Epistemic EFX Allocations Exist for Monotone Valuations

AAAI-25

Nidhi Rathi

max planck institut informatik

ADFOCS - 2025

Joint work with **Hannaneh Akrami**

UNI BONN

Problem Definition

Given:

- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$

Problem Definition

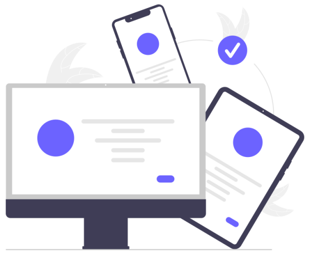
Given:

- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$

Problem Definition

Given:

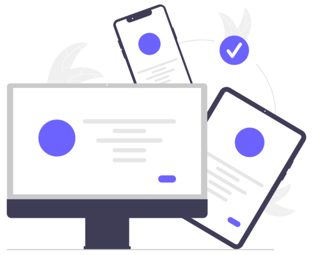
- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$



Problem Definition

Given:

- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$

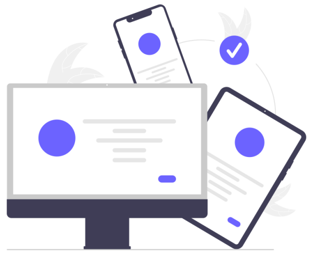


$$v_i(S \cup \{g\}) \geq v_i(S) \text{ for all } S \subset \mathcal{M} \text{ and } g \in \mathcal{M} \setminus S$$

Problem Definition

Given:

- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$



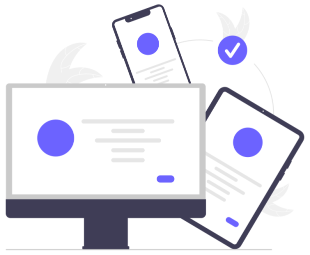
$$v_i(S \cup \{g\}) \geq v_i(S) \text{ for all } S \subset \mathcal{M} \text{ and } g \in \mathcal{M} \setminus S$$

Goal: Find a **fair** allocation of the goods to the agents.

Problem Definition

Given:

- $\mathcal{N}$: set of n agents
- $\mathcal{M}$: set of m indivisible goods
- Monotone valuation functions $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$



$v_i(S \cup \{g\}) \geq v_i(S)$ for all $S \subset \mathcal{M}$ and $g \in \mathcal{M} \setminus S$

Goal: Find a **fair** allocation of the goods to the agents.

A partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ of $\mathcal{M}$

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Envy-freeness up to any good (EFX): [Caragiannis, Kurokawa, Moulin, Procaccia, Shah, Wang 2016]

- $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all agents i, j and all goods $g \in X_j$.

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Envy-freeness up to any good (EFX):

- $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all agents i, j and all goods $g \in X_j$.

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Envy-freeness up to any good (EFX):

- $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all agents i, j and all goods $g \in X_j$.

Strong Envy: i strongly envies j, if there exists a good $g \in X_j$ such that $v_i(X_i) < v_i(X_j \setminus \{g\})$.

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Envy-freeness up to any good (EFX):

- $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all agents i, j and all goods $g \in X_j$.

Strong Envy: i strongly envies j, if there exists a good $g \in X_j$ such that $v_i(X_i) < v_i(X_j \setminus \{g\})$.

Do EFX allocations always exist?

Fairness

Envy-freeness: [Foley 1967]

- $v_i(X_i) \geq v_i(X_j)$ for all agents i, j.
- Does not always exist.

Envy-freeness up to any good (EFX):

- $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all agents i, j and all goods $g \in X_j$.

Strong Envy: i strongly envies j, if there exists a good $g \in X_j$ such that $v_i(X_i) < v_i(X_j \setminus \{g\})$.

Do EFX allocations always exist? 

EFX

Special cases:

- $n = 2$ [Plaut, Roughgarden 2018]
- $n = 3$ [Chaudhury, Garg, Mehlhorn 2020], [Berger, Cohen, Feldman, Fiat 2021], [Akrami, Alon, Chaudhury, Garg, Mehlhorn, Mehta 2023]
- Identical valuations [Plaut, Roughgarden 2018]
- Binary valuations [Barman, Krishnamurthy, Vaish 2018]
- Bi-valued valuations [Amanatidis, Birmpas, Filos-Ratsika, Hollender, Voudouris 2021]

Relaxations of EFX:

- EFX with charity [Caragiannis, Gravin, Huang 2019], [Chaudhury, Kavitha, Mehlhorn, Sgouritsa 2020], ...
- α-EFX [Plaut, Roughgarden 2018], [Amanatidis,Markakis,Ntokos 2020] ...
- Epistemic EFX [Caragiannis, Garg, **Rathi**, Sharma, Varricchio 2023]

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
	100	100	100	1	1	1	1
	1	1	1	100	100	100	1
	1	51	51	1	1	1	100

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
	100	100	100	1	1	1	1
	1	1	1	100	100	100	1
	1	51	51	1	1	1	100

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
	100	100	100	1	1	1	1
	1	1	1	100	100	100	1
	1	51	51	1	1	1	100

EFX ✓

EEFX ✓

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
	100	100	100	1	1	1	1
	1	1	1	100	100	100	1
	1	51	51	1	1	1	100

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
	100	100	100	1	1	1	1
	1	1	1	100	100	100	1
	1	51	51	1	1	1	100

EFX ✗ $v_3(X_1 \setminus \{g_1\}) > v_3(X_3)$

Epistemic EFX

A bundle $B \subseteq \mathcal{M}$ is **EEFX-feasible** for agent i, if there exists a partition of $\mathcal{M} \setminus B$ into $n-1$ bundles $A_1, \ldots, A_{n-1}$, such that i would not strongly envy any A_j upon receiving B.

Epistemic EFX (EEFX):

- X_i is EEFX-feasible for i, for all agents i.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
Agent 1	100	100	100	1	1	1	1
Agent 2	1	1	1	100	100	100	1
Agent 3	1	51	51	1	1	1	100

EFX ✗ $v_3(X_1 \setminus \{g_1\}) > v_3(X_3)$

EEFX ✓

Epistemic EFX

- EEFX allocations exist and can be computed in polynomial time for additive valuations. [Caragiannis, Garg, **Rathi**, Sharma, Varricchio 2023]
- Do EEFX allocations exist for more general valuations?

The concept of Epistemic EF was introduced by Aziz et al. [2018]
- proved that Epistemic EF may not always exist

Epistemic EFX

- EEFX allocations exist and can be computed in polynomial time for additive valuations. [Caragiannis, Garg, **Rathi**, Sharma, Varricchio 2023]
- Do EEFX allocations exist for more general valuations?

Theorem 1 [This work]

Epistemic EFX allocations exist for monotone valuations.

The concept of Epistemic EF was introduced by Aziz et al. [2018]
- proved that Epistemic EF may not always exist

Our Technique

- Consider any *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$ for an agent i.
  - $\mathcal{P}$: EEFX-feasibility of B for i

Our Technique

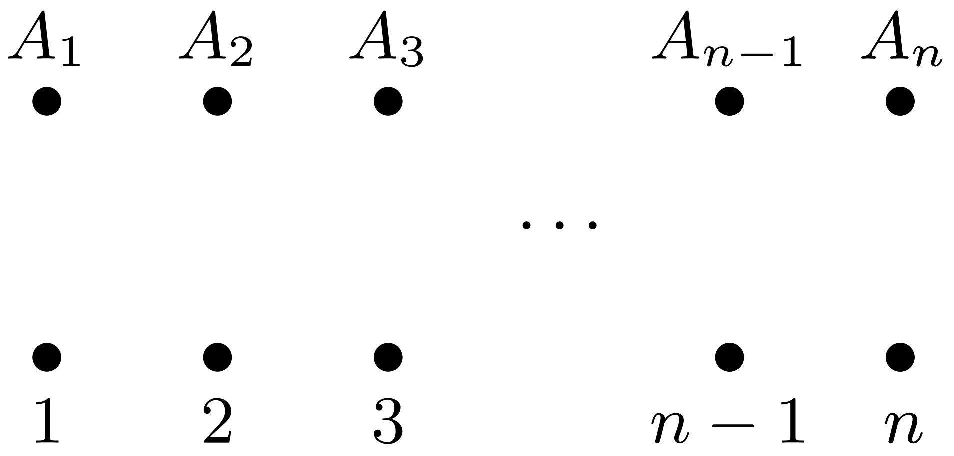
- Consider any *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$ for an agent i.
  - $\mathcal{P}$: EEFX-feasibility of B for i

- For any partition of $\mathcal{M}$ into n bundles $A_1, A_2, \ldots, A_n$:
  - bipartite graph $(A, \mathcal{N})$,
  - edge between i and A_j, iff i finds A_j desirable.

Our Technique

- Consider any *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$ for an agent i.
  - $\mathcal{P}$: EEFX-feasibility of B for i

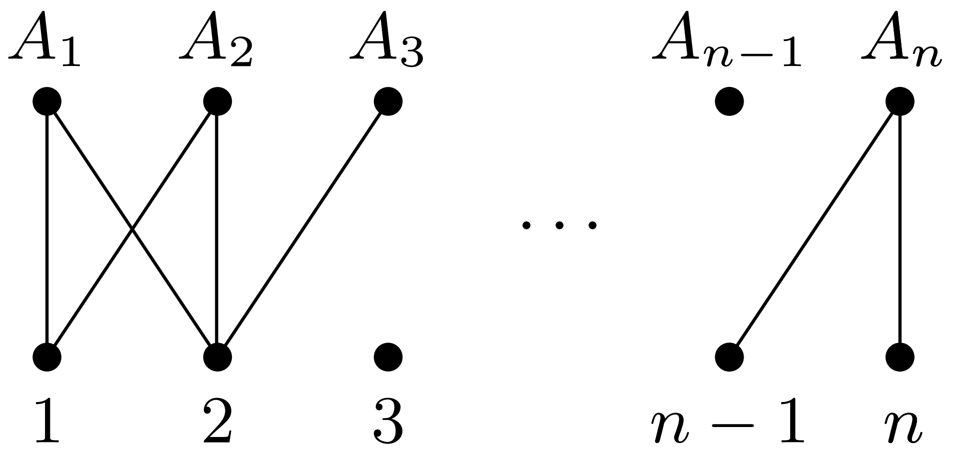
- For any partition of $\mathcal{M}$ into n bundles $A_1, A_2, \ldots, A_n$:
  - bipartite graph $(A, \mathcal{N})$,
  - edge between i and A_j, iff i finds A_j desirable.



Our Technique

- Consider any *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$ for an agent i.
  - $\mathcal{P}$: EEFX-feasibility of B for i

- For any partition of $\mathcal{M}$ into n bundles $A_1, A_2, \ldots, A_n$:
  - bipartite graph $(A, \mathcal{N})$,
  - edge between i and A_j, iff i finds A_j desirable.

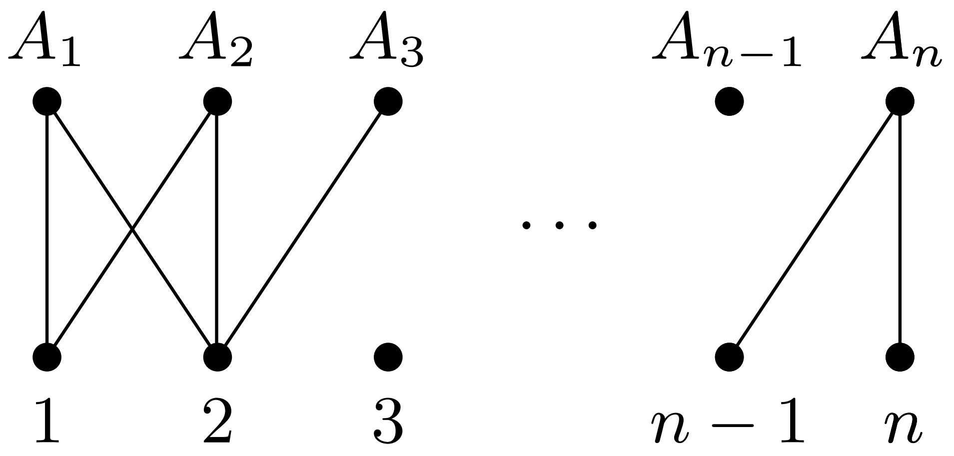


Our Technique

- Consider any *desirable property* $\mathcal{P}$ of a bundle $B \subseteq \mathcal{M}$ for an agent i.
  - $\mathcal{P}$: EEFX-feasibility of B for i

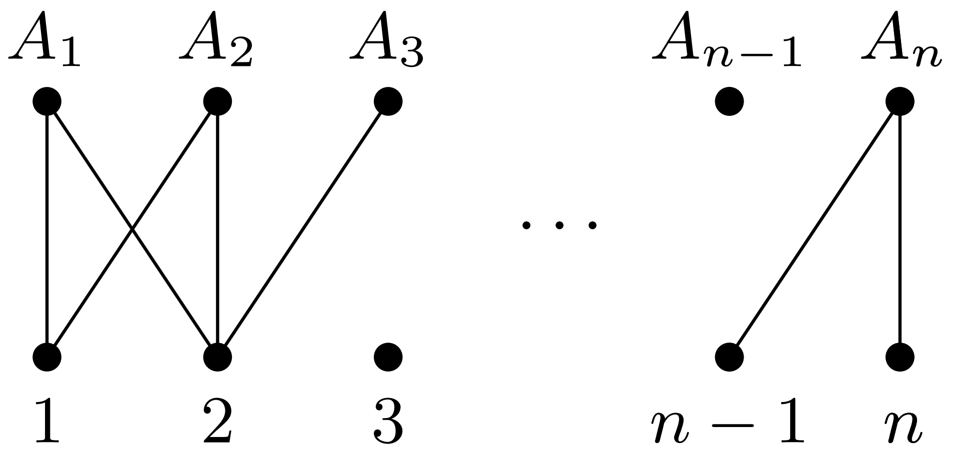
- For any partition of $\mathcal{M}$ into n bundles $A_1, A_2, \ldots, A_n$:
  - bipartite graph $(A, \mathcal{N})$,
  - edge between i and A_j, iff i finds A_j desirable.

- If a perfect matching exists, a desirable allocation exists.



Our Technique

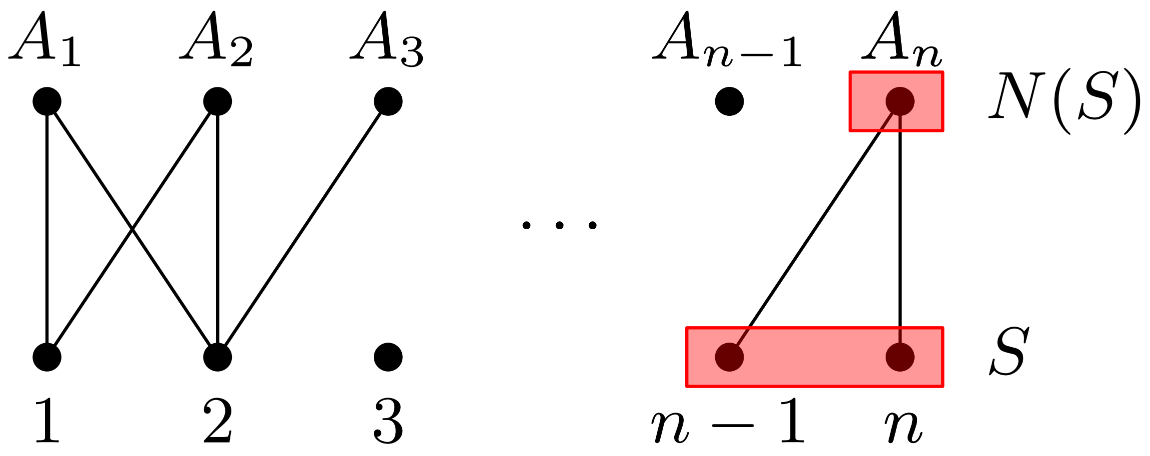
- If a perfect matching does not exists, there is a Hall's violater set.



Our Technique

- If a perfect matching does not exists, there is a Hall's violater set.

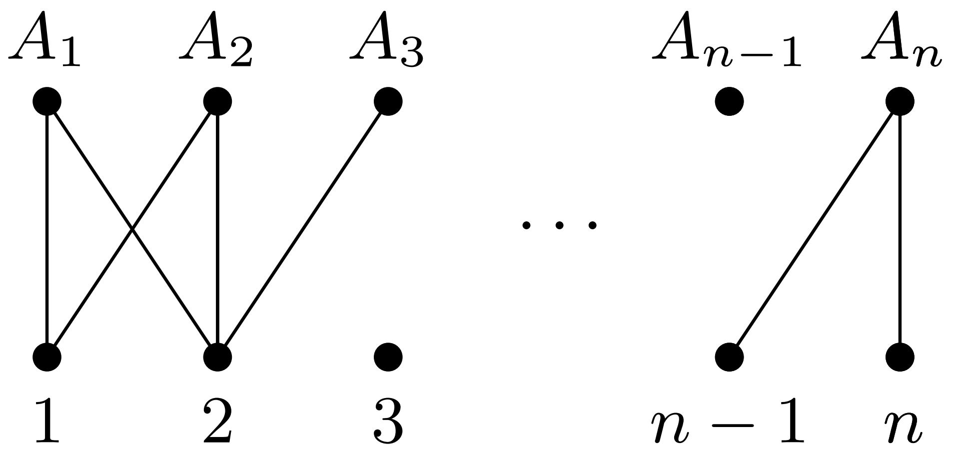
There exists a subset of agents S, s.t. $|N(S)| < |S|$.



Our Technique

- If a perfect matching does not exists, there is a Hall's violater set.

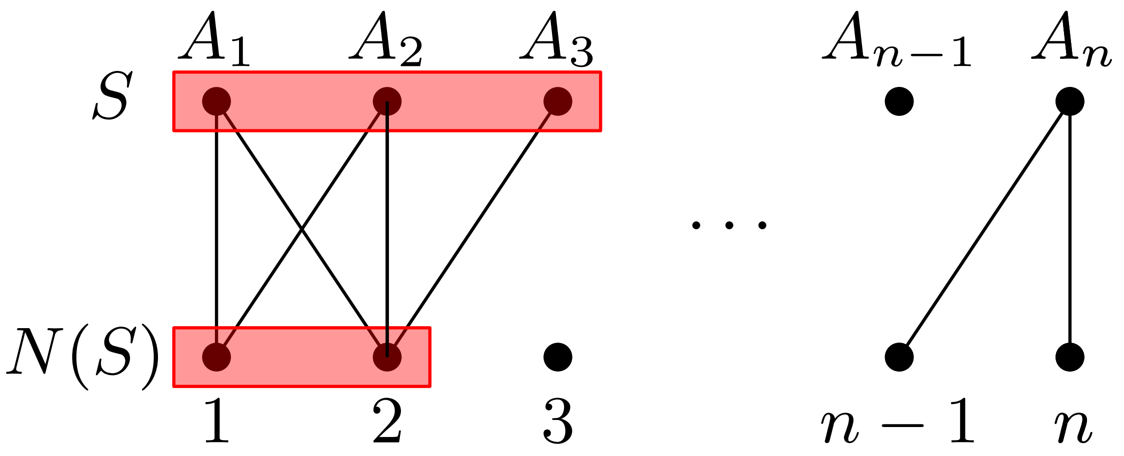
There exists a subset of ~~agents~~ bundles S, s.t. $|N(S)| < |S|$.



Our Technique

- If a perfect matching does not exists, there is a Hall's violater set.

There exists a subset of ~~agents~~ bundles S, s.t. $|N(S)| < |S|$.

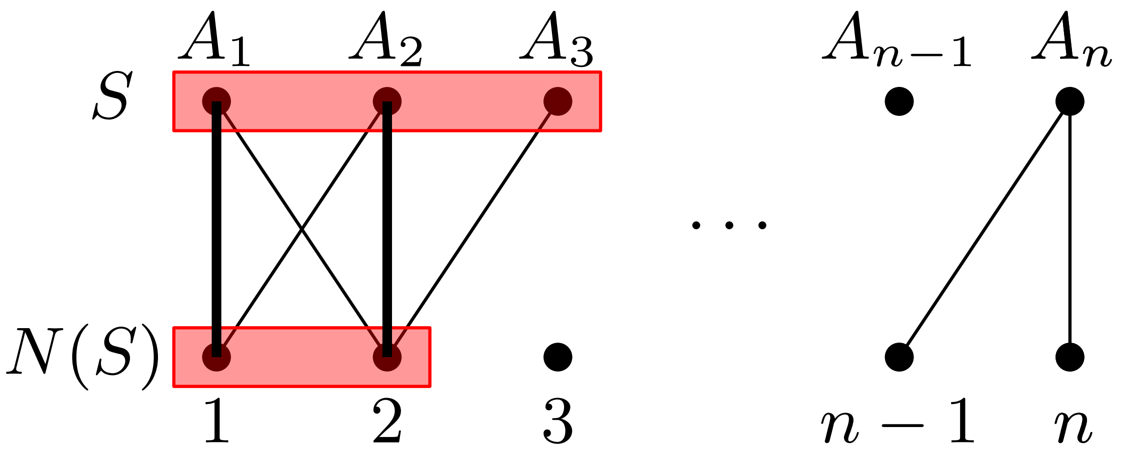


Our Technique

- If a perfect matching does not exists, there is a Hall's violater set.

There exists a subset of ~~agents~~ bundles S, s.t. $|N(S)| < |S|$.

If S is minimal, then there exists a perfect matching covering $N(S)$.



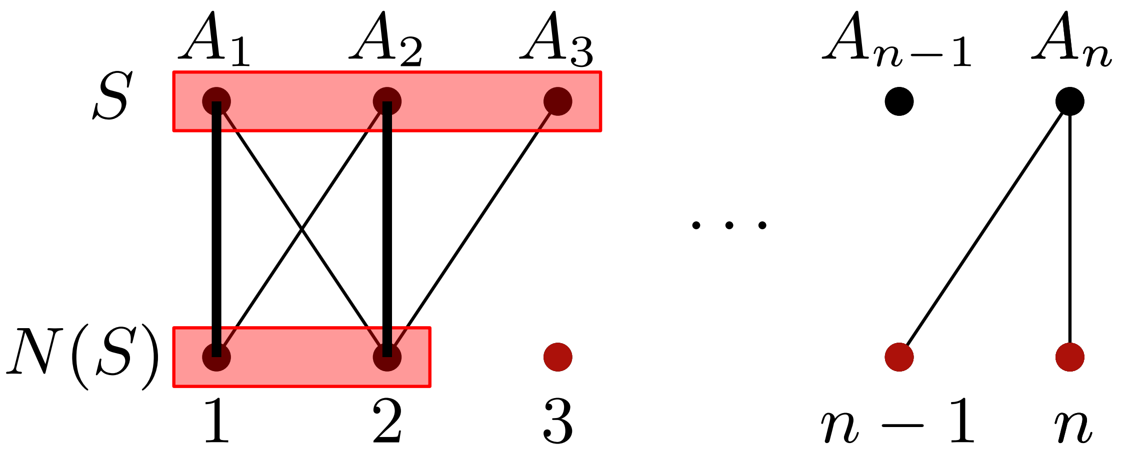
Our Technique

- If a perfect matching does not exists, there is a Hall's violater set.

There exists a subset of ~~agents~~ bundles S, s.t. $|N(S)| < |S|$.

If S is minimal, then there exists a perfect matching covering $N(S)$.

All the matched bundles are undesirable to the unmatched agents.



Our Technique

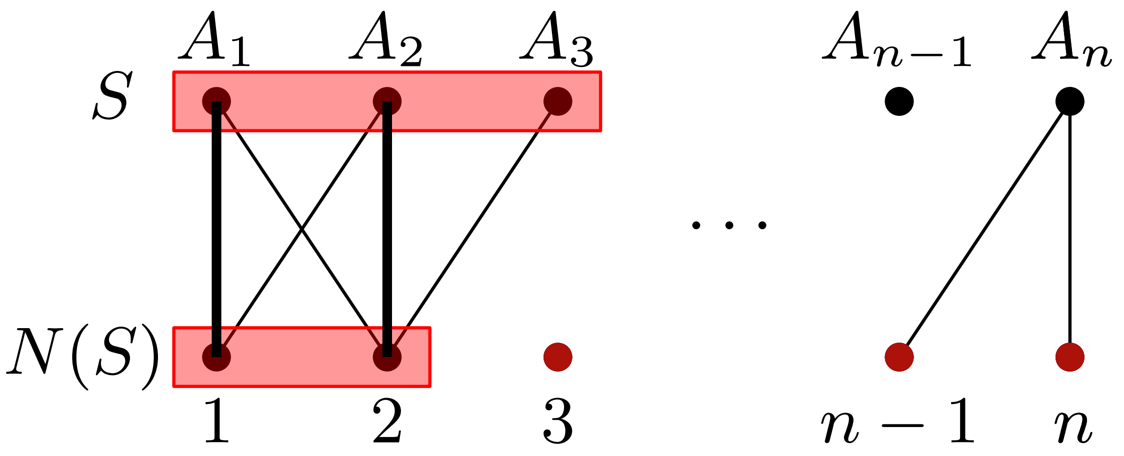
- If a perfect matching does not exists, there is a Hall's violater set.

There exists a subset of ~~agents~~ bundles S, s.t. $|N(S)| < |S|$.

If S is minimal, then there exists a perfect matching covering $N(S)$.

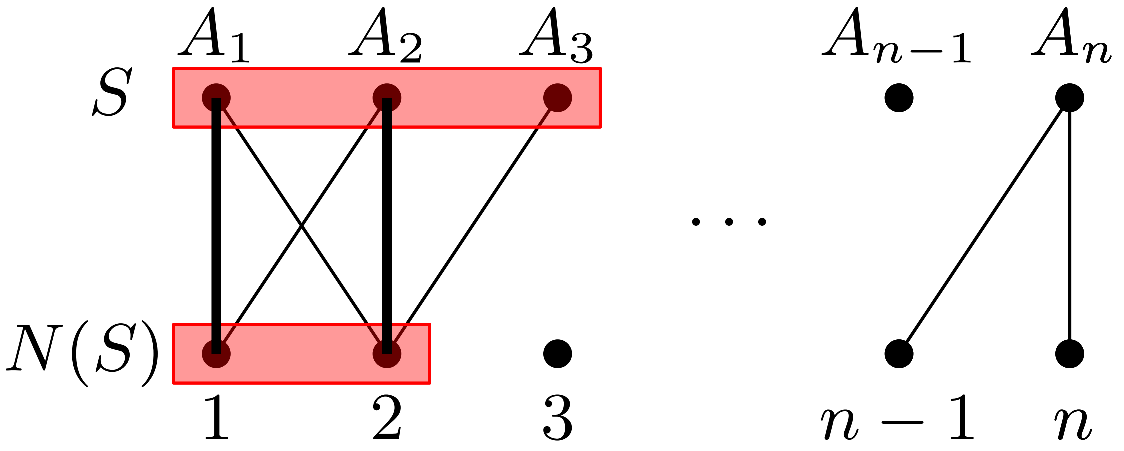
All the matched bundles are undesirable to the unmatched agents.

Idea: Allocate the matching and solve the problem for a smaller instance.



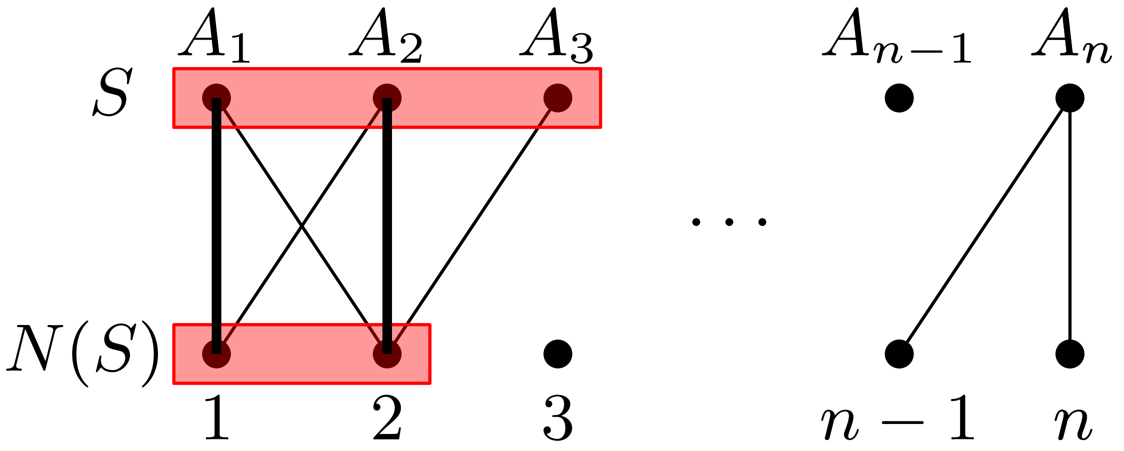
Our Technique

Idea: Allocate the matching and solve the problem for a smaller instance.



Our Technique

Idea: Allocate the matching and solve the problem for a smaller instance.

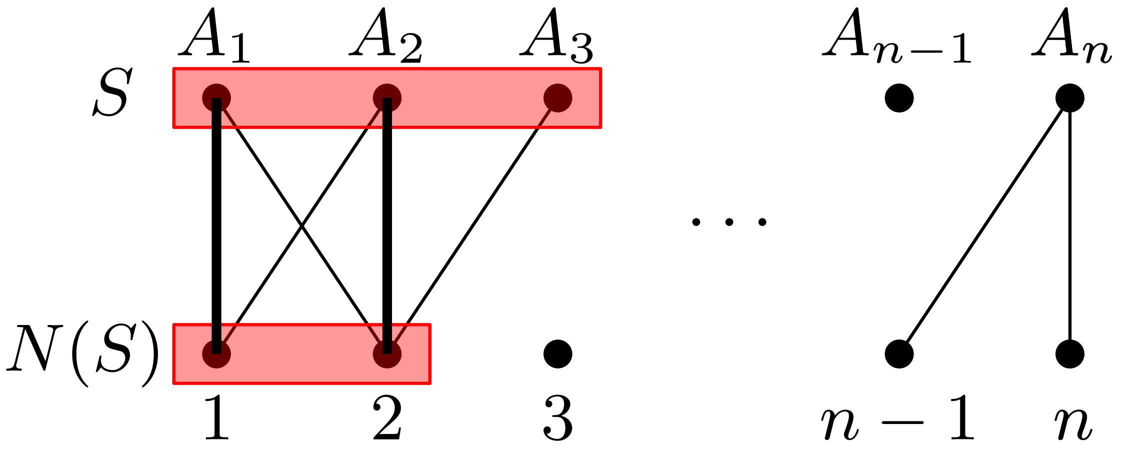


We need to ensure two conditions:

- $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Our Technique

Idea: Allocate the matching and solve the problem for a smaller instance.



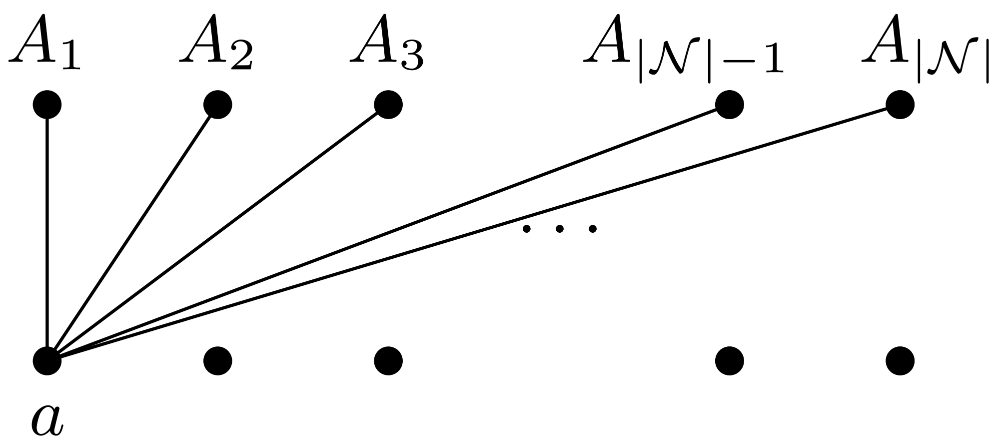
We need to ensure two conditions:

- $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

We ensure them for $\mathcal{P}$ being EEFX-feasibility.

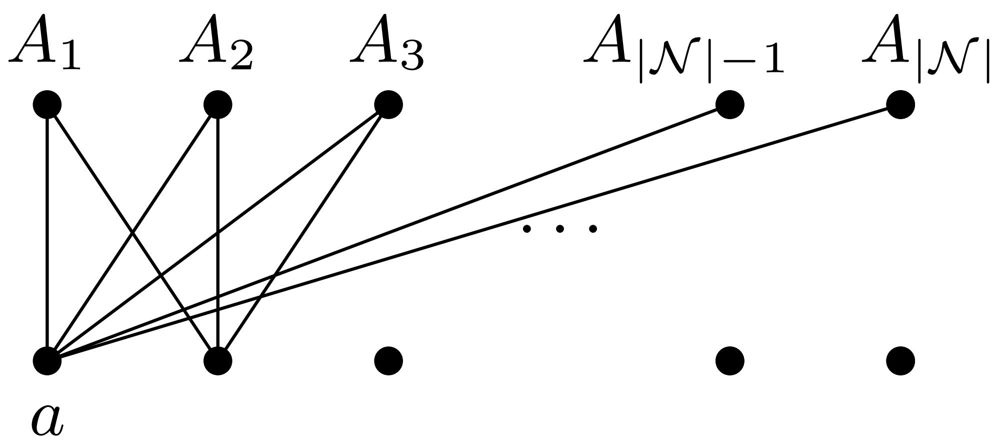
Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M



Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M



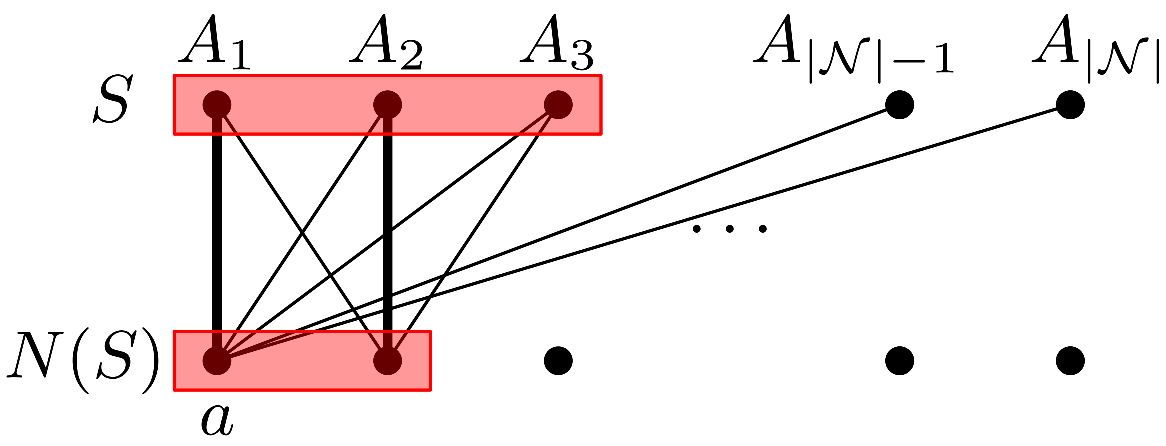
Our Algorithm

While $\mathcal{N} \neq \emptyset$:

Pick $a \in \mathcal{N}$

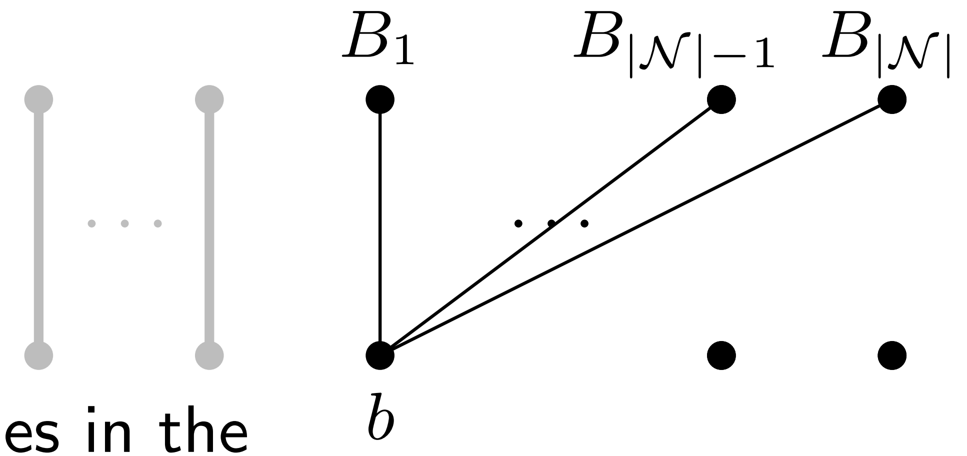
Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]

Find and remove a "good" matching M



Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M

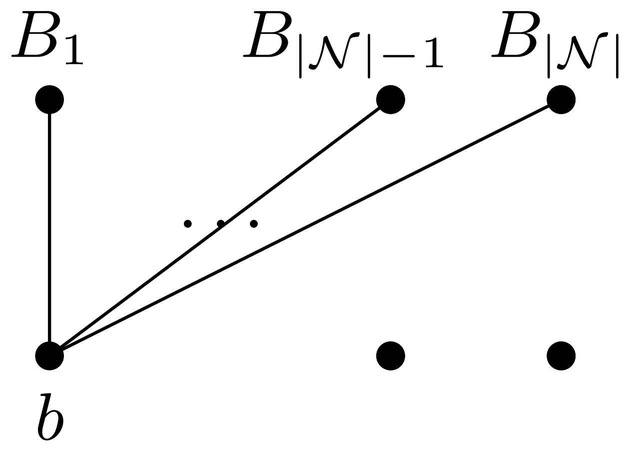


We need to ensure two conditions:

- $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

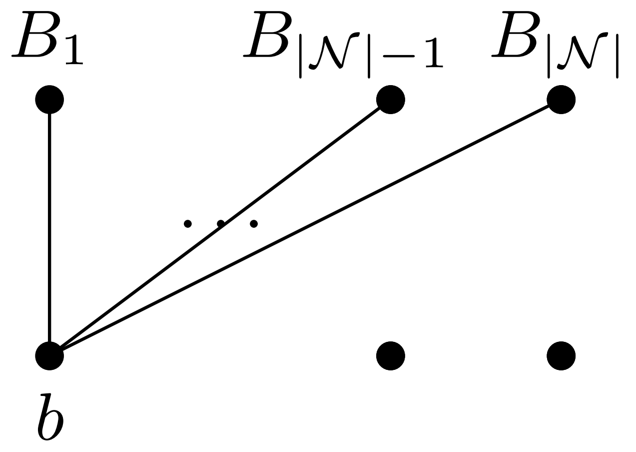
Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M



Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M

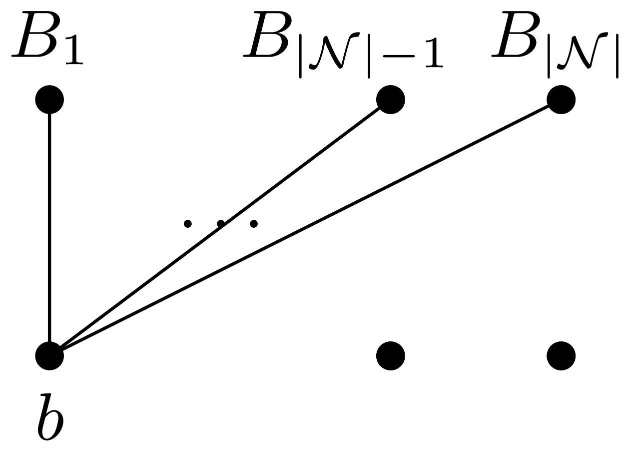


We need to ensure two conditions:

- $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M

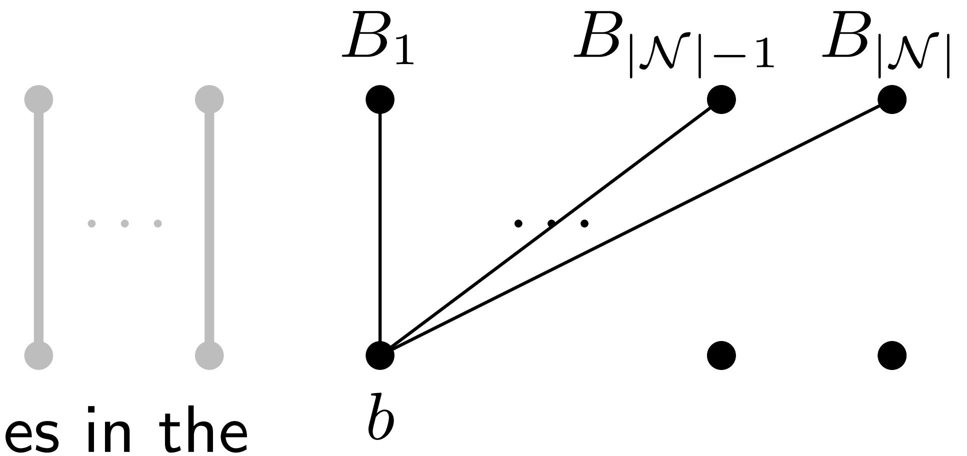


We need to ensure two conditions:

- ✓ $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M

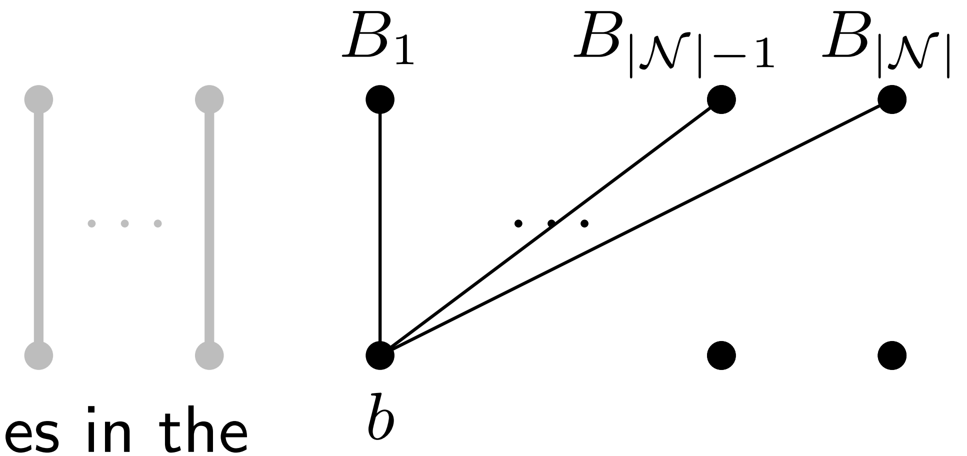


We need to ensure two conditions:

- ✓ $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Our Algorithm

While $\mathcal{N} \neq \emptyset$:
  Pick $a \in \mathcal{N}$
  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]
  Find and remove a "good" matching M



We need to ensure two conditions:

- $N(S) \neq \emptyset$.
- After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Lemma: If A is not EEFX-feasible for i and B is EEFX-feasible for i in an instance with $n-1$ agents after removing A, B is EEFX- feasible in the original instance.

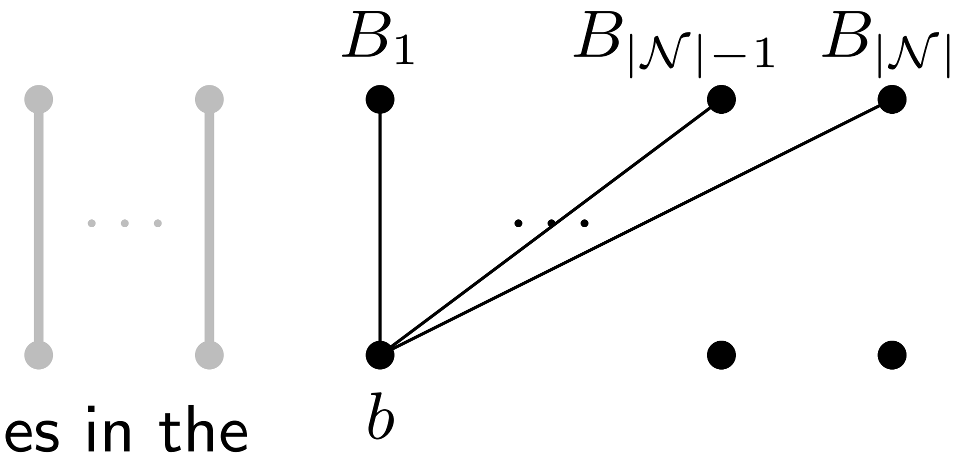
Our Algorithm

While $\mathcal{N} \neq \emptyset$:

  Pick $a \in \mathcal{N}$

  Let $(A_1, \ldots, A_{|\mathcal{N}|})$ be EFX when $|\mathcal{N}|$ agents have valuations v_a [Plaut, Roughgarden 2018]

  Find and remove a "good" matching M



We need to ensure two conditions:

- ✓ $N(S) \neq \emptyset$.
- ✓ After removing the matching, desirable bundles in the smaller instance are desirable in the original instance.

Lemma: If A is not EEFX-feasible for i and B is EEFX-feasible for i in an instance with $n-1$ agents after removing A, B is EEFX- feasible in the original instance.

Our Results

Theorem 1

Epistemic EFX allocations exist for monotone valuations.

Our Results

Theorem 1

Epistemic EFX allocations exist for monotone valuations.

Theorem 2

Exponentially many queries are required to find an EEFX allocation, even for identical submodular valuations.

Reduction from [Plaut, Roughgarden 2020]

Our Results

Theorem 1

Epistemic EFX allocations exist for monotone valuations.

Theorem 2

Exponentially many queries are required to find an EEFX allocation, even for identical submodular valuations.

Reduction from [Plaut, Roughgarden 2020]

Theorem 3

The problem of finding an EEFX allocation is PLS-hard even for identical submodular valuations.

Reduction from [Goldberg, Høgh, Hollender]

Future Direction

- PTAS for EEFX?
- EEFX + EF1 allocations?
- EEFX + other fairness/efficiency notions?

Future Direction

- PTAS for EEFX?
- EEFX + EF1 allocations?
- EEFX + other fairness/efficiency notions?