

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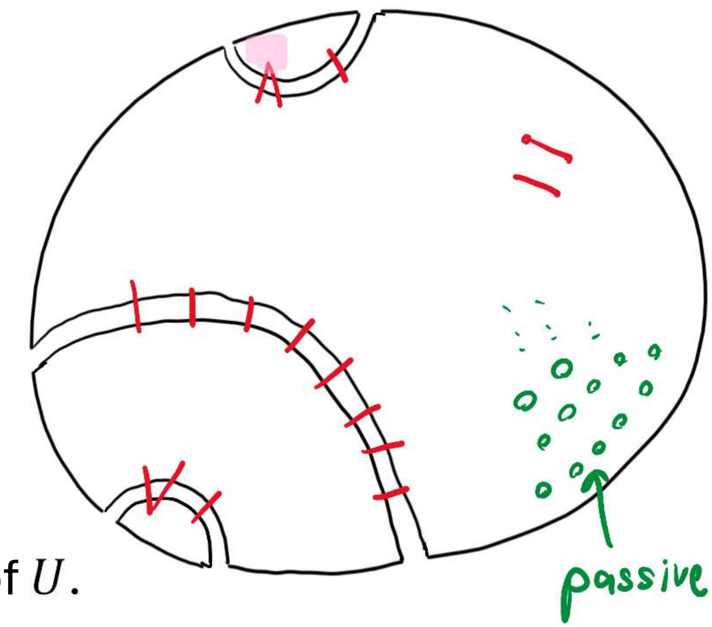

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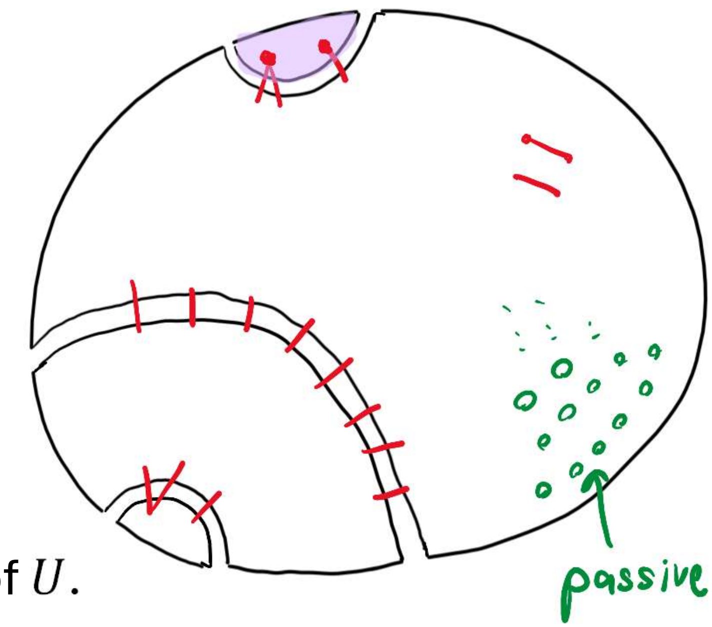

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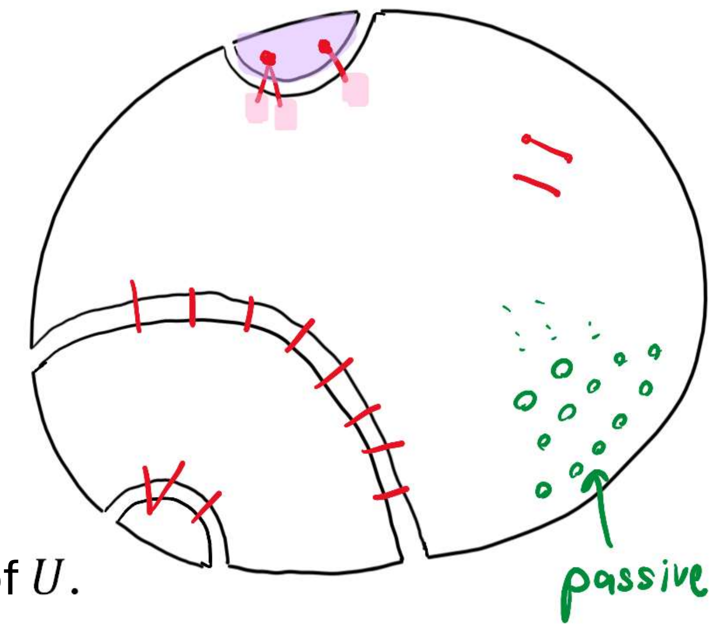

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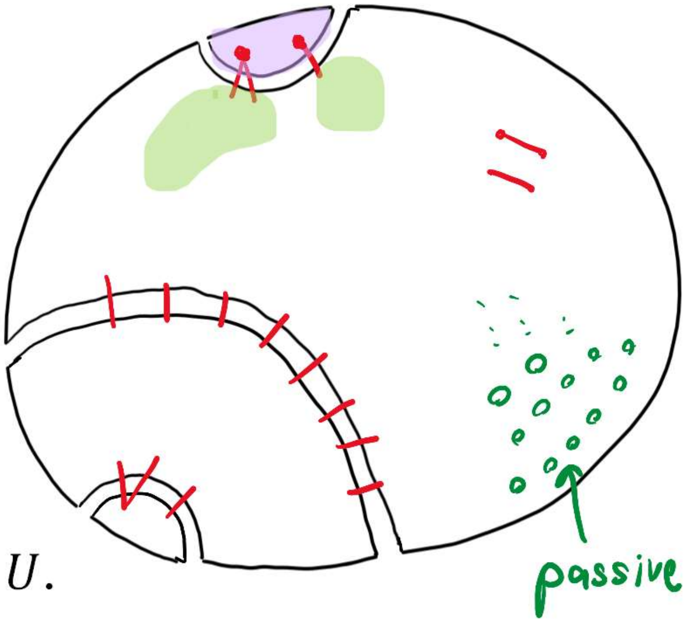

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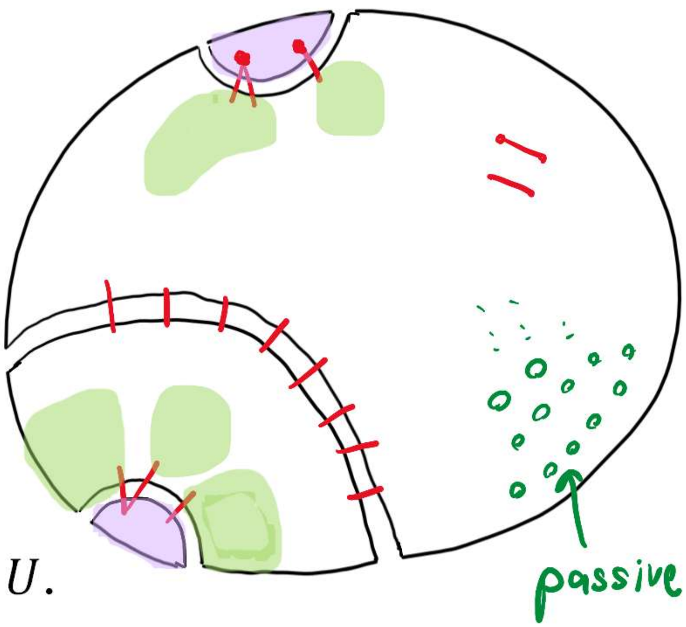

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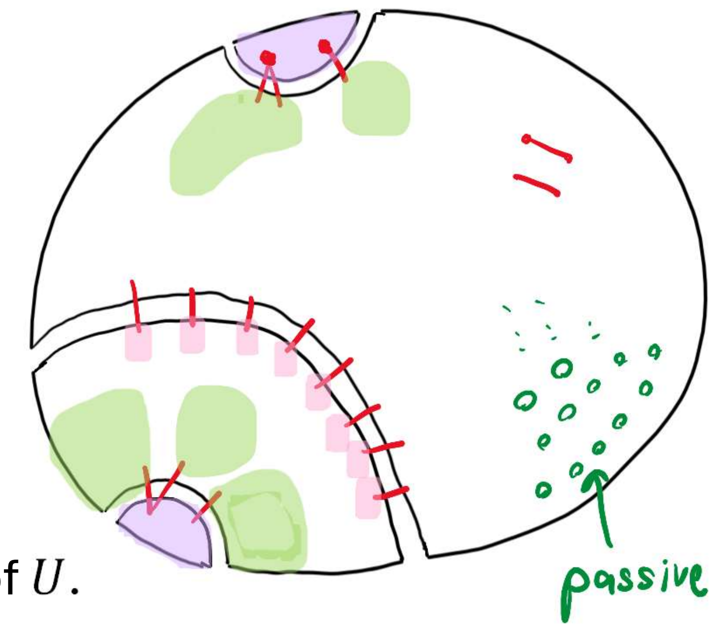

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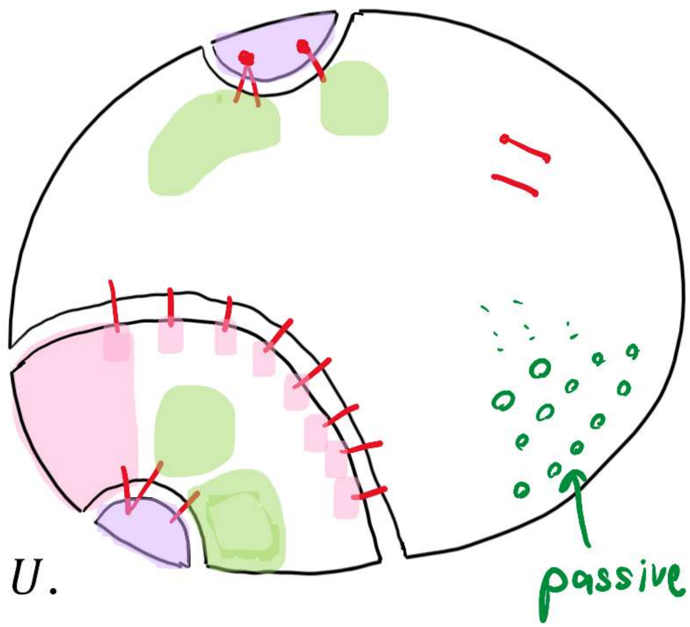

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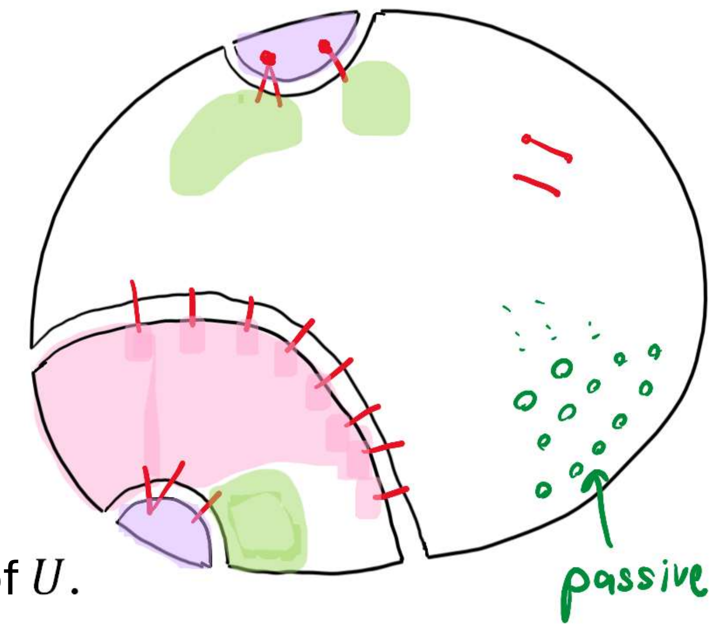

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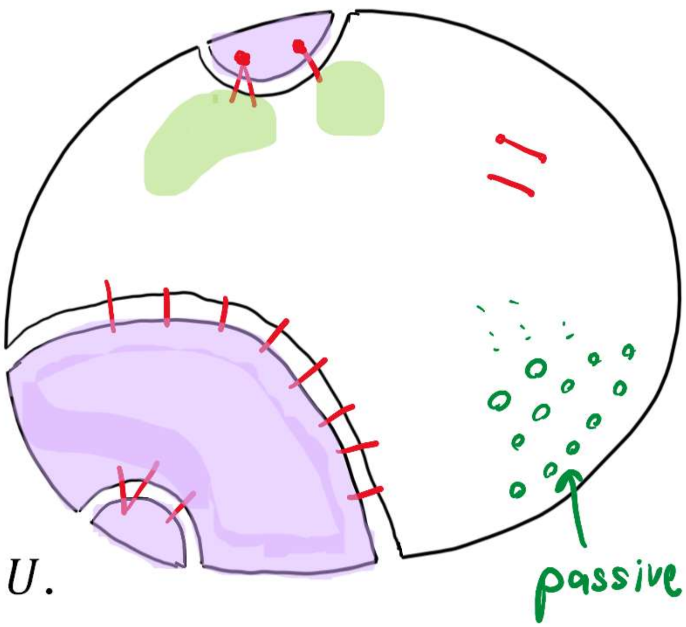

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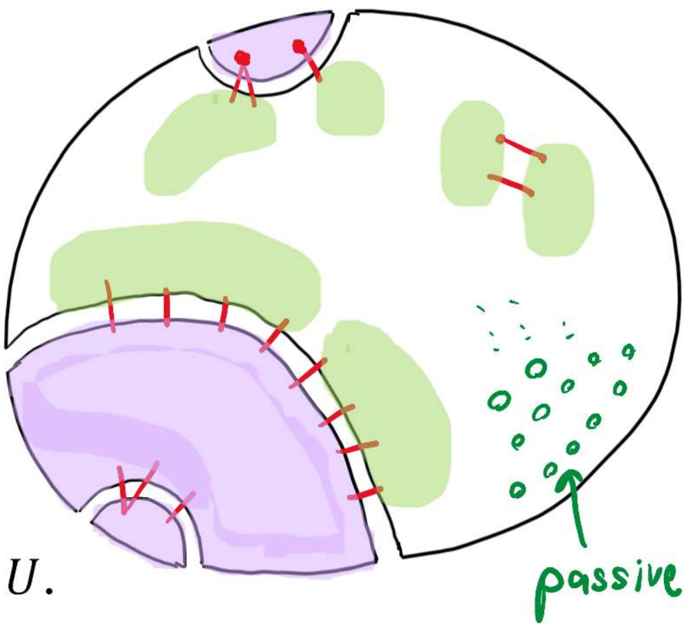

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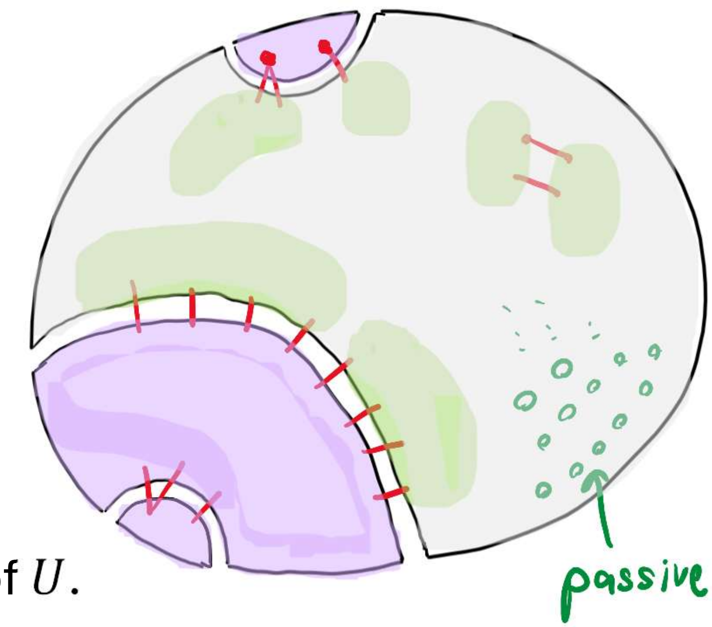

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- **Claim**: Components of $G - D$ = **isolated regions**, **union of passive regions**

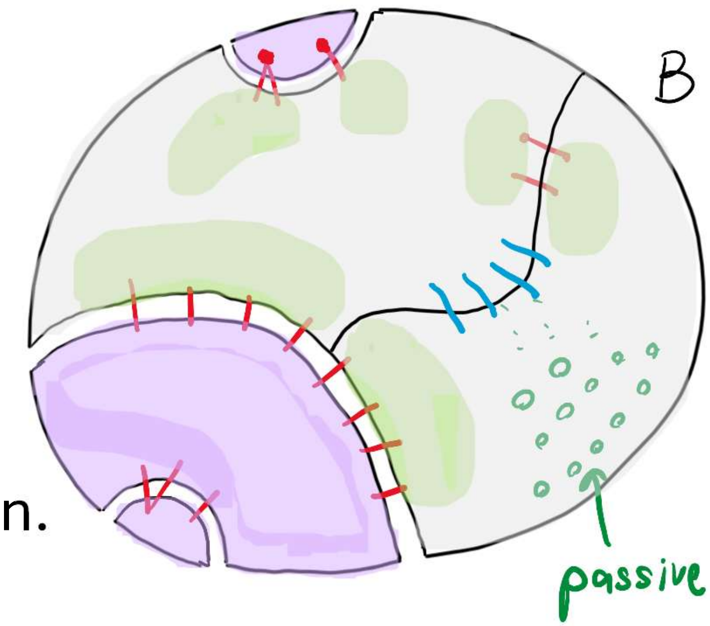
Correctness

Claim: all **passive regions** are in the same component of $G - D$

Proof: Suppose not. $\exists B \subseteq V$ where $\deg(B) \leq \deg(V)/2$

- B is a component in $G - D$
- B = union of some passive region.

- $|E(B, V - B)| \geq \phi \deg(B)$
  - G is ϕ-expander
- $\mathrm{del}(B) < \phi \deg(B)/2$
  - B = union of passive comp.
- $\exists e \in E(B, V - B) \setminus D$.
- B is **not** a component in $G - D$. Contradiction.



Update Time

Update time: $\tilde{O}(d/\phi)$

- Total volume explored $\leq 4d/\phi$
  - Only explore from active region U: $\deg(U) \leq 2\mathrm{del}(U)/\phi$
  - $\Sigma_U \mathrm{del}(U) = 2d$
- Data structures for merging regions
  - Union find

Connectivity Oracles under Edge Faults

1. Preprocess(G) where $G = (V, E)$
2. Update(D) where $D \subset E$ has size $|D| = d$

   Then, get representation of connected components of $G - D$
   Can answer connectivity queries in $G - D$ in $\tilde{O}(1)$ time.

Update time: $\tilde{O}(d/\phi)$ on ϕ-expanders. ✔
What about general graphs?

Plan

Expanding Balanced Separator

↓

Separator-Expanding Hierarchy

↓

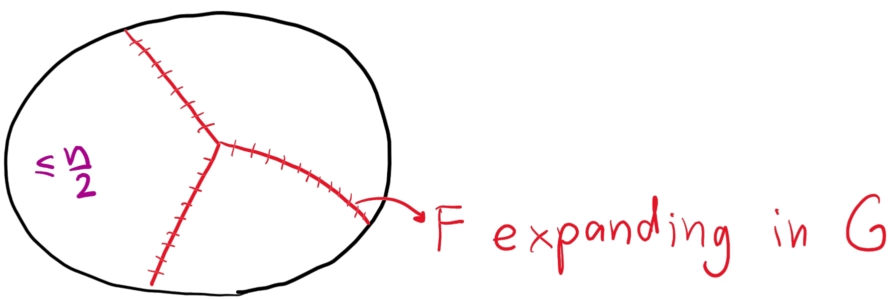
Connectivity Oracles on **General Graphs**

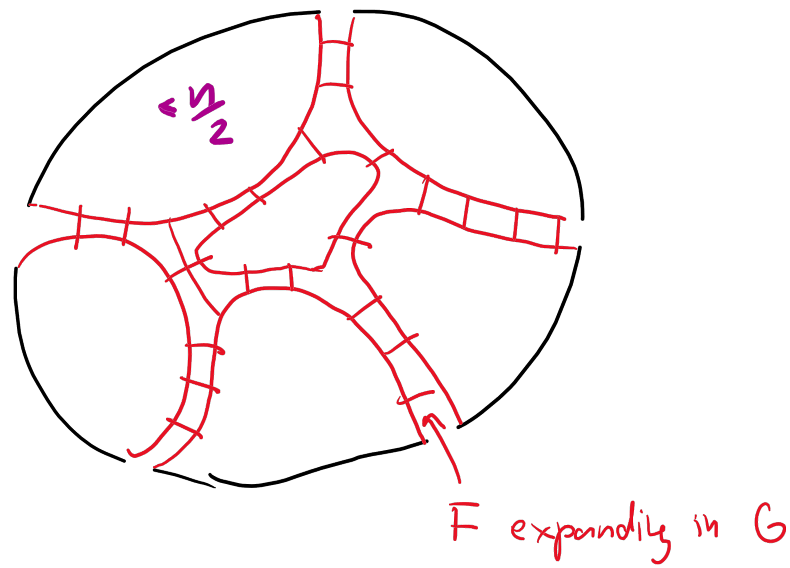
Part 2
Expanding Balanced Separator

Expanding Balanced Separator

Theorem: every graph G contains an edge set $F \subseteq E$ such that

- F is $1/4$-expanding in G
- each component C in $G - F$ has $\leq n/2$ vertices.





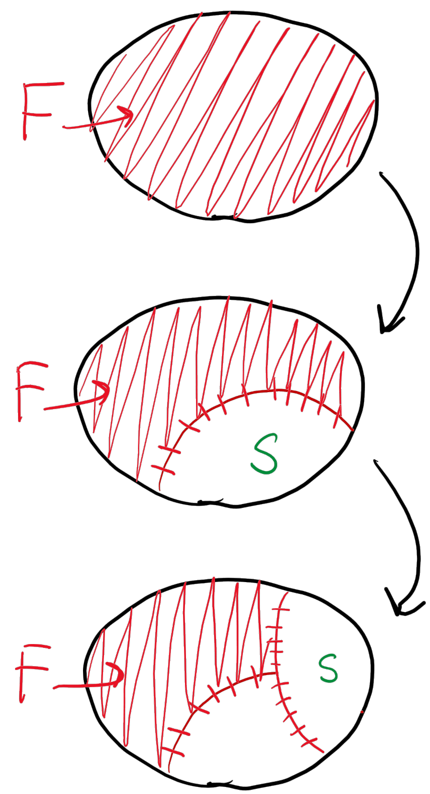
Expanding Balanced Separator: Construction

Theorem: every graph G contains an edge set $F \subseteq E$ such that

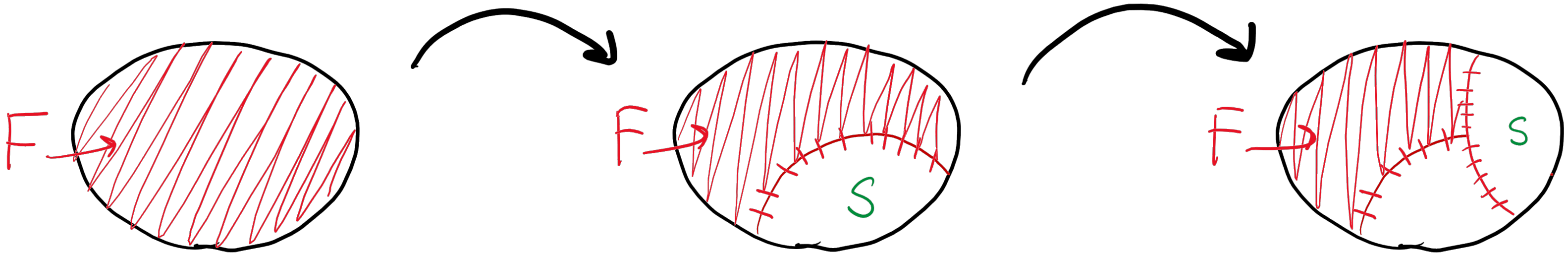
- F is 1/4-expanding in G
- each component C in $G - F$ has $\leq n/2$ vertices.

Algo:

- $F \leftarrow E$
- While F is not 1/4-expanding
  - $\exists\, S$ s.t. $|\partial_G(S)| < 1/4 \min\{\deg_F(S), \deg_F(V-S)\}$
  - Assume $|S| \leq n/2$ by symmetry
  - $F \leftarrow F \cup \partial_G(S) - \big(F \cap (S \times S)\big)$



Expanding Balanced Separators: Proof



- **Invariant:** each component U in $G - F$ has $\leq n/2$ vertices.
- Once stop → F is $1/4$-expanding.
- Why stop?
- **Claim:** $|F|$ strictly decreases after each iteration
  - increase by $\leq |\partial_G S|$
  - decrease by $\geq |F \cap (S \times S)| \geq \frac{\deg_F(S) - |\partial_G S|}{2} > 1.5|\partial_G S|$

Expanding Balanced Separator

Theorem: for every G, $\exists$ edge set $F \subseteq E$ such that

- F is $1/4$-expanding in G
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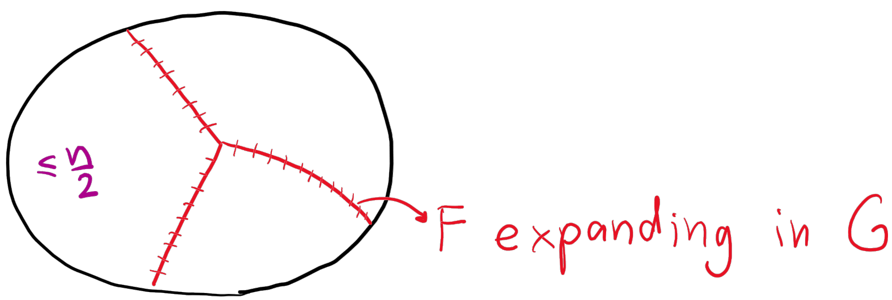
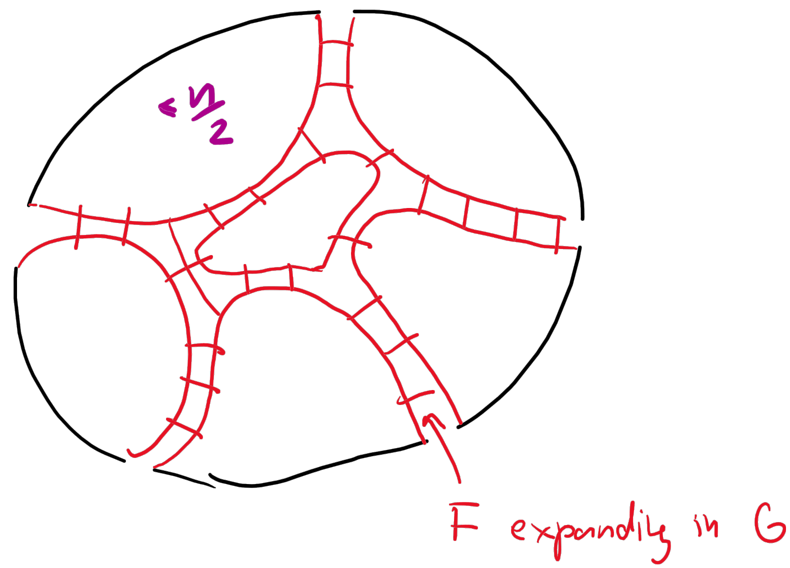
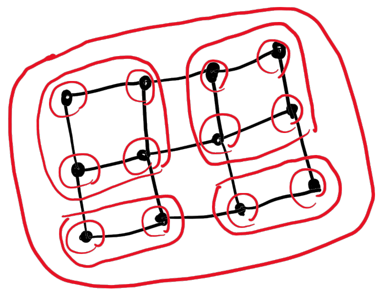




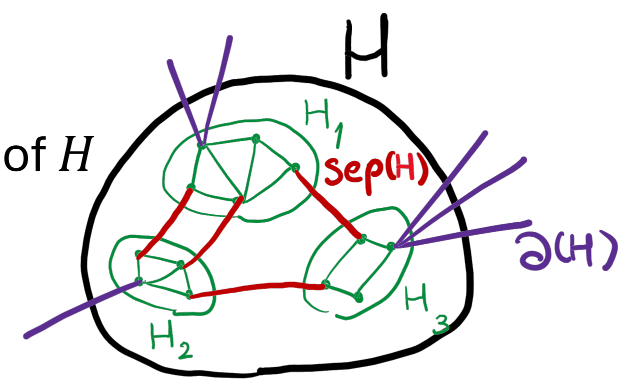
Part 3
Separator-Expanding Hierarchy

Hierarchy

- A hierarchy $\mathcal{H}$ of $G = (V, E)$ is a laminar family of induced graphs:
  - root = G
  - leaf = a vertex
  - non-leaf = $G[S]$ for some S



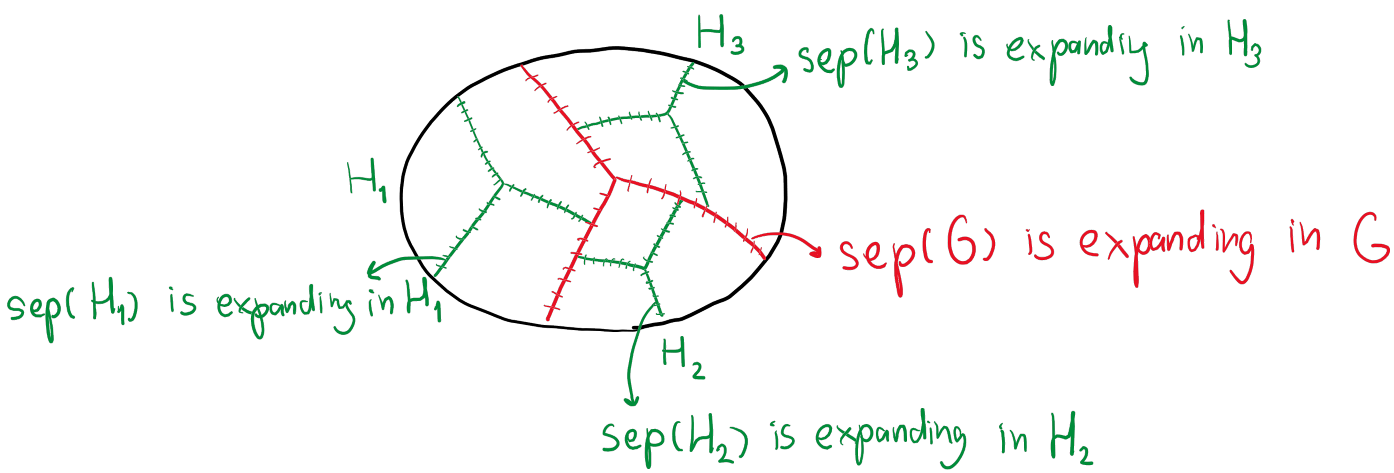
- For each *cluster* $H \in \mathcal{H}$,
  - **Separator of *H*** is $\mathrm{sep}(H)$:= edges crossing children of H
  - **Boundary of *H*** is $\partial(H)$:= edges leaving H



Separator-Expanding Hierarchy

Def: a ϕ-separator-expanding (ϕ-SE) hierarchy of $G = (V, E)$ is

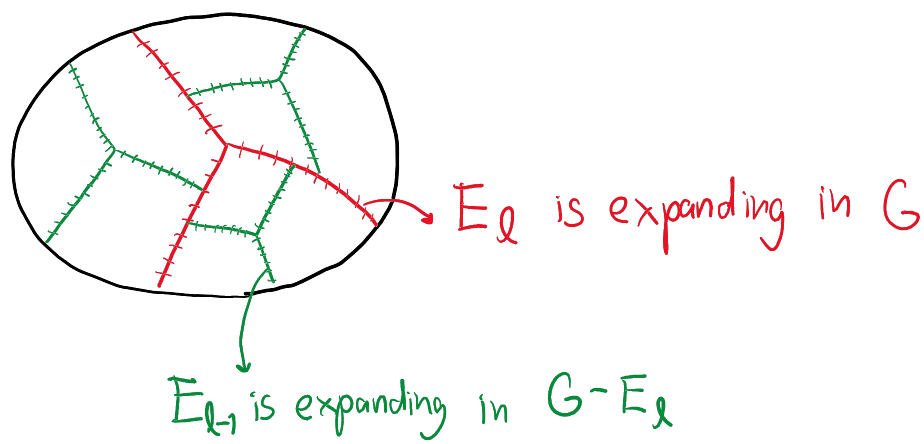
- a hierarchy $\mathcal{H}$ of G where, for each cluster $H \in \mathcal{H}$,
- $\mathrm{sep}(H)$ is ϕ-expanding in H.



Separator-Expanding Hierarchy: Partition View

Def: a ϕ-separator-expanding (ϕ-SE) hierarchy of $G = (V, E)$ is

- a partition $E_0, \ldots, E_\ell$ of E s.t.
- E_i is ϕ-expanding in $G - E_{>i}$



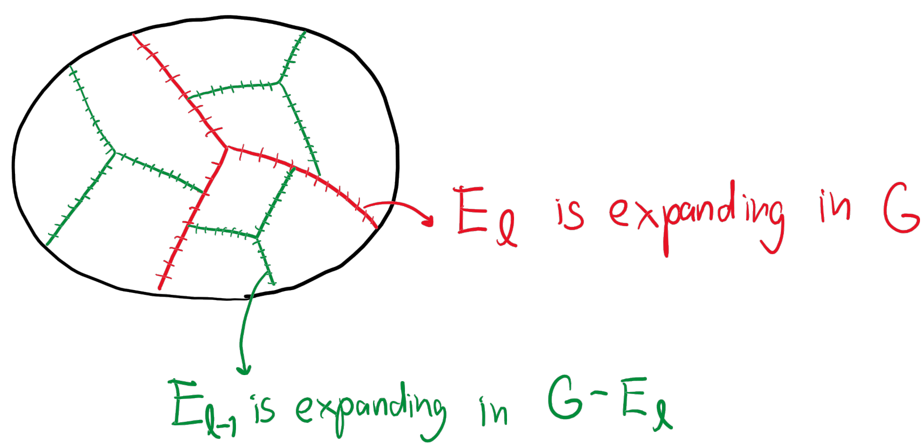
Quiz

In a ϕ-SE hierarchy $E_0, \ldots, E_\ell$

Def: A level-i cluster is a component in $G - E_{>i}$.

Q: A level-0 cluster H a ϕ-expander. Why?

- H only contains edges from E_0.
- E_0 is expanding in $G - E_{>0}$.
- $E(H)$ is ϕ-expanding in H
- $\Rightarrow H$ is a ϕ-expander



Construction

Theorem: for every G, $\exists$ edge set $F \subseteq E$ such that
- F is $1/4$-expanding in G
- each component U in $G - F$ has $\leq n/2$ vertices.

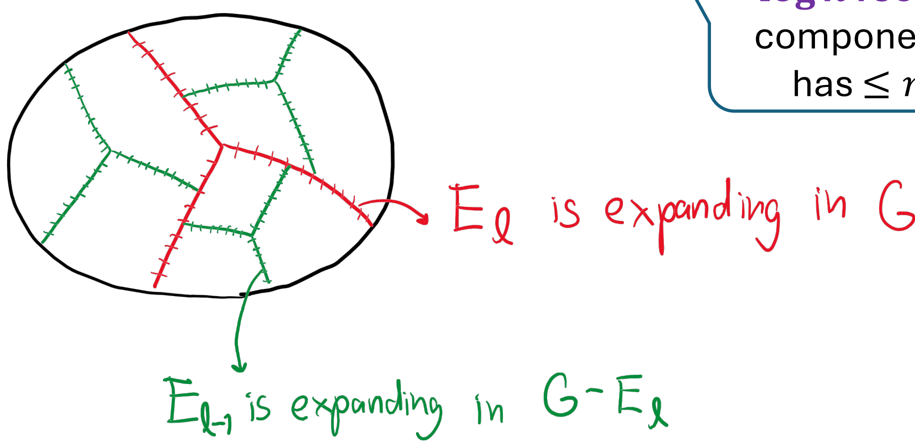
Thm: Every graph has ¼-SE hierarchy with $\log n$ levels

- **[LPS'25]** implicit in **[R'02]**
- Improves **[PT'07]** $\Omega(1/\log n)$-SE hierarchy with $\log n$ levels

Algo:

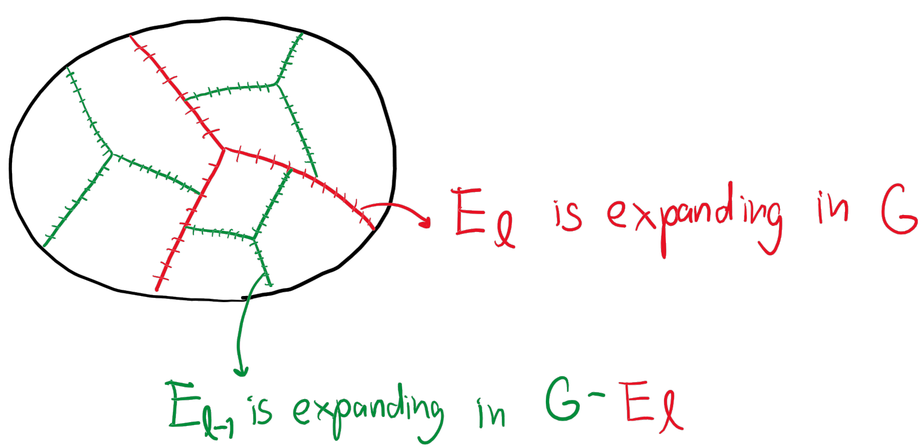
1. $E_\ell \leftarrow$ ¼-expanding balanced separator.
2. Recurse on each component U of $G - E_\ell$

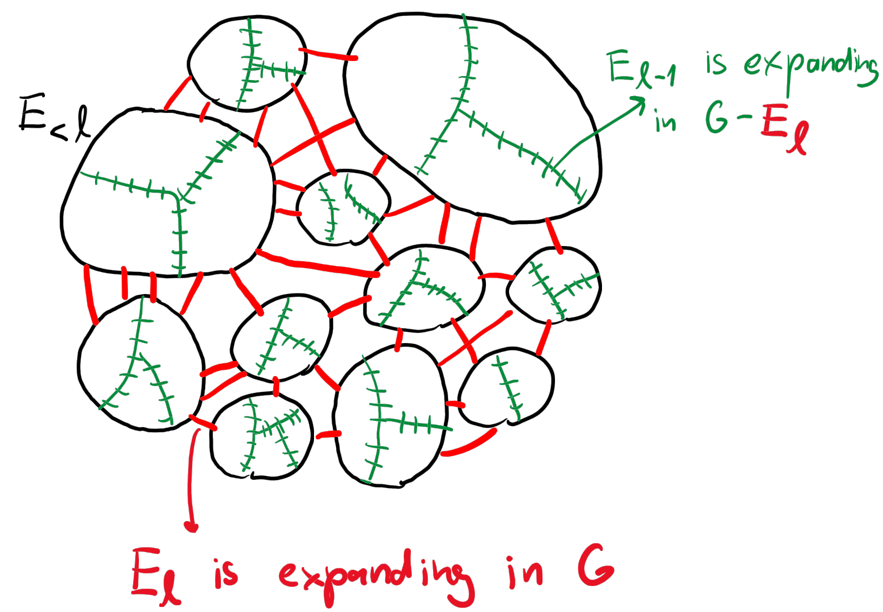
$\log n$ recursion levels
component U in $G - E_\ell$ has $\leq n/2$ vertices



Summary

Thm: Every graph has ¼-SE hierarchy with $\log n$ levels





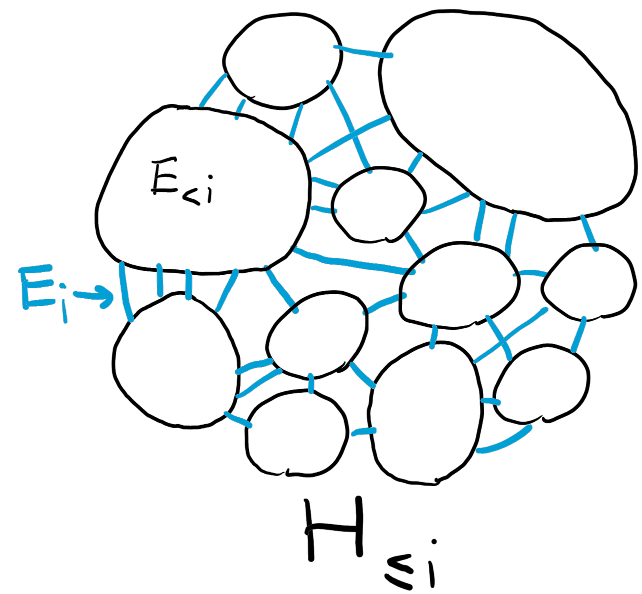
Part 4

Oracle Construction on

General Graphs

Bottom-Up Strategy

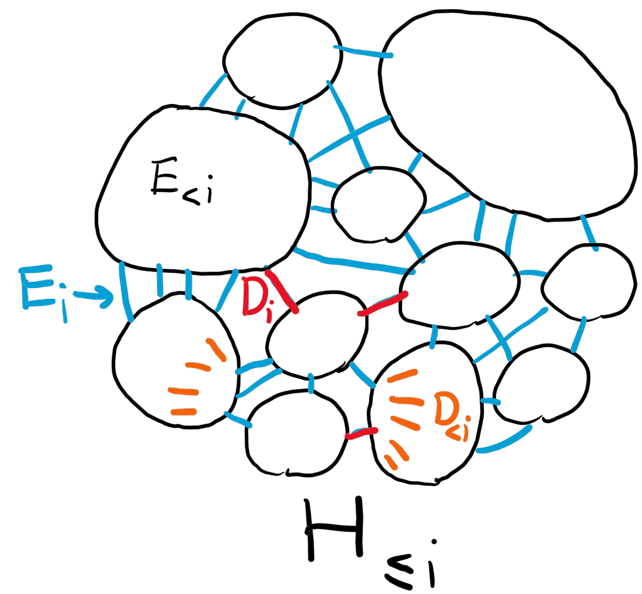
On a level-i cluster $H_{\le i}$



Bottom-Up Strategy

On a level-i cluster $H_{\leq i}$

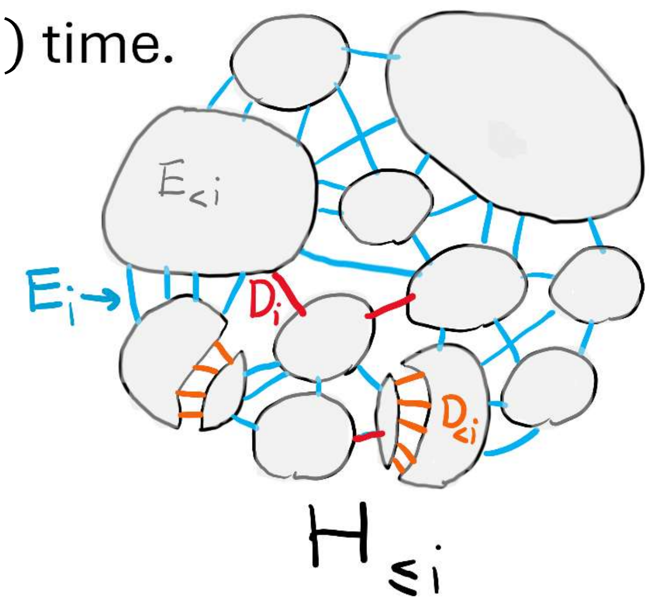
- Consider deleted edges in $D \cap H_{\leq i}$.
- **Assume**: components of children clusters of $H_{\leq i}$ are updated.



Bottom-Up Strategy

On a level-i cluster $H_{\le i}$

- Consider deleted edges in $D \cap H_{\le i}$.
- **Assume**: components of children clusters of $H_{\le i}$ are updated.
- **Task:** update components of $H_{\le i}$ in $\tilde{O}(|D \cap H_{\le i}|)$ time.



Bottom-Up Strategy

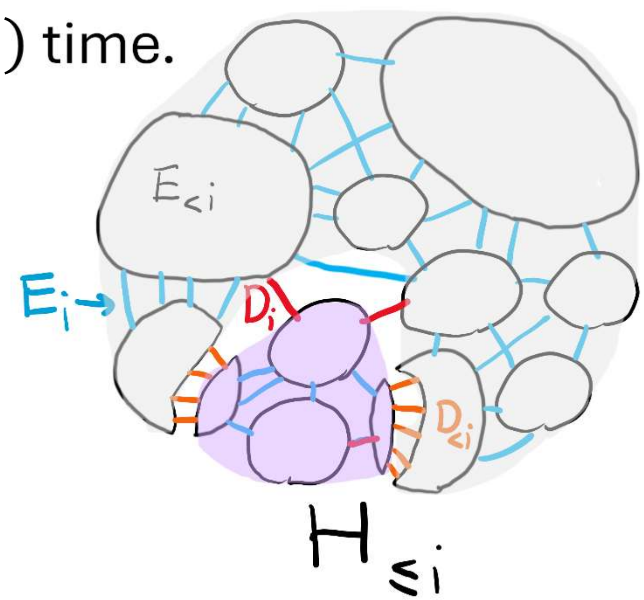
On a level-i cluster $H_{\le i}$

- Consider deleted edges in $D \cap H_{\le i}$.
- **Assume**: components of children clusters of $H_{\le i}$ are updated.
- **Task:** update components of $H_{\le i}$ in $\tilde{O}(|D \cap H_{\le i}|)$ time.

After updating the top-level cluster G, DONE.

$\Rightarrow$ **Total update time**: $\tilde{O}(d) \cdot \ell = \tilde{O}(d)$

Remain to solve the task on each level-i cluster.



Oracle on Level-i Cluster $H_{\leq i}$

- Init passive *region* = components of all children clusters (contracted to vertices)
- A region U is active if
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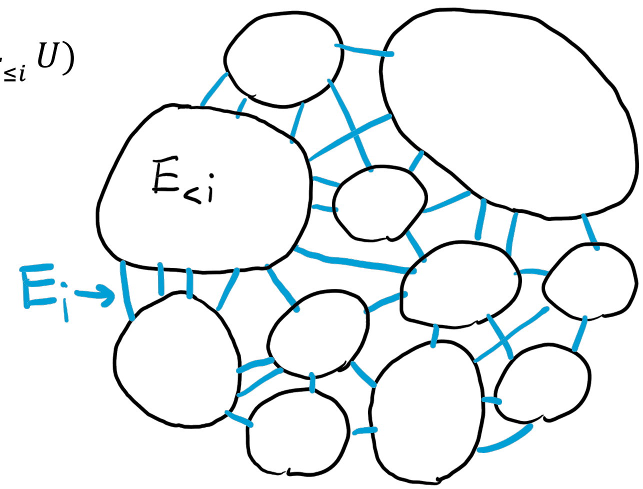

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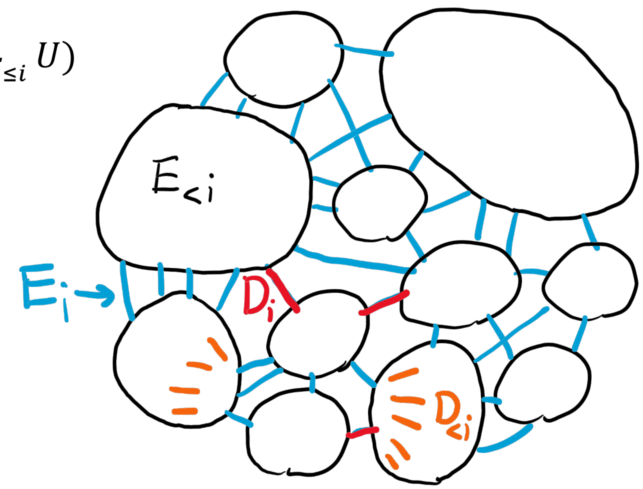

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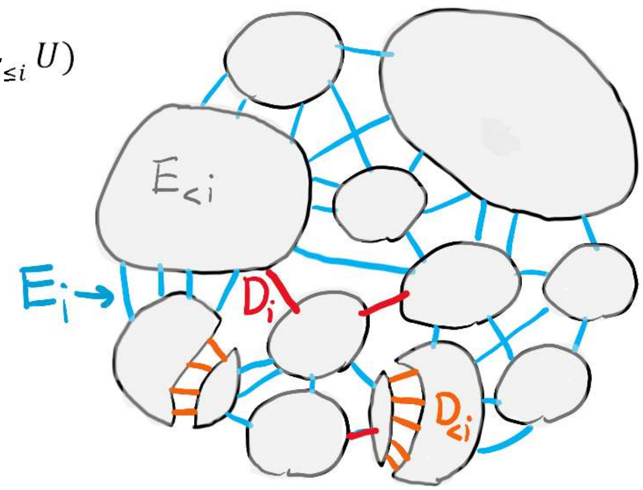

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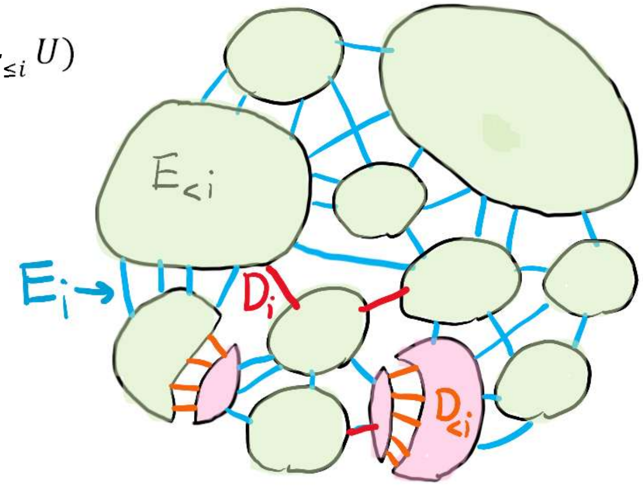

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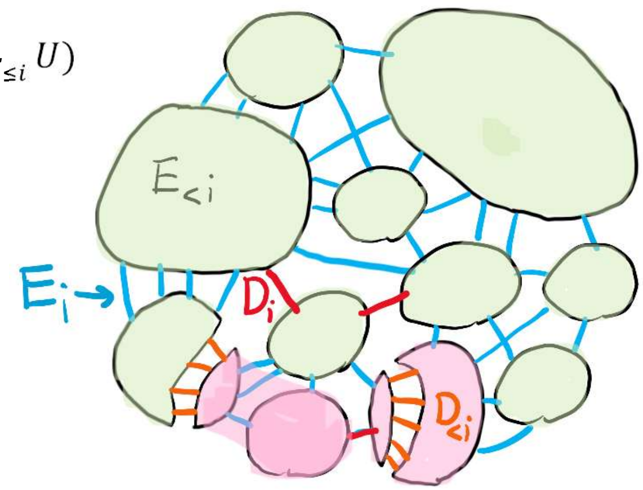

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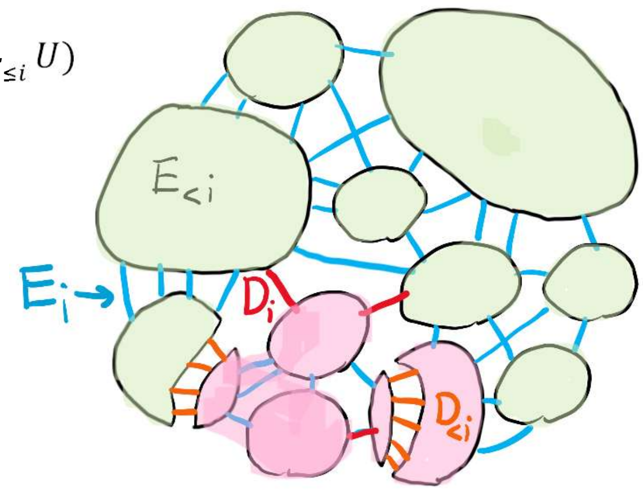

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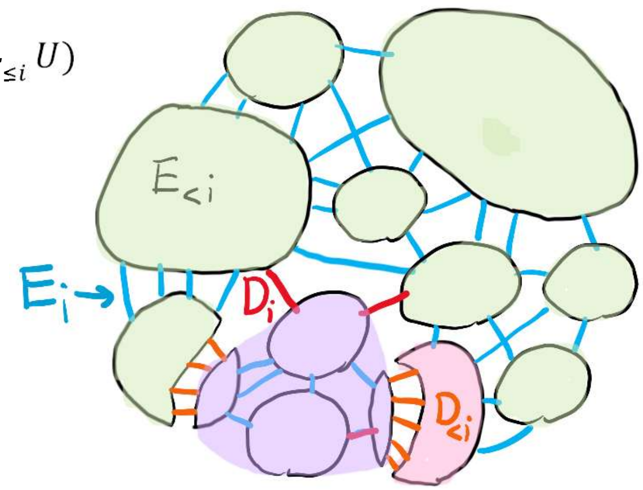

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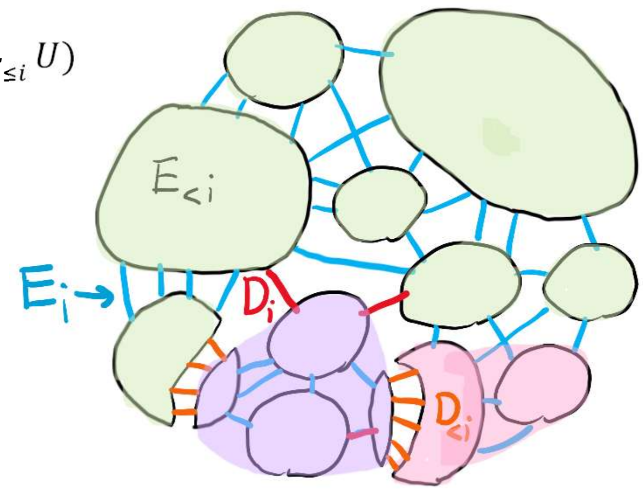

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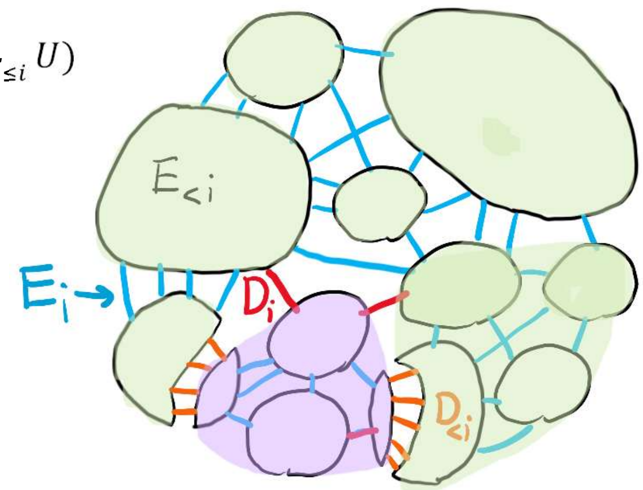

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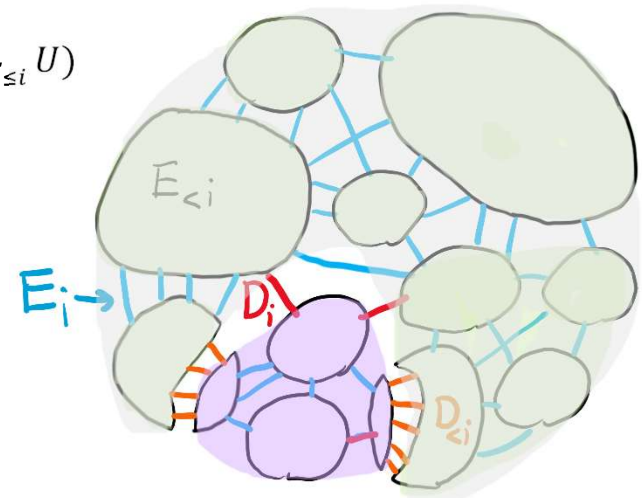

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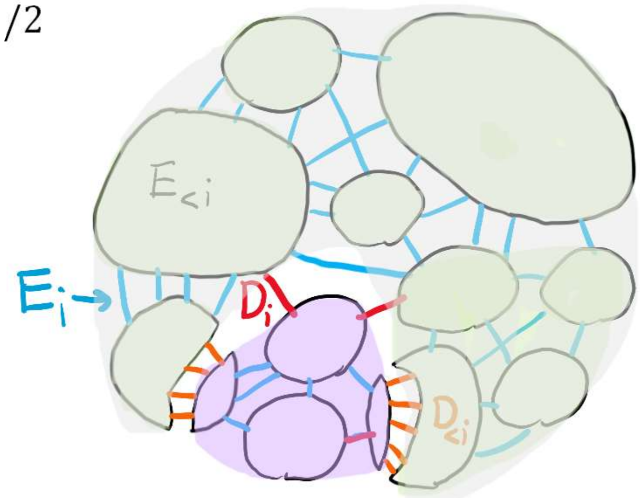
Correctness

Claim: all **passive regions** are in the same component of $H_{\leq i} - D$

Proof: Suppose not. $\exists B \subseteq H_{\leq i}$ where $\deg_i(B) \leq \deg_i(H_{\leq i})/2$

- B is a component in $H_{\leq i} - D$
- B = union of some passive region.



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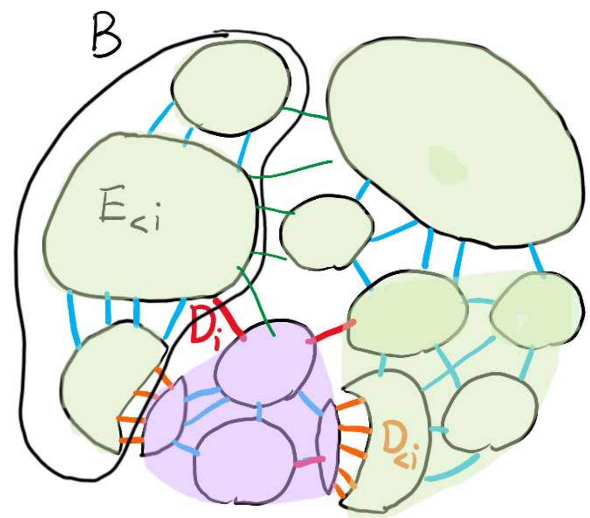
- $|E_{\leq i}(B, V - B)| \geq \phi \deg_i(B)$
  - E_i is ϕ-expanding in $H_{\leq i}$
- $\mathrm{del}_{\leq i}(B) < \phi \deg_i(B)/2$
  - B = union of passive comp.
- $\exists e \in E_{\leq i}(B, V - B) \setminus D_{\leq i}$.
  - B is **not** a component in $H_{\leq i} - D$. Contradiction.



Update Time on Level-i Cluster $H_{\le i}$

Let $d' = |D \cap H_{\le i}|$ be the number of deletions in $H_{\le i}$.

Update time: $\tilde{O}(d'/\phi)$

- Total level-i volume explored $\le 4d'/\phi$
  - Only explore from active region U: $\deg_i(U) \le 2\mathrm{del}_{\le i}(U)/\phi$
  - $\Sigma_U \mathrm{del}_{\le i}(U) = 2d$
- Data structures for merging regions
  - Union find

Connectivity Oracles under Edge Faults

1. Preprocess(G) where $G = (V, E)$
2. Update(D) where $D \subset E$ has size $|D| = d$

   Then, get representation of connected components of $G - D$
   Can answer connectivity queries in $G - D$ in $\tilde{O}(1)$ time.

Update time: $\tilde{O}(d)$ on any general graph ✔

Summary

Summary

- **Robust of expansion:**
  - Suppose F is ϕ-expanding in G.
  - After deleting d edges, can disconnect $\approx d/\phi$ F-edges
- Connectivity Oracles under Edge Faults [Patraşcu Thorup'07]
- **Separator-Expanding Hierarchy**
  - Top-down construction using **Expanding Balanced Separator**
  - A key tool for generalizing algorithms on expanders to general graphs