Lecture 3:
Boundary-Separator-Expanding Hierarchies

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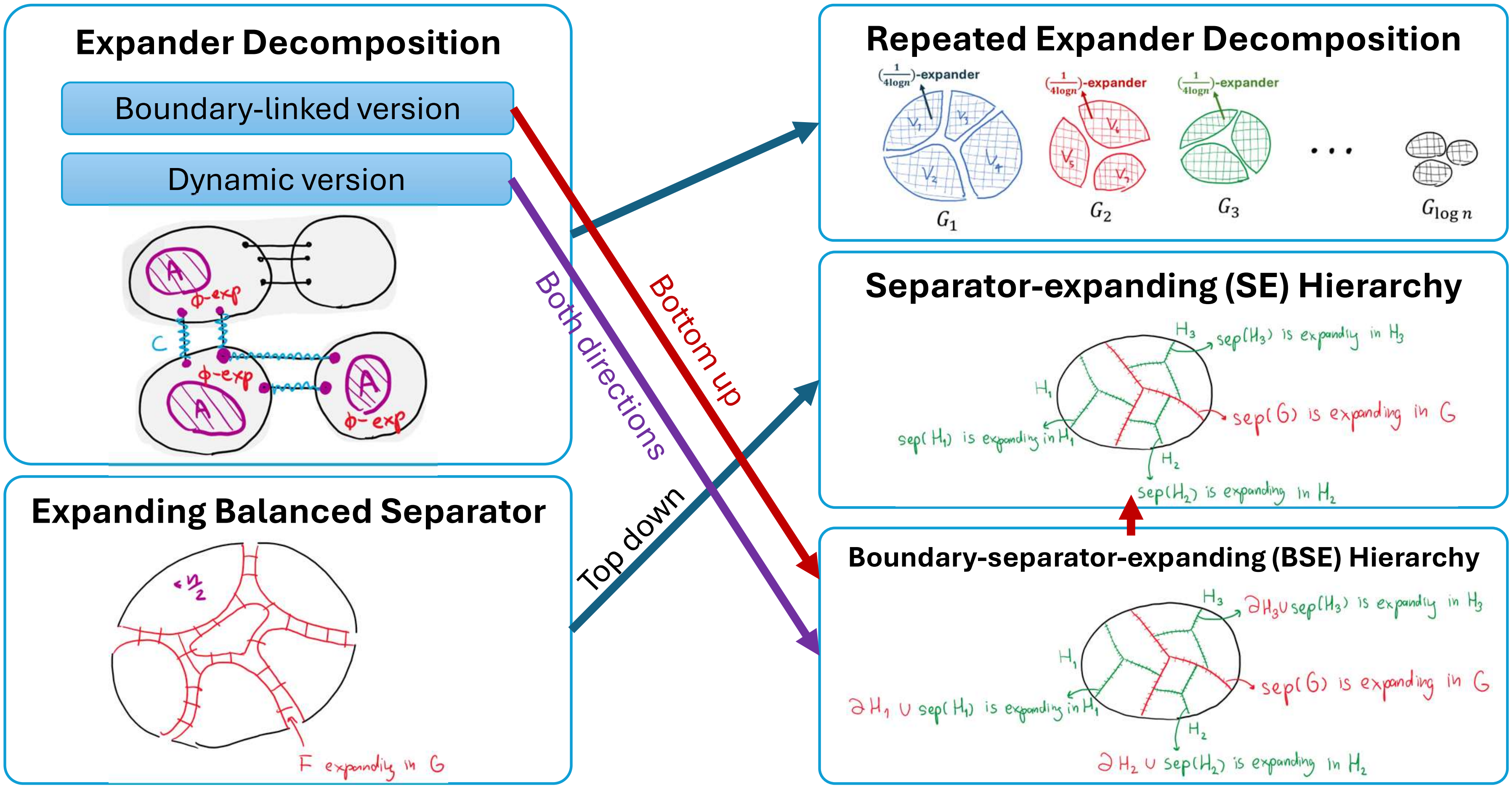
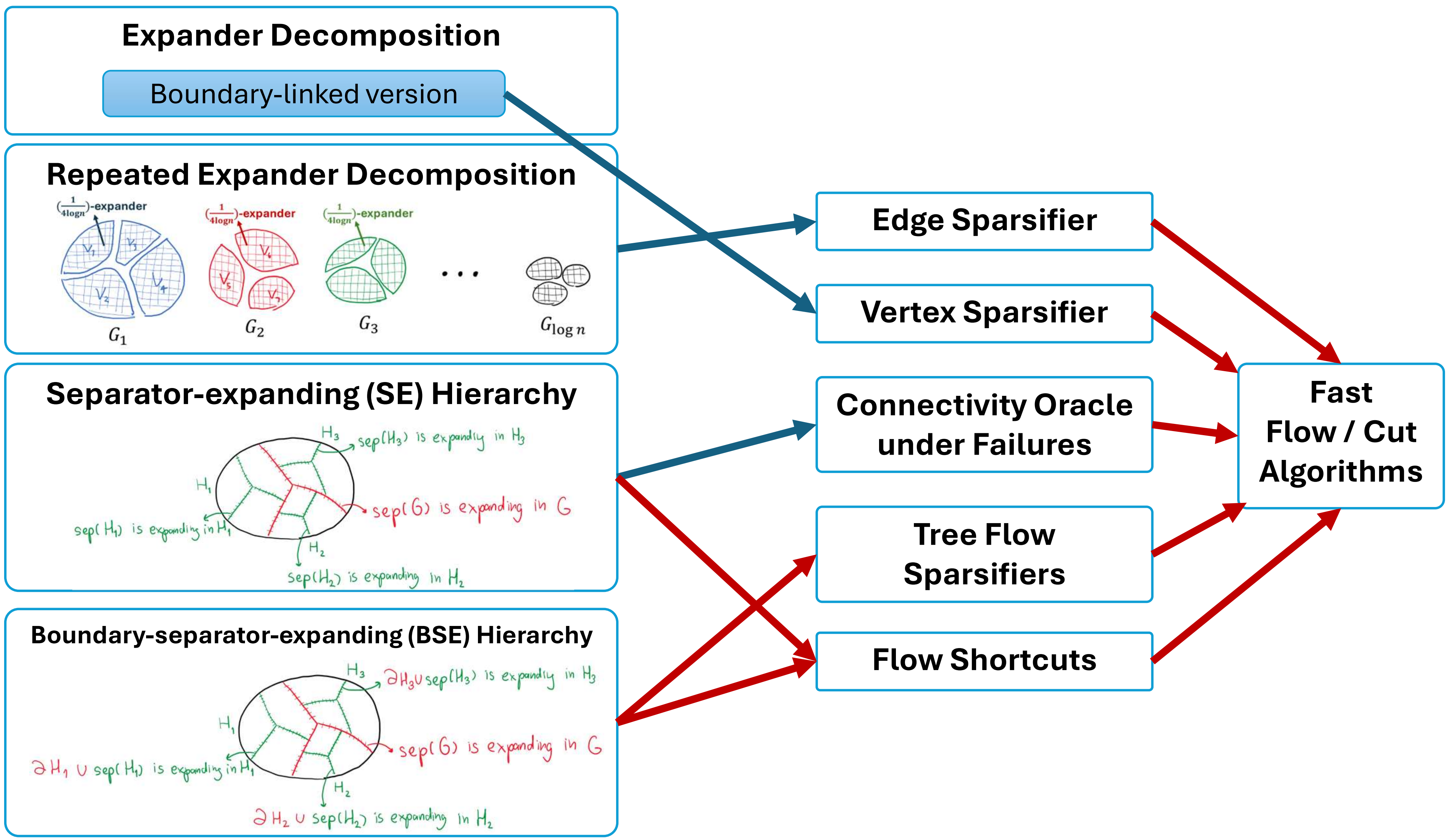
ADFOCS

Recap key concepts



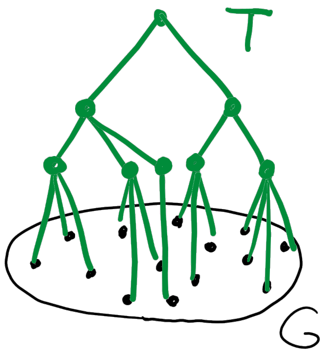


Expander Decomposition
Boundary-linked version
Repeated Expander Decomposition
$(\frac{1}{4\log n})$-expander
G_1
G_2
G_3
$G_{\log n}$
Separator-expanding (SE) Hierarchy
sep(G) is expanding in G
Boundary-separator-expanding (BSE) Hierarchy
Edge Sparsifier
Vertex Sparsifier
Connectivity Oracle under Failures
Tree Flow Sparsifiers
Flow Shortcuts
Fast Flow / Cut Algorithms

Part 1

Tree Cut/Flow Sparsifiers

Tree Cut Sparsifier



Def: A **tree cut sparsifier** T of G with quality q:

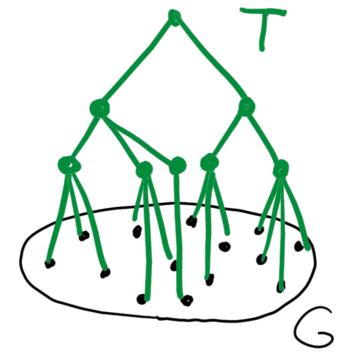
1. A capacitated tree where leaf set $= V(G)$
2. For any $X, Y \subset V$

$$\text{mincut}_G(A,B) \le \text{mincut}_T(A,B) \le q \,\text{mincut}_G(A,B)$$

A **single** tree that approximates all 2^n cuts of G!

- Related: a Gomory-Hu tree exactly preserves n^2 pair-wise mincuts of G.

Tree Flow Sparsifier



Def: A **tree cut sparsifier** T of G with quality q:

1. A capacitated tree where leaf set $= V(G)$
2. For any $X, Y \subset V$
$$\text{mincut}_G(A,B) \leq \text{mincut}_T(A,B) \leq q\ \text{mincut}_G(A,B)$$

Def: A **tree flow sparsifier** T of G with quality q:

2. For any $\deg_G$-respecting demand D
   - If D is routable in $G \Rightarrow D$ is routable in T
   - If D is routable in $T \Rightarrow D$ is routable in G with congestion q

Again, they are the same objects (up to $\log n$ factor).
Tree flow sparsifiers are stronger.

Today’s goal

Every graph admits a tree cut sparsifier with quality $O(\log^2 n)$ and depth $O(\log n)$

State of the art

- **Upper bound:** quality $O(\log n \log\log n)$ and depth $O(\log n)$ [Räcke, Shah’14]
- **Lower bound**: quality $\Omega(\log n)$ even on a grid graph.

Plan

1. Applications of tree cut/flow sparsifiers
2. Boundary-separator-expanding (BSE) hierarchies
3. Construct (1) using (2)
4. Constructions of BSE hierarchies
   - Simple construction (implemented in dynamic/distributed models)
   - Better construction (generalized to directed/length-constrained expansion)

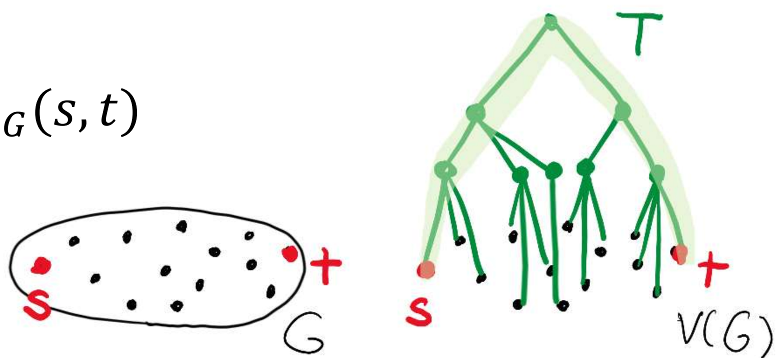
Part 2
Applications of Tree Cut Sparsifiers

Approximate Minimum Cut

Given a tree cut sparsifier T of G with quality q and depth d.

For any (s, t), we can q-approx. (s, t)-mincut in $O(d)$ time.

- **Algo:** return the minimum capacity c^* in (s, t)-path in T
- **Analysis:**
  - $c^* = \text{mincut}_T(s, t)$
  - $\text{mincut}_G(s, t) \leq \text{mincut}_T(s, t) \leq q \; \text{mincut}_G(s, t)$



Vertex Sparsifiers

Recall Lecture 1: Given $G = (V, E)$ and terminal set $U \subseteq V$.

There is a graph H s.t.

- for all $X, Y \subseteq U$,
  $$\mathrm{mincut}_G(X, Y) \leq \mathrm{mincut}_H(X, Y) \leq 4 \log n \cdot \mathrm{mincut}_G(X, Y)$$
- $|E(H)| = O(\deg_G(U))$

Weak if U contains high degree vertices

Vertex Sparsifiers

Will show: Given $G = (V, E)$ and terminal set $U \subseteq V$.

There is a graph H s.t.

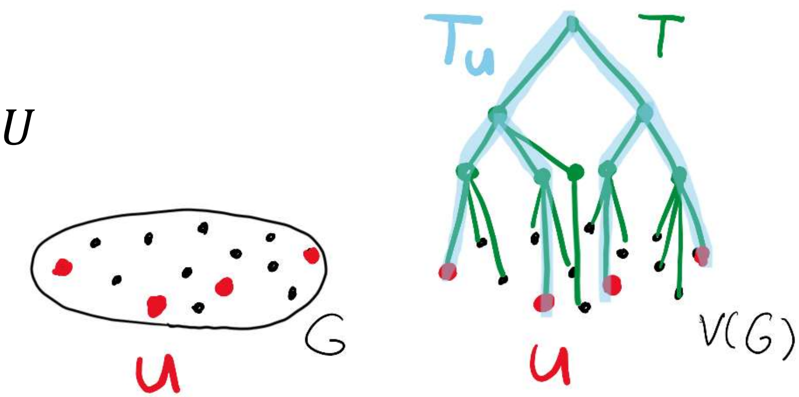
- for all $X, Y \subseteq U$,
  $$\mathrm{mincut}_G(X,Y) \leq \mathrm{mincut}_H(X,Y) \leq O(\log^2 n) \cdot \mathrm{mincut}_G(X,Y)$$
- $|E(H)| = O(|U| \log n)$

Vertex Sparsifiers from Tree Cut Sparsifiers

Given a tree cut sparsifier T of G with quality $q = O(\log^2 n)$ and depth $d = O(\log n)$.

$T_U \leftarrow$ union of root-to-leaf paths in T for all $v \in U$

- $|E(T_U)| = O(|U| \log n)$
- **Thm:** for all $X, Y \subseteq U$,
  $$\text{mincut}_G(X,Y) \leq \text{mincut}_{T_U}(X,Y) \leq O(\log^2 n) \cdot \text{mincut}_G(X,Y)$$
- **Proof:**
  - $\text{mincut}_{T_U}(X,Y) = \text{mincut}_T(X,Y)$ for $X, Y \subseteq U$
  - $\text{mincut}_T(X,Y) \approx_q \text{mincut}_G(X,Y)$



Vertex Sparsifiers

Will show: Given $G = (V, E)$ and terminal set $U \subseteq V$.

There is a graph H s.t.

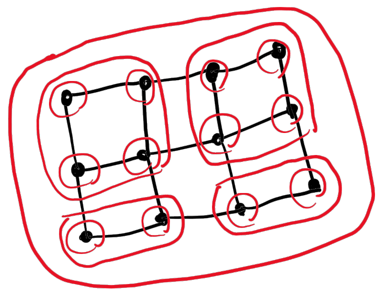
- for all $X, Y \subseteq U$,
  $$\text{mincut}_G(X, Y) \leq \text{mincut}_H(X, Y) \leq O(\log^2 n) \cdot \text{mincut}_G(X, Y)$$ ✔
- $|E(H)| = O(|U| \log n)$

Part 3

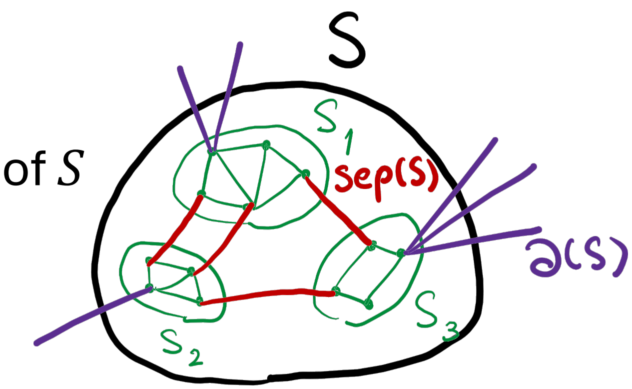
Boundary-Separator-Expanding Hierarchies

Hierarchy

- A hierarchy $\mathcal{H}$ of $G = (V, E)$ is a laminar family of induced graphs:
  - root = G
  - leaf = a vertex
  - non-leaf = $G[S]$ for some S



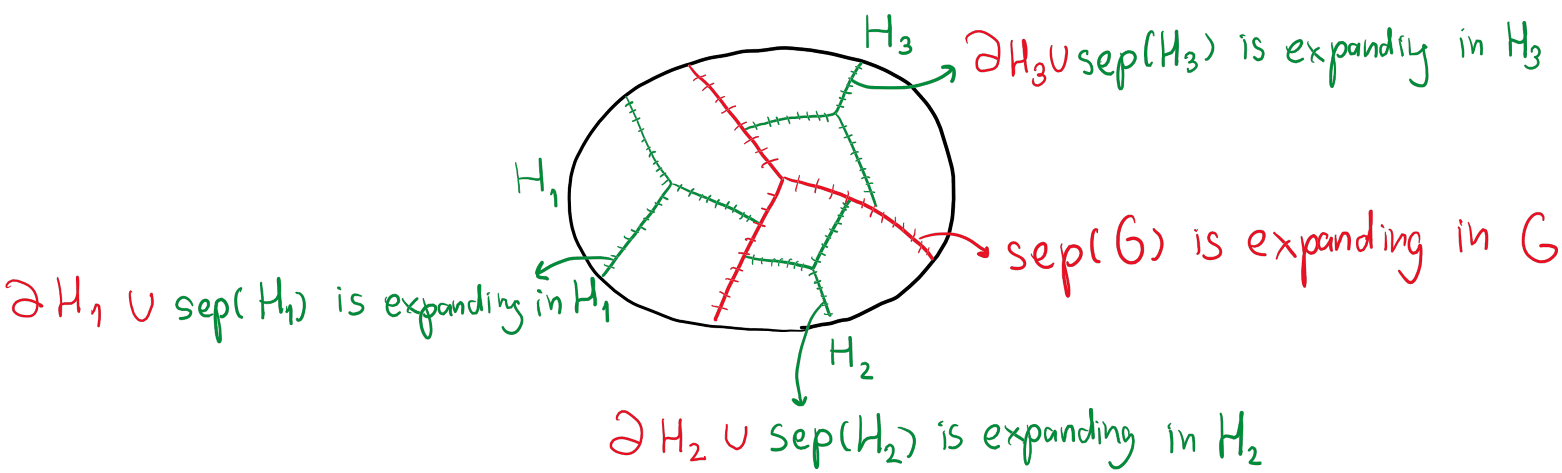
- For each *cluster* $H \in \mathcal{H}$,
  - **Separator of S** is $\text{sep}(S) \coloneqq$ edges crossing children of S
  - **Boundary of S** is $\partial(S) \coloneqq E(S, V - S)$



BSE Hierarchy

Def: a ϕ-boundary-separator-expanding (ϕ-BSE) hierarchy of G is

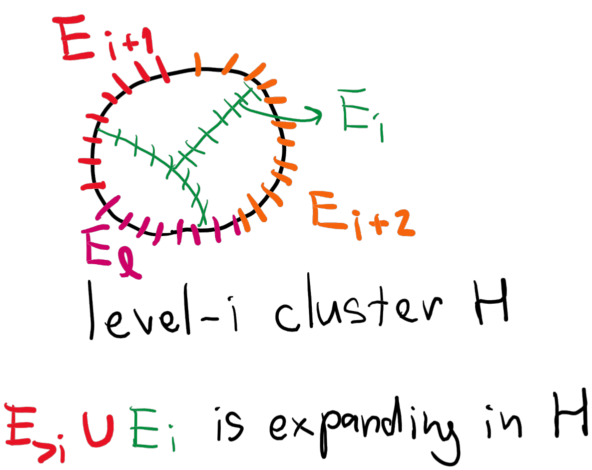
- a hierarchy $\mathcal{H}$ s.t. for every cluster $H \in \mathcal{H}$,
- $\partial H \cup \mathrm{sep}(H)$ is ϕ-expanding in H.

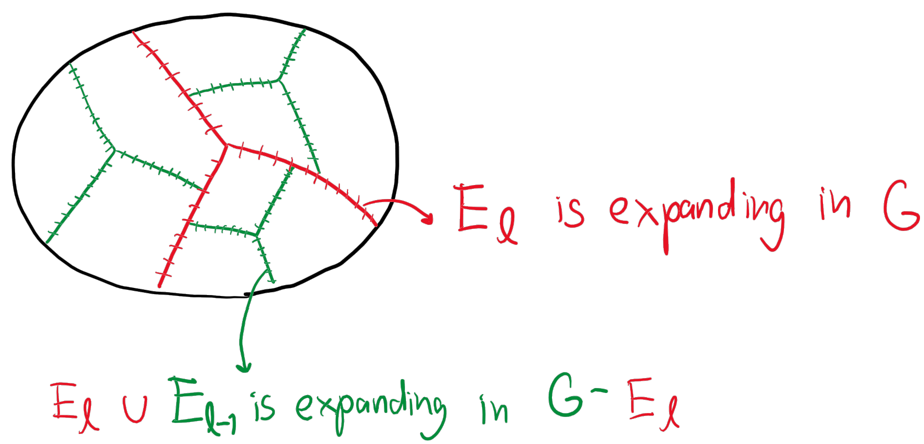


BSE Hierarchy: Partition View

Def: a ϕ-boundary-separator-expanding (ϕ-BSE) hierarchy of G is

- a partition $E_0, \ldots, E_\ell$ of $E(G)$ s.t.
- $E_{\geq i}$ is ϕ-expanding in $G - E_{>i}$



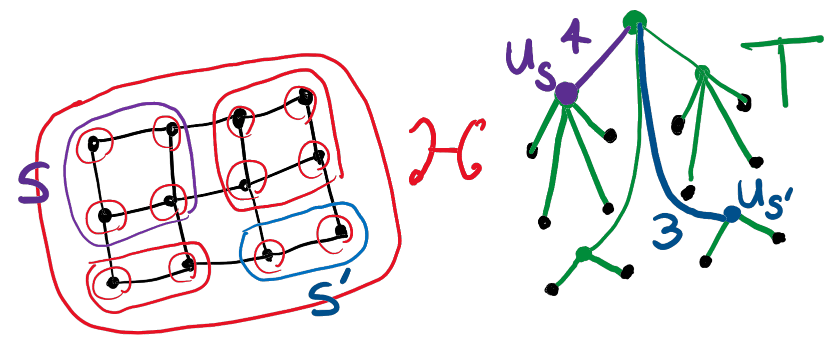


Part 4

BSE hierarchy → Tree flow sparsifier

BSE hierarchy → Tree flow sparsifier

- $\mathcal{H}$: ϕ-BS-expanding hierarchy with ℓ levels.
- T: tree corresponding to $\mathcal{H}$
  - cluster $S \leftrightarrow$ tree node u_S
  - $\text{cap}_T(u_S, \text{parent}(u_S)) = |\partial_G(S)|$



Thm: [Räcke'02]

T is tree flow sparsifier of G with quality ℓ/ϕ.

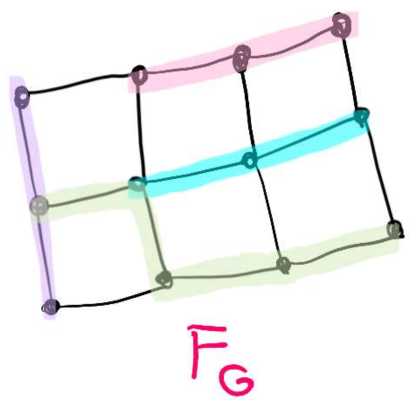
Remain to prove: For any $\deg_G$-respecting demand D
1. If D is routable in $G \Rightarrow D$ is routable in T
2. If D is routable in $T \Rightarrow D$ is routable in G with congestion $q = \ell/\phi$

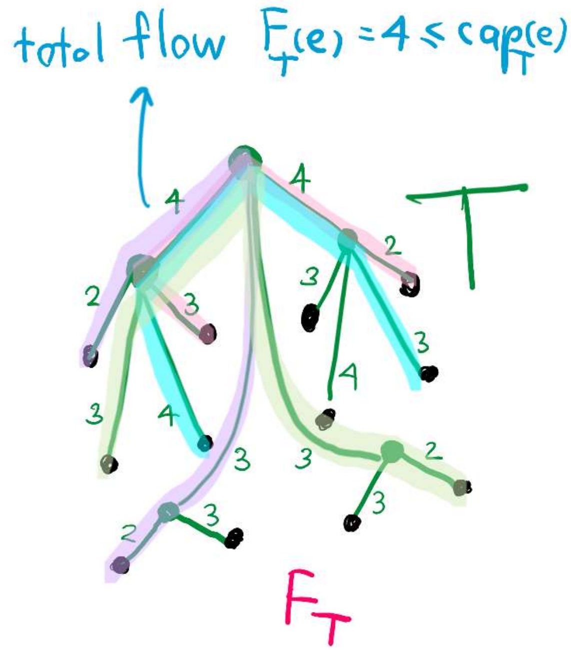
Routable in G ⇒ Routable in T

- Let D be a $\deg_G$-respecting demand routable in G with congestion 1
- **Goal**: Construct F_T routing D in T with congestion 1

- $F_T \leftarrow$ the unique way to route D in T.
- For each tree edge $e = (u_H, \text{parent}(u_H))$
  - $F_T(e) =$ total demand of D out of H
  - $\text{cap}_T(e) = |\partial H|$
  - $F_T(e) \leq \text{cap}_T(e)$
    - As D is routable in G with congestion 1
- So, $\text{cong}(F_T) \leq 1$

Did not need that $\mathcal{H}$ is a ϕ-BSE hierarchy in this argument





Routable in T $\Rightarrow$ Routable in G with congestion ℓ/ϕ

- Let D be a $\deg_G$-respecting demand routable in T with congestion 1
- **Goal**: Construct F_G routing D in G with congestion ℓ/ϕ

Bottom-up Strategy:

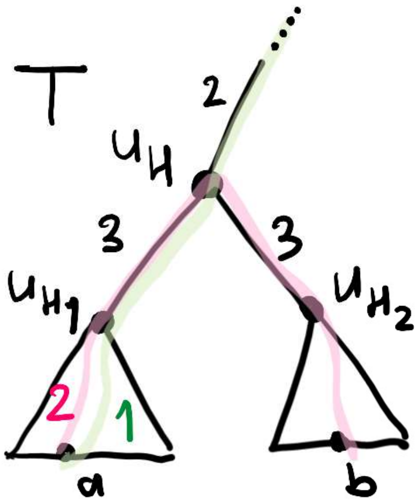
- For each cluster $H \in \mathcal{H}$, define F_H **inside $\boldsymbol{H}$**
  - F_H "finishes" routing demand in D between children of H.
  - F_H routes demand out of H to boundary ∂H ("forward" to parent cluster)
  - F_H has congestion $1/\phi$
- F_G $\leftarrow$concatenate F_H overall clusters H
  - F_G successfully routes D
  - F_G has congestion ℓ/ϕ

Requirement of F_H on cluster H

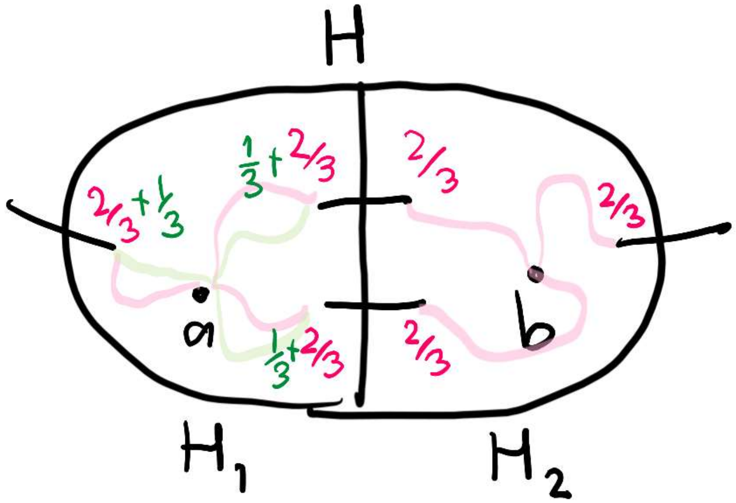
- Let $H_1, \ldots, H_s$ be children of cluster H

- **Pre-condition:**
  - Demand out of H_i has been routed uniformly to ∂H_i
- **Post-condition:**
  - Demand between children of H is successfully routed.
  - Demand out of H is routed uniformly to ∂H

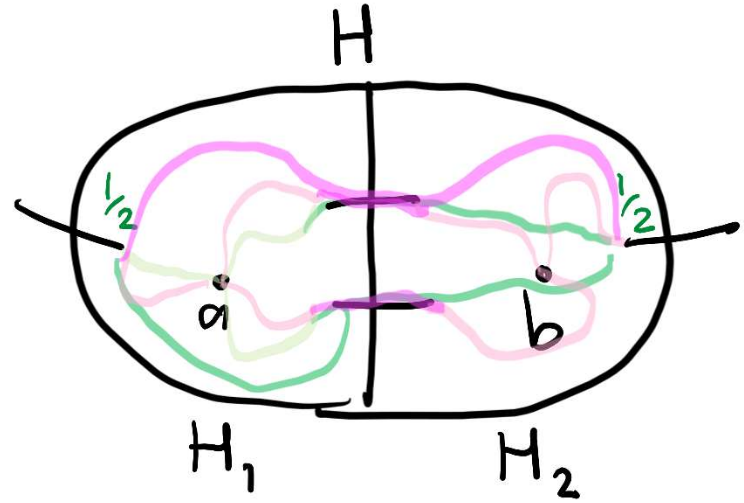
Requirement of F_H on cluster H (with pictures)



Demand
$D(a,b) = 2$
$D(a,c) = 1$ where $c \notin H$



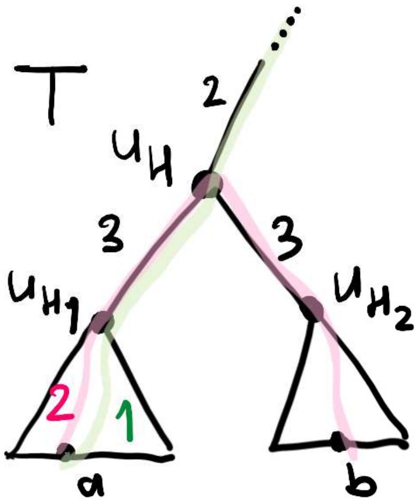
Pre-condition:
demand out of H_i is on ∂H_i uniformly



- F_H routes demand between children
- F_H routes demand out of H to ∂H uniformly

So, Post-condition:
demand out of H is on ∂H uniformly

BSE → Existence of F_H with low congestion



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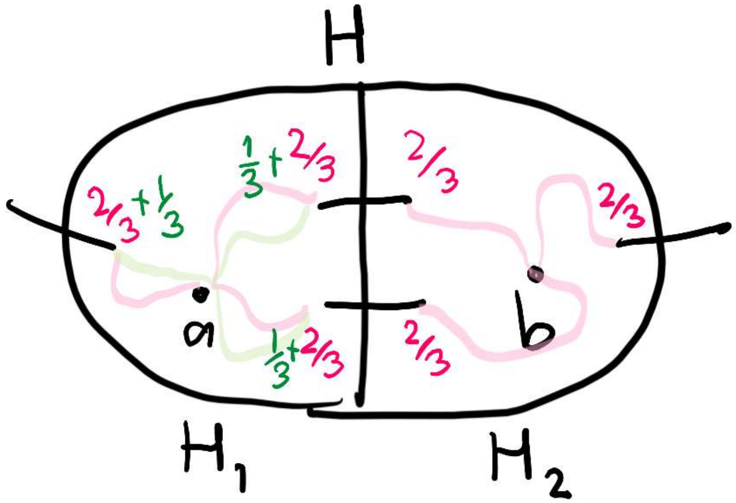

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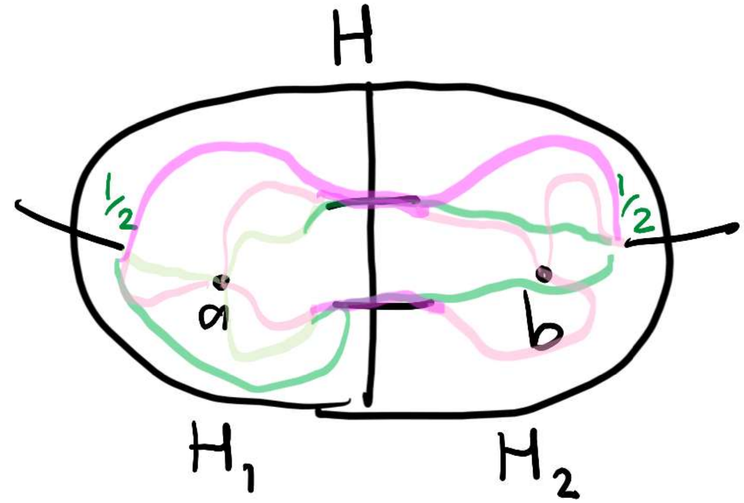

- F_H routes demand between children
- F_H routes demand out of H to ∂H uniformly

So, Post-condition:
demand out of H is on ∂H uniformly

To satisfy **Post-condition** given **Pre-condition**,
this induces a demand D_H respecting $\partial H \cup \text{sep}(H)$.
$\partial H \cup \text{sep}(H)$ is ϕ-expanding in $H \Rightarrow \exists F_H$ routing D_H in H with congestion $1/\phi$

Routable in T $\Rightarrow$ Routable in G with congestion ℓ/ϕ

- Let D be a $\deg_G$-respecting demand routable in T with congestion 1
- **Goal**: Construct F_G routing D in G with congestion ℓ/ϕ

Bottom-up Strategy:

- ✔ For each cluster $H \in \mathcal{H}$, define F_H **inside** $\boldsymbol{H}$
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  - F_H has congestion $1/\phi$
- ✔ F_G $\leftarrow$concatenate F_H overall clusters H
  - F_G route all demand pairs in D
  - F_G has congestion ℓ/ϕ

BSE hierarchy → Tree flow sparsifier

- $\mathcal{H}$: ϕ-BS-expanding hierarchy with ℓ levels.
- T: tree corresponding to $\mathcal{H}$
  - cluster $S \leftrightarrow$ tree node u_S
  - $\mathrm{cap}_T\left(u_S, \mathrm{parent}(u_S)\right) = |\partial_G(S)|$

Thm: [Räcke'02]

T is tree flow sparsifier of G with quality ℓ/ϕ.

Part 5
Simple Bottom-Up Construction of BSE hierarchies

Based on [Goranci Raecke S Tan'21]

BSE Hierarchy

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- a hierarchy $\mathcal{H}$ s.t. for every cluster $H \in \mathcal{H}$,
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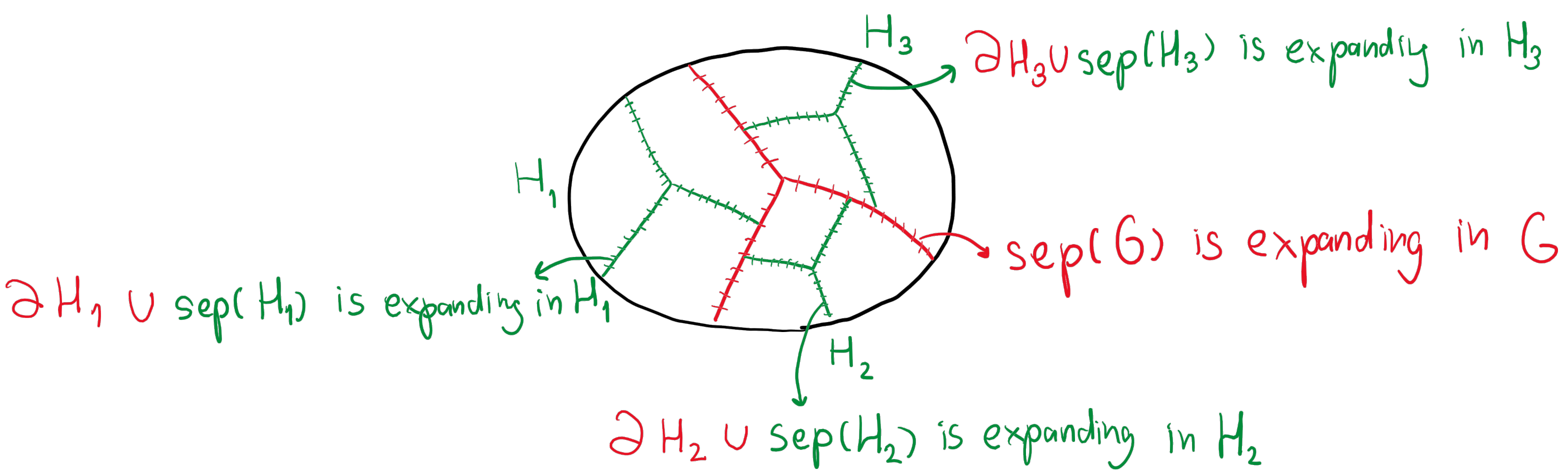


Our Goal

Let G be a graph.

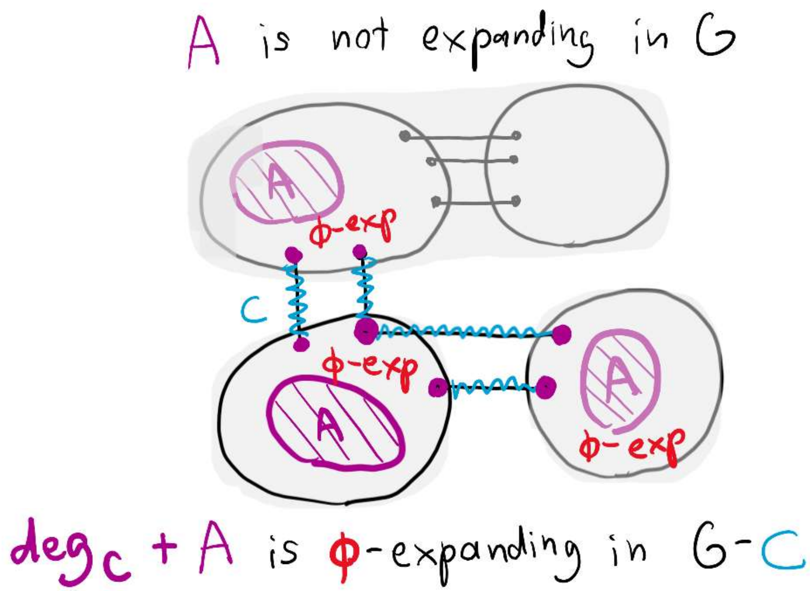
Theorem: ∃ a $(1/n^{o(1)})$-BSE hierarchy of G with $\sqrt{\log n}$ levels.

Corr: ∃ a tree flow sparsifier of G with quality $n^{o(1)}$

Recall: boundary-linked ϕ-expander decomposition

Theorem: Given $G = (V, E)$, A, $\phi \leq 1/4 \log n$, there exists $C \subseteq E$

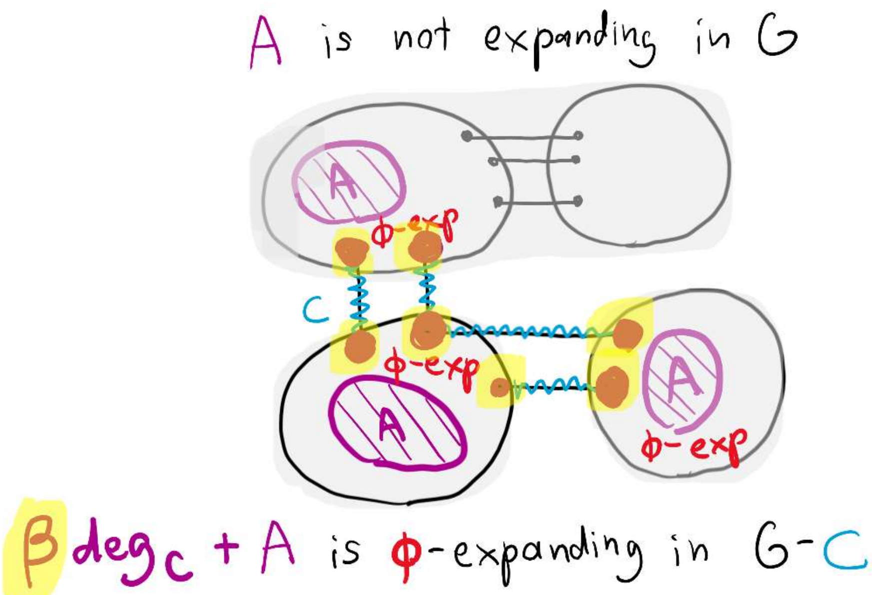
- $|C| \leq (2\phi \log n) \cdot |A|$
- $A + \deg_C$ is ϕ-expanding in $G - C$



Recall: β-boundary-linked ϕ-expander decomposition

Theorem: Given $G = (V, E)$, A, $\beta \leq 1/4\phi \log n$, there exists $C \subseteq E$

- $|C| \leq (2\phi \log n) \cdot |A|$
- $A + \beta \deg_C$ is ϕ-expanding in $G - C$



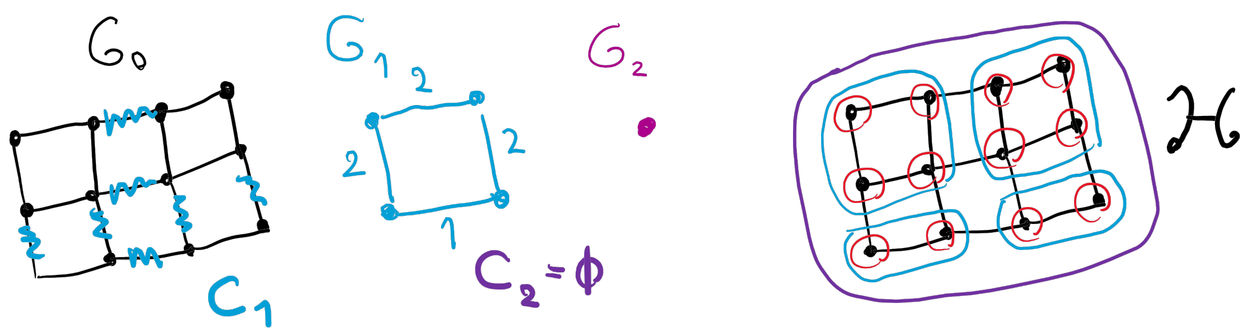
Think:

$\phi = 1/(\log n)^{\Theta(\sqrt{\log n})}$ and $\beta = 1/4\phi \log n$.

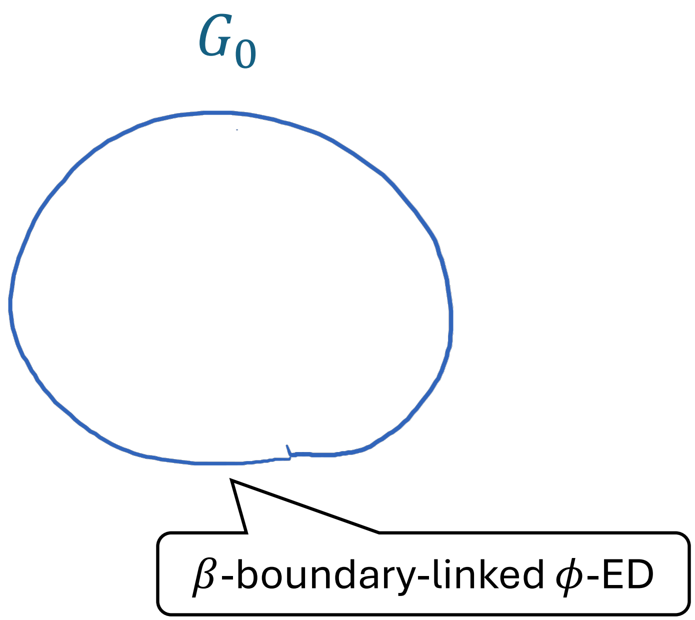
- $A \cap U$ is ϕ-expanding in $G[U]$.
- ∂U is $(\frac{\phi}{\beta} = \frac{1}{4 \log n})$-expanding in $G[U]$
- The boundary is much more expanding

Construction: **Contract and Recurse**

- Init: $G_0 \leftarrow G$, $\phi = 1/(\log n)^{\Theta(\sqrt{\log n})}$, $\beta = 1/4\phi \log n$.
- For $i \geq 1$
  - $C_i \leftarrow \beta$-boundary-linked ϕ-ED of G_{i-1}
  - $G_i \leftarrow$ contract components of $G_{i-1} - C_i$ (remove self loops)
  - if $E(G_i) = \emptyset$, break
- Return $\mathcal{H} \leftarrow \{\text{super-nodes in all } G_i\}$

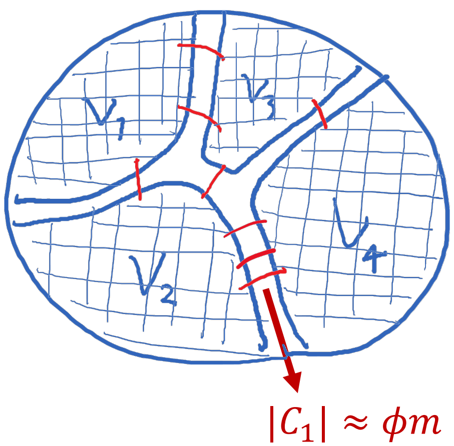


Illustration

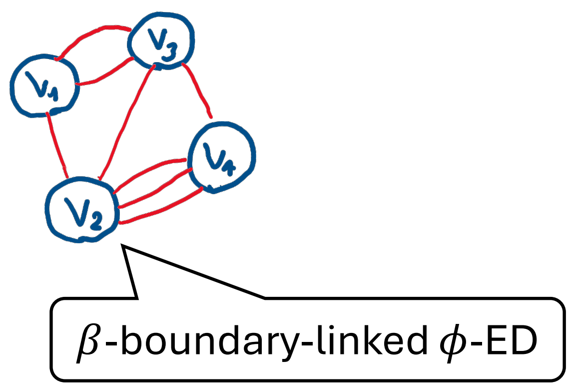


Illustration

G_0



G_1: contracting G_0



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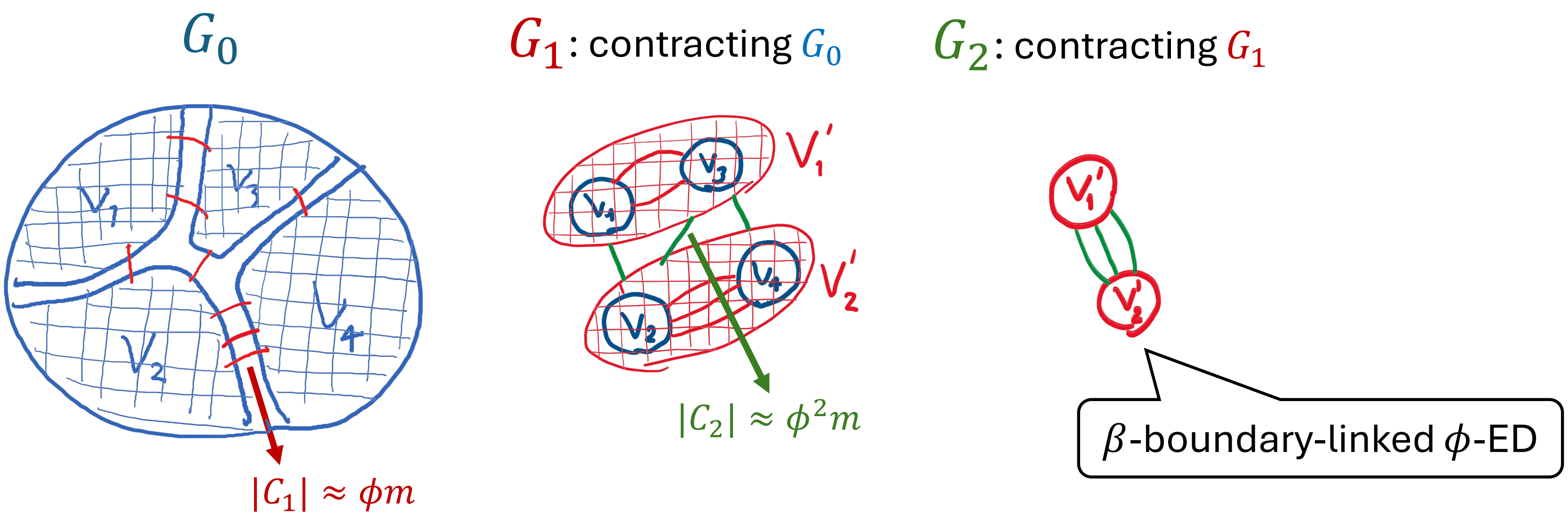

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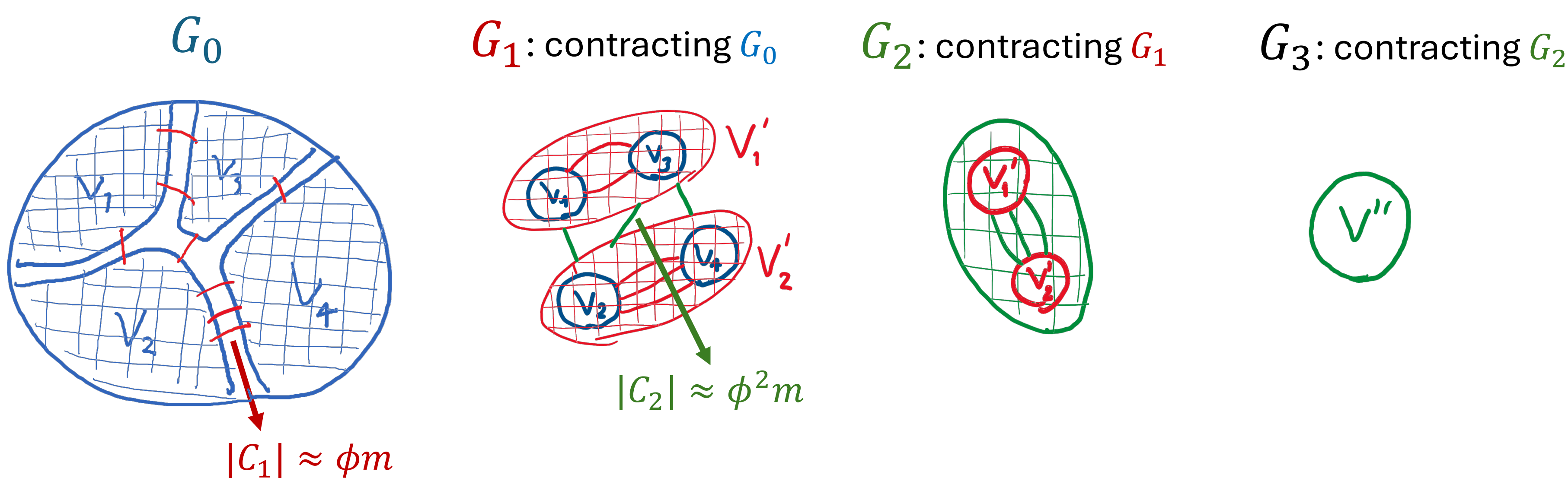
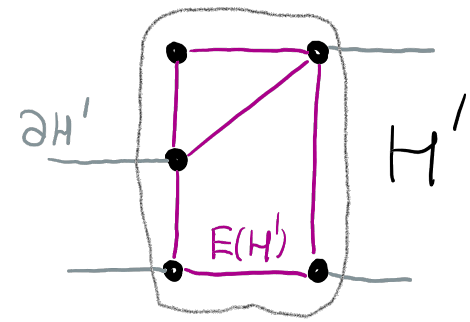
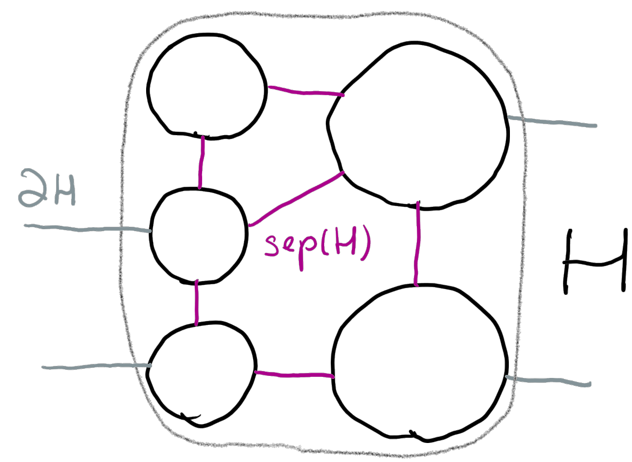


Contracted Cluster H' vs. Cluster H

- Let H' be a component in $G_i - C_{i+1}$.
- **Key:** $\beta(\partial H') + E(H')$ is ϕ-expanding in H'
  - C_{i+1} is a β-boundary-linked ϕ-ED of G_i



- H: uncontract supernodes of H'
  - $\partial H = \partial H'$
  - $\mathrm{sep}(H) = E(H')$
- H is a **level-*i*** cluster of $\mathcal{H}$ (component of $G - C_{i+1}$).



Construction: **Contract and Recurse**

- Init: $G_0 \leftarrow G$, $\phi = 1/(\log n)^{\Theta(\sqrt{\log n})}$, $\beta = 1/4\phi \log n$.
- For $i \geq 1$
  - $C_i \leftarrow \beta$-boundary-linked ϕ-ED of G_{i-1}
  - $G_i \leftarrow$ contract components of $G_{i-1} - C_i$ (remove self loops)
  - if $E(G_i) = \emptyset$, break
- Return $\mathcal{H} \leftarrow \{$super-nodes in all $G_i\}$

Analysis

- There are $\ell = \sqrt{\log n}$ levels as $|C_i| \leq (2\phi \log n)^i m$.
- **To show:** boundary and separator of cluster H are expanding in H

Analysis plan

Let H be a **level-$\boldsymbol{i}$** cluster (component of $G - C_{i+1}$).

Step 1: ∂H is $\frac{1}{(4\log n)^i}$ expanding in H

Step 2: $\partial H \cup \mathrm{sep}(H)$ is $\frac{\phi}{(4\log n)^i}$ expanding in H

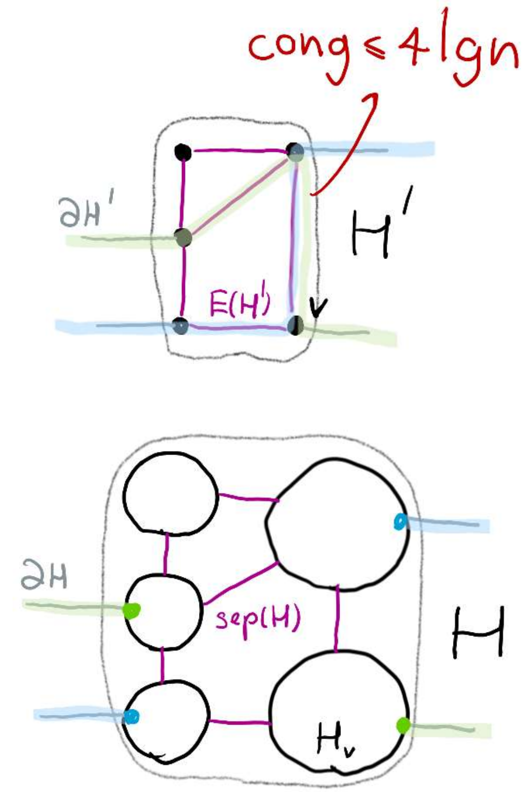
So, $\mathcal{H}$ is $\left(\frac{\phi}{(4\log n)^\ell} = \frac{1}{(\log n)^{O\left(\sqrt{\log n}\right)}}\right)$-BSE hierarchy of G.

Boundary is Expanding

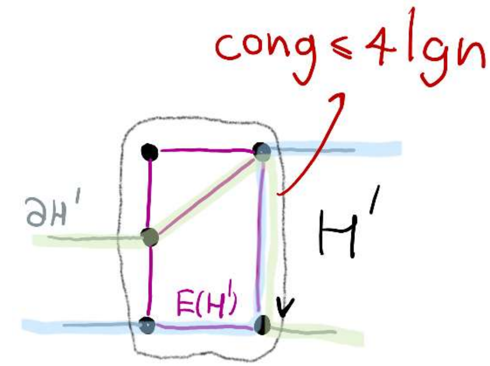
D: ∂H-respecting demand

Task: route D in H with congestion $(4\log n)^{i+1}$

1. Route D in contracted H'
   $\exists F'$ routing D in H with congestion $4\log n$
   - $\beta(\partial H')$ is ϕ-expanding in H'
   - $\partial H'$ is $(\phi/\beta = 1/4\log n)$-expanding in H'
   - D respects $\partial H = \partial H'$
2. Route inside supernode of H'



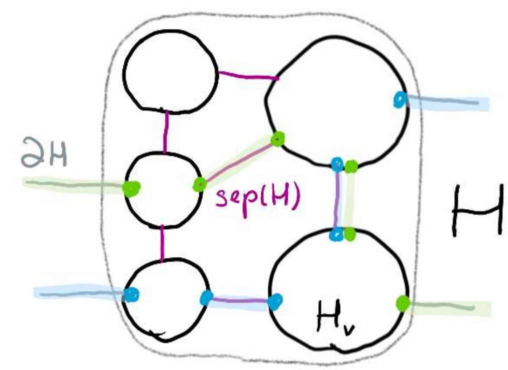
Boundary is Expanding



D: ∂H-respecting demand

Task: route D in H with congestion $(4 \log n)^{i+1}$

1. Route D in contracted H'
   $\exists F'$ routing D in H with congestion $4 \log n$
2. Route inside supernode of H'
   - H_v: a level-$(i-1)$ child of H
   - ∂H_v-respecting demand is routable in H_v with cong $(4 \log n)^i$ (by induction)
   - "To connect F' inside H_v" induces a $(4 \log n) \partial H_v$-respecting demand D_v
   - D_v is routable with congestion $(4 \log n)^{i+1}$



Analysis plan

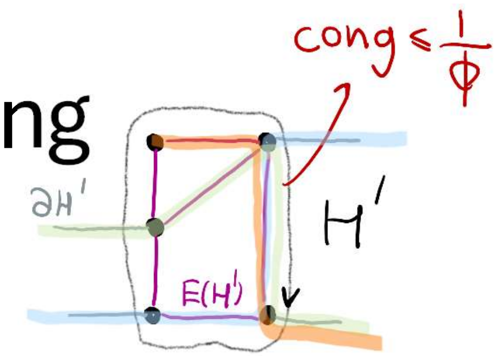
Let H be a **level-$\boldsymbol{i}$** cluster (component of $G - C_{i+1}$).

✔ **Step 1**: ∂H is $\frac{1}{(4\log n)^i}$ expanding in H

Step 2: $\partial H \cup \mathrm{sep}(H)$ is $\frac{\phi}{(4\log n)^i}$ expanding in H

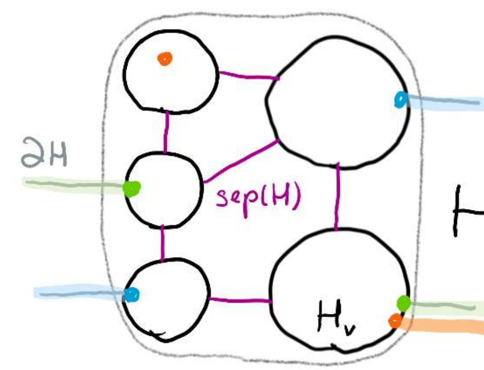
So, $\mathcal{H}$ is $\left(\frac{\phi}{(4\log n)^\ell} = \frac{1}{(\log n)^{O(\sqrt{\log n})}}\right)$-BSE hierarchy of G.

Boundary and Separator are Expanding



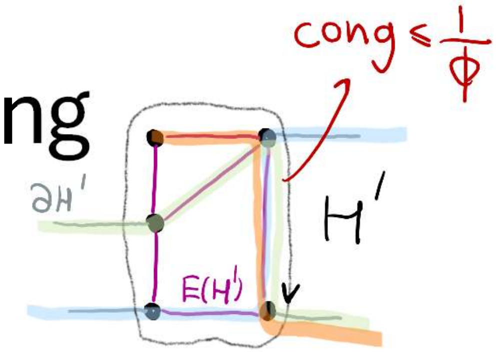
D: $\partial H \cup \text{sep}(H)$-respecting demand

Task: route D in H with congestion $(4 \log n)^i / \phi$



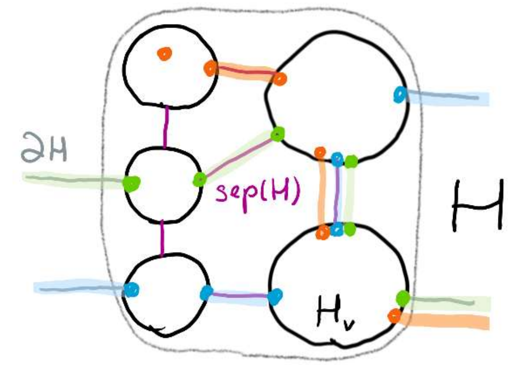
1. Route D in contracted H'
   $\exists F'$ routing D in H with congestion $1/\phi$
   - $\partial H' \cup E(H')$ is ϕ-expanding in H'
   - D respects $\partial H \cup \text{sep}(H) = \partial H' \cup E(H')$
2. Route inside supernode of H'

Boundary and Separator are Expanding



D: $\partial H \cup \text{sep}(H)$-respecting demand

Task: route D in H with congestion $(4 \log n)^i / \phi$



1. Route D in contracted H'
   $\exists F'$ routing D in H with congestion $1/\phi$
2. Route inside supernode of H'
   - H_v: a level-$(i-1)$ child of H
   - ∂H_v-respecting demand is routable in H_v with cong $(4 \log n)^i$ (by induction)
   - "To connect F' inside H_v" induces a $(1/\phi)\partial H_v$-respecting demand D_v
   - D_v is routable with congestion $(4 \log n)^i / \phi$

Analysis plan

Let H be a **level-$\boldsymbol{i}$** cluster (component of $G - C_{i+1}$).

✔ **Step 1**: ∂H is $\frac{1}{(4\log n)^i}$ expanding in H

✔ **Step 2**: $\partial H \cup \mathrm{sep}(H)$ is $\frac{\phi}{(4\log n)^i}$ expanding in H

✔ So, $\mathcal{H}$ is $\left(\frac{\phi}{(4\log n)^\ell} = \frac{1}{(\log n)^{O\left(\sqrt{\log n}\right)}}\right)$-BSE hierarchy of G.

Summary

Let G be a graph.

✔ **Theorem:** ∃ a $(1/n^{o(1)})$-BSE hierarchy of G with $\sqrt{\log n}$ levels.
Corr: ∃ a tree flow sparsifier of G with quality $n^{o(1)}$

Simplicity of this construction leads to

- Dynamic construction [GRST'21]
  ⇒ dynamic max flow [GRST'21], dynamic mincut [JST'24, EHL'25], static exact max flow [BCKLMPS'24]
- Distributed construction [HRG'22]
  ⇒ Universally Optimal Distributed Algorithms

Part 6
Construction of BSE Hierarchies via Dynamic Expander Decomposition

Based on [Haeupler Long Röyskö S'26]

Next Goal

Let G be a graph.

Theorem: ∃ a $(\frac{1}{16\ \mathrm{lo}})$-BSE hierarchy of G with $\log m$ levels.

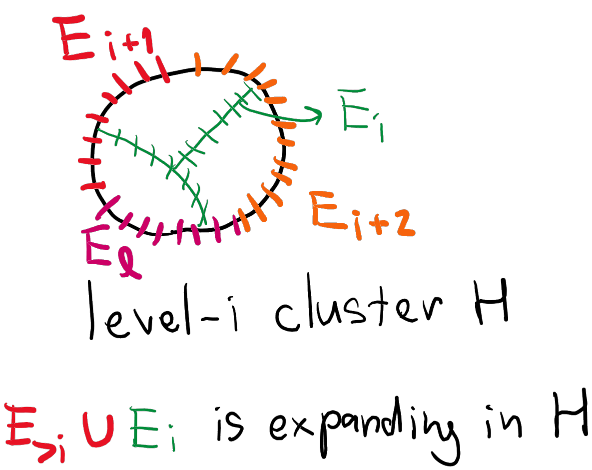
Corr: ∃ a tree cut sparsifier of G with quality $O\left(\log^2 n\right)$

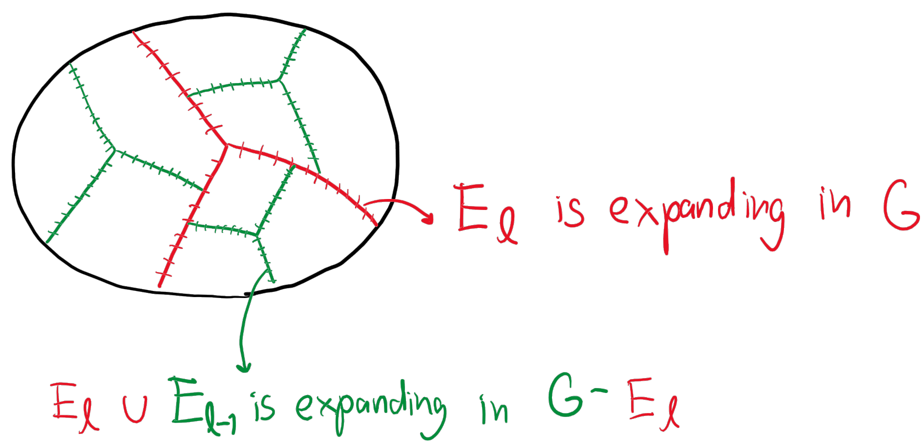
$*$Do not get tree flow sparsifier. The argument only bounds cut expansion in the BSE hierarchy

Recall: BSE Hierarchy: Partition View

Def: a ϕ-boundary-separator-expanding (ϕ-BSE) hierarchy of G is

- a partition $E_0, \ldots, E_\ell$ of $E(G)$ s.t.
- $E_{\geq i}$ is ϕ-expanding in $G - E_{>i}$





Ingredient: Dynamic Expander Decomposition

- **Fixed** graph $G = (V, E)$ and ϕ
- Objects that **only grows**
  - $D \subseteq E$: set of edge deletions
  - A: a node weighting

Thm: $\mathrm{DynED}(G, \phi, A, D)$ maintains an incremental set $C \supseteq D$

- A is ϕ-expander in $G - C$
- $|C| - |D| \leq \phi |A| \log n$

Note: Even when A and D grow, C might not grow

BSE Hierarchy Construction

Thm: $\mathrm{DynED}(G, \phi, A, D)$ maintains $C \supseteq D$
- A is ϕ-expander in $G - C$
- $|C| - |D| \leq \phi |A| \log n$

- Init:
  - $C_0 \leftarrow E, C_i \leftarrow \emptyset$ for $i \geq 1$
  - $\phi = \frac{1}{16 \log n}$ and $\ell = \frac{\log m}{\log(1/4 \log \quad)}$

- For $0 \leq i \leq \ell$, maintain *until there is no update*

$$C_{i+1} \leftarrow \mathrm{DynED}(G, \phi, A_i \coloneqq \deg_{C_i}, D_i \coloneqq C_{i+2})$$

- Return $E_i = C_i - C_{i+1}$ for all $i \leq \ell$.

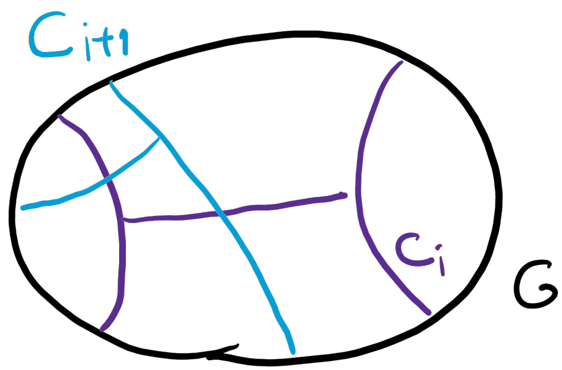
Let's first understand what's going on

$C_{i+1} \leftarrow \mathrm{DynED}_i := \mathrm{DynED}(G, \phi, A_i := \deg_{C_i}, D_i := C_{i+2})$

Bidirectional interactions between levels

Not trivial! This is not quite bottom-up nor top-down.

When C_{i+1} grows, $D_{i-1} := C_{i+1}$ at level $i-1$ grows too



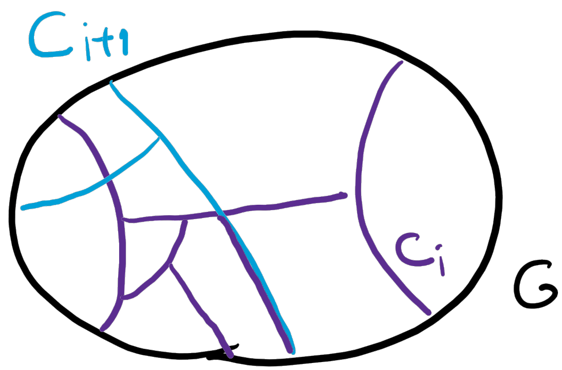
Interaction with level $i-1$

$C_{i+1} \leftarrow \mathrm{DynED}_i \coloneqq \mathrm{DynED}(G, \phi, A_i \coloneqq \deg_{C_i}, D_i \coloneqq C_{i+2})$

Bidirectional interactions between levels

Not trivial! This is not quite bottom-up nor top-down.

When C_{i+1} grows,
$D_{i-1} \coloneqq C_{i+1}$ at level $i-1$
grows too



After D_{i-1} grows,
$C_i \leftarrow \mathrm{DynED}_{i-1}$ at level $i-1$ grows too
$A_i \coloneqq \deg_{C_i}$ grows
This might cause C_{i+1} to grow again

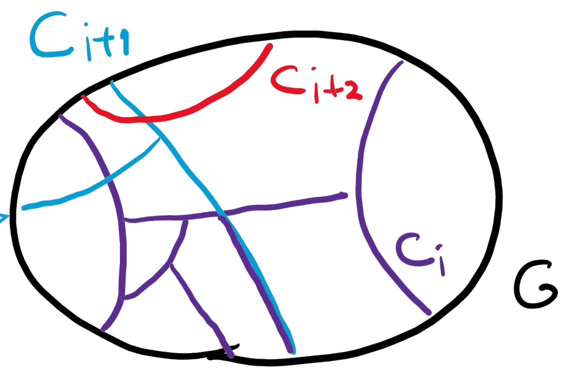
Interaction with level $i-1$

$C_{i+1} \leftarrow \mathrm{DynED}_i \coloneqq \mathrm{DynED}(G, \phi, A_i \coloneqq \mathrm{deg}_{C_i}, D_i \coloneqq C_{i+2})$

Bidirectional interactions between levels

Not trivial! This is not quite bottom-up nor top-down.

When C_{i+1} grows,
$A_{i+1} \coloneqq \mathrm{deg}_{C_{i+1}}$ at level $i+1$ grows
C_{i+2} might grow



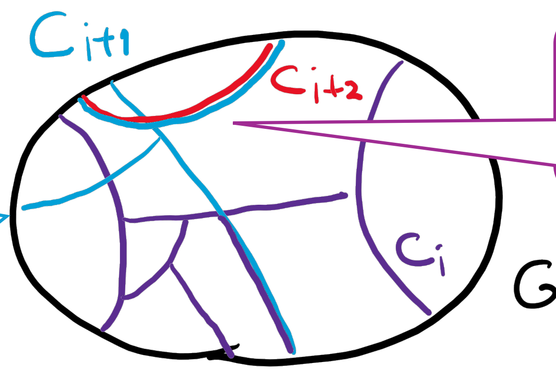
Interaction with level $i + 1$

$C_{i+1} \leftarrow \mathrm{DynED}_i := \mathrm{DynED}(G, \phi, A_i := \deg_{C_i}, D_i := C_{i+2})$

Bidirectional interactions between levels

Not trivial! This is not quite bottom-up nor top-down.

When C_{i+1} grows,
$A_{i+1} := \deg_{C_{i+1}}$ at level $i+1$ grows
C_{i+2} might grow

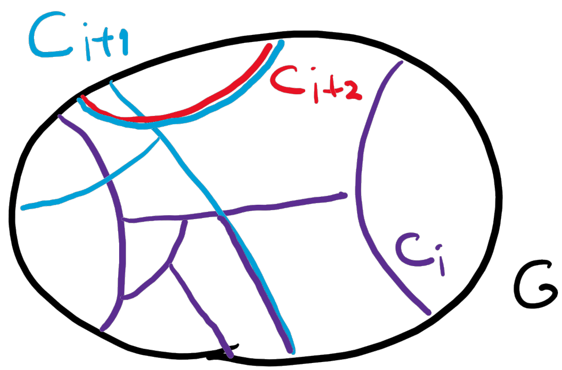


After C_{i+2} grows,
$D_i := C_{i+2}$ grows
$C_{i+1} \supseteq D_i$ must grow again

Interaction with level $i+1$

$C_{i+1} \leftarrow \mathrm{DynED}_i \coloneqq \mathrm{DynED}(G, \phi, A_i \coloneqq \deg_{C_i}, D_i \coloneqq C_{i+2})$

But $|C_i|$ get smaller and smaller



- **Homework:** $|C_i| \leq (4\phi \log n)^i m$ by non-trivial induction.
- So, $C_{\ell+1=O(\log m)} = \emptyset$.

BSE Hierarchy Construction

Thm: $\mathrm{DynED}(G, \phi, A, D)$ maintains $C \supseteq D$
- A is ϕ-expander in $G - C$
- $|C| - |D| \leq \phi |A| \log n$

- Init:
  - $C_0 \leftarrow E$, $C_i \leftarrow \emptyset$ for $i \geq 1$
  - $\phi = \frac{1}{16 \log n}$ and $\ell = \frac{\log m}{\log(1/4 \log n)}$

Correct assuming $C_{\ell+1} = \emptyset$.
$C_{i+1} \supseteq C_{i+2}$ for all i. (They are nested).
$C_i = E_{\geq i}$.
C_i is expanding in $G - C_{i+1}$
$\Rightarrow E_{\geq i}$ is expanding in $G - E_{>i}$.

- For $0 \leq i \leq \ell$, maintain *until there is no update*

$$C_{i+1} \leftarrow \mathrm{DynED}(G, \phi, A_i \coloneqq \deg_{C_i}, D_i \coloneqq C_{i+2})$$

- Return $E_i = C_i - C_{i+1}$ for all $i \leq \ell$.

How did we use $C_{\ell+1} = \emptyset$?
Otherwise, E_ℓ is not expanding in G

Conclude

Let G be a graph.

Theorem: ∃ a $(\frac{1}{16 \text{ lo}})$-BSE hierarchy of G with $\log m$ levels.

Corr: ∃ a tree cut sparsifier of G with quality $O(\log^2 n)$

*Do not get tree flow sparsifier. The argument only bounds cut expansion in the BSE hierarchy

Remark

- Previous $\tilde{\Omega}(1)$-BSE hierarchy only work with edge-expansion in undirected graphs. [R'02,BKR'03,HHR'02'RS'14,RST'14]
- Our construction generalizes to other expansions.
  - Combinatorial max flow [BBST'24]: directed expansion
  - Fault-tolerant distance oracle [HLRS'26]: length-constrained expansion
- **Open:** Explore power of BSE-hierarchy for other expansion notions

Summary

Summary

- Tree flow sparsifiers and applications
- BSE hierarchy → Tree flow sparsifiers
- Constructions of BSE hierarchies
  1. Based on Boundary-linked ED: Contract and Recurse
     - Implemented in dynamic/distributed models
     - Quality $n^{o(1)}$
  2. Based on Dynamic ED
     - Generalized to directed/length-constrained expansion
     - Quality $\log^2 n$