Fine-grained Complexity of Ambiguity Testing of Automata

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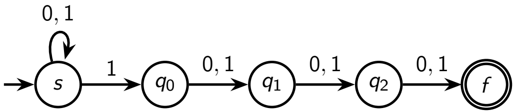
Finite Automata

Finite Automaton



$$\mathcal{A} = (Q, \delta, \Sigma, \{s\}, \{f\})$$

$$\mathcal{L}(\mathcal{A}) = (0+1)^*1(0+1)^3$$

- A word $w \in \Sigma^*$ is accepted by $\mathcal{A}$ if it labels an accepting run

DFA

✓ efficient for complementation, minimization, acceptance, containment, equivalence ...

NFA (this talk: no ε-transitions)

✗ containment and equivalence are PSPACE-complete

✓ exponentially more succint

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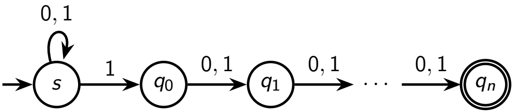

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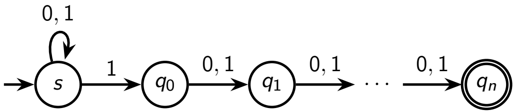

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DFA

✓ efficient for complementation, minimization, acceptance, containment, equivalence ...

- words label a unique accepting run

NFA (this talk: no ε-transitions)

✗ containment and equivalence are PSPACE-complete

✓ exponentially more succint

- words can label multiple accepting runs

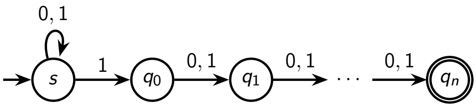
Ambiguity of Finite Automata

Definition

An NFA $\mathcal{A}$ is **unambiguous** if every word has $\leqslant 1$ accepting run. $\rightarrow$ UFA

UFA

- ✓ UFA is more succint than DFA
- ✓ containment and equivalence are still polytime in UFA



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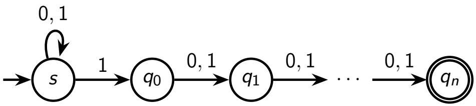


Given an NFA $\mathcal{A}$, is it unambiguous?

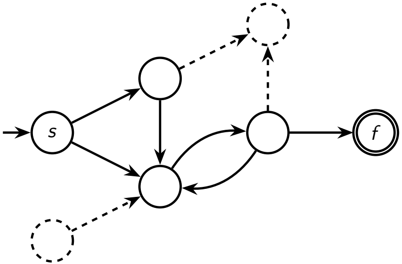
- Is decidable in n^2 time. [1991, 2001]
- Tight under OV or k-cycle. [our work]

$n = |Q| + |\delta| =$ size of $\mathcal{A}$

Reminder

Assumptions

- single starting state s and single accepting state f
- $\mathcal{A}$ is *trim*, i.e. every state is reachable from s and co-reachable from f



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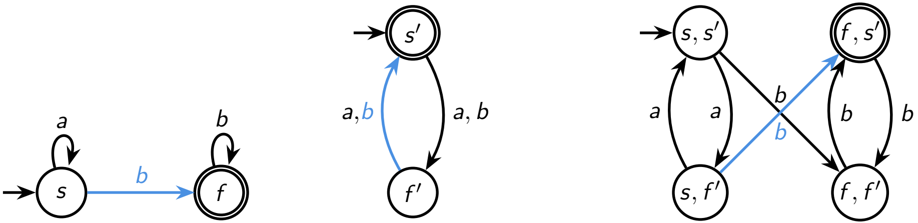
Intersection of NFAs

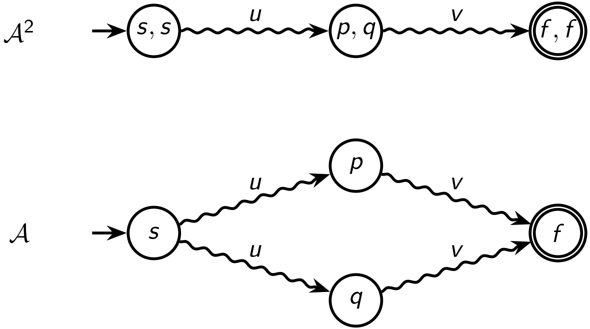


$\mathcal{L}(\mathcal{A}_1) = a^*b^*$ $\qquad$ $\mathcal{L}(\mathcal{A}_2) = (\Sigma\Sigma)^*$ $\qquad$ $\mathcal{L}(\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2)$

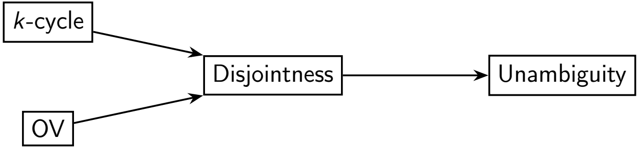
Unambiguity Test

Lemma

$\mathcal{A}$ is unambiguous iff every state in trimmed $\mathcal{A}^2 = \mathcal{A} \cap \mathcal{A}$ is of the form (q, q) where $q \in Q$.



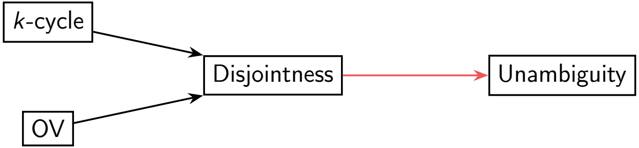
Lower-bounds



Problem (Disjointness)

Given two DFAs $\mathcal{D}_1$ and $\mathcal{D}_2$, decide whether $\mathcal{L}(\mathcal{D}_1 \cap \mathcal{D}_2) = \varnothing$.

Lower-bounds



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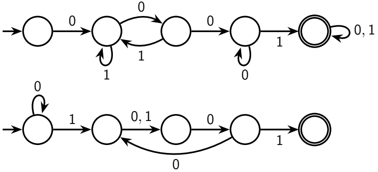
Disjointness $\to$ Unambiguity

- Given $\mathcal{D}_1$ and $\mathcal{D}_2$, let $\mathcal{A} \coloneqq \mathcal{D}_1 \cup \mathcal{D}_2$.
- $\mathcal{A}$ is unambiguous $\Leftrightarrow \mathcal{L}(\mathcal{D}_1 \cap \mathcal{D}_2) = \varnothing$



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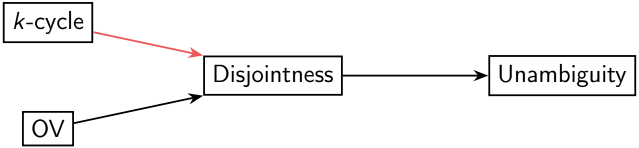


Problem (Disjointness)

Given two DFAs $\mathcal{D}_1$ and $\mathcal{D}_2$, decide whether $\mathcal{L}(\mathcal{D}_1 \cap \mathcal{D}_2) = \varnothing$.

***k*-cycle**

Given a k-partite graph $G = (V_1 \uplus \cdots \uplus V_k, E)$, decide if it contains a cycle of length k.
Hypothesis: $\forall \varepsilon > 0, \exists k \in \mathbb{N}$ not in time $\mathcal{O}(|E|^{2-\varepsilon})$

k-Cycle $\to$ Disjointness

Construction Given k-partite graph $G = (V_1 \uplus \cdots \uplus V_k, E)$, need to detect k-cycle.

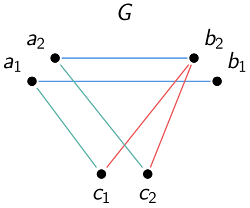
- $\mathcal{D}_1$ recognize paths $V_1 \to V_2 \cdots \to V_k \to V_1$,
- $\mathcal{D}_2$ recognize word of length $k+1$ starting and ending with the same vertex $v \in V_1$.

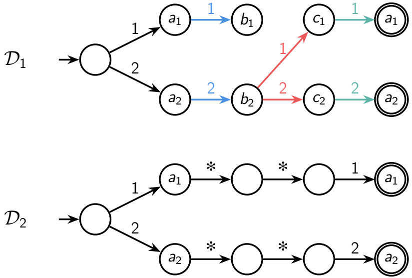
$\to$ Then any word $w \in \mathcal{L}(\mathcal{D}_1) \cap \mathcal{L}(\mathcal{D}_2)$ encodes a k-cycle in G.

Example

$k = 3$

$n = 2$





Size $\mathcal{O}(n+m)$ states and transitions

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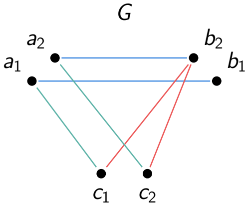

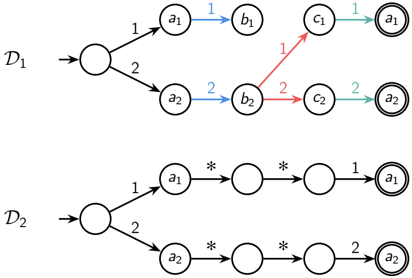

Issue

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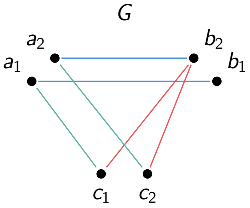
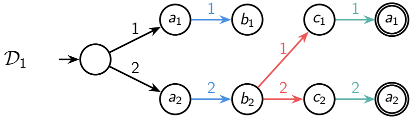
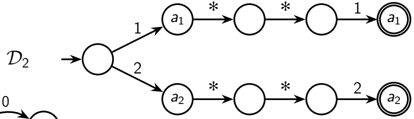
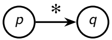


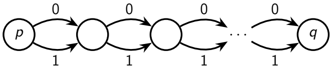




Fix



$\equiv$



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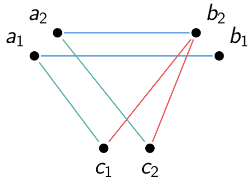
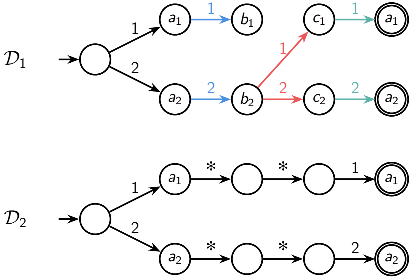
G





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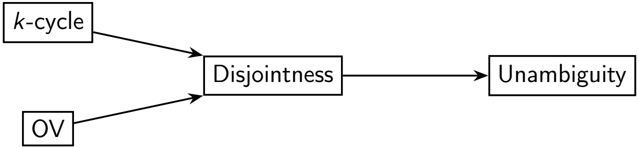

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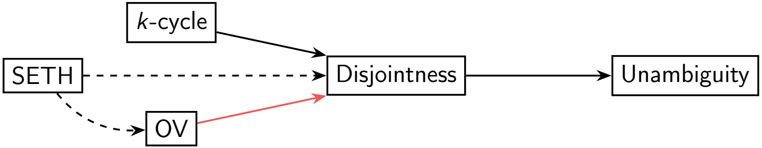

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Orthogonal Vector

Given sets $A, B \subseteq \{0,1\}^d$ of size n decide if

$$\exists a \in A, \exists b \in B : \ a \perp b.$$

Hypothesis: $\forall \varepsilon > 0, \exists c > 0$ not in time $\mathcal{O}(n^{2-\varepsilon})$ for $d = c \log n$

OV $\to$ Disjointness

Construction Given $A, B \subseteq \{0,1\}^d$ of size n, want to check whether

$$\exists a \in A, \exists b \in B : \ \forall \ell \in [d], a[\ell] = 0 \text{ or } b[\ell] = 0$$

Write $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$.
Let $\mathcal{D}_1$ and $\mathcal{D}_2$ recognize words $c_A c_B r_1 \ldots r_d$ such that:

- $c_A, c_B \in [n]$,
- $r_1, \ldots, r_d \in \{A, B\}$, and

$$\begin{array}{lll} (\text{for } \mathcal{D}_1) & \forall \ell \in [d] & : (r_\ell = A) \Rightarrow (a_{c_A}[\ell] = 0) \\ (\text{for } \mathcal{D}_2) & \forall \ell \in [d] & : (r_\ell = B) \Rightarrow (b_{c_B}[\ell] = 0) \end{array}$$

$\to$ Then any word $w \in \mathcal{L}(\mathcal{D}_1) \cap \mathcal{L}(\mathcal{D}_2)$ encodes a pair of orthogonal vectors in $A \times B$.

OV $\rightarrow$ Disjointness

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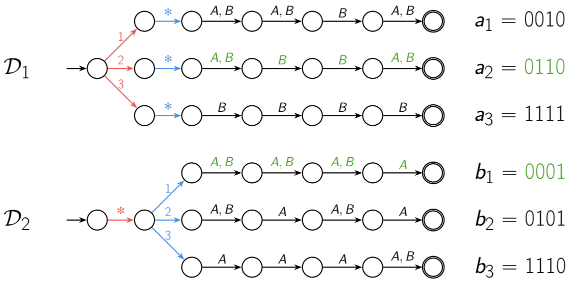
Example

$n = 3$

$d = 4$



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Recap

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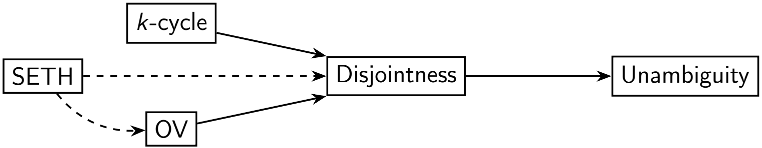

Thank you!