Lecture 4: Testing convex position

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Convex position

**Definition (completeness):** A pointset $P \subseteq \mathbb{R}^d$ is in **convex position** if every point is a vertex of the convex hull (i.e. extreme point)

- In addition, we assume the points are in **general position**
  - No $d + 1$ points on the same hyperplane.

**Definition (soundness):** A pointset $P \subseteq \mathbb{R}^d$ is $\epsilon$-far from convex position if $\forall Q \subseteq P$, $|Q| \leq \epsilon n$, $P \setminus Q$ is not in convex position.
Caratheodory’s theorem

**Theorem**

Let $P$ be a set of $n$ points in $\mathbb{R}^d$ and $p \in CH(P)$. Then, $p$ can be written as a convex combination of at most $d + 1$ points of $P$.

**Corollary:** There exist $d + 2$ points of $P$ not in convex position.
Lemma

Let $P$ be a set of $n$ points in $\mathbb{R}^d$, $p \in \text{CH}_{\text{int}}(P)$ and $P \cup \{p\}$ in general position. There exist $W \subseteq P$, $U \subseteq P \setminus W$ such that $|W| = d$ and $|U| \geq \frac{n}{d+1}$ and $p \in \text{CH}_{\text{int}}(W \cup \{q\})$ for each $q \in U$.

Proof:
Useful lemmas

Lemma

Let $P$ be a set of $n$ points in $\mathbb{R}^d$, which are $\epsilon$-far from being in convex position and let $k = \frac{\epsilon n}{d+1}$. There exist $W_i \subseteq P$, $W_i \subseteq P$, $i \in [k]$ such that:

1. $|W_i| = d + 1$
2. For $i \neq j$: $W_i \cap W_j = \emptyset$
3. For $q \in U_i$: $W_i \cup \{q\}$ not in convex position.
4. $|U_i| \geq (1 - \epsilon) \frac{n}{d+1}$

Proof:
Algorithm and analysis

**Algorithm**

\[ \text{CONVEX}\text{TESTER} (P, \epsilon) \]

\[
\text{let } s = 16 \left( 4^{d+1} \sqrt{n^d/\epsilon} + 2d + 2 \right) 
\]

Choose a set \( S \subseteq P \) of size \( s \) uniformly at random

**Analysis:**

- If \( S \) is in convex position then **accept**
- Else **reject**
Algorithm and analysis

Lemma

Let $\Omega$ be a set of $n$ elements and $W_1, \ldots, W_k$ disjoint size $\ell$ subsets of $\Omega$. Also let $W$ be a set of $s \geq \frac{2n}{(2k)^{1/\ell}}$ elements chosen u.a.r from $\Omega$. Then,

$$\Pr[\exists j \in [k] : W_j \subseteq W] \geq \frac{1}{4}$$

Claim: The query complexity of the algorithm is $O(d^{d+1}\sqrt{n^d/\epsilon})$.

Proof:
Canonical testers:

Lemma

Let $A$ be a property tester for convex position of point sets with query complexity $q(\epsilon, n)$. Then, there is a property tester $A'$ for convex position that samples a set $S$ of size $q(\epsilon, n)$ u.a.r and accepts iff $S$ is in convex position.
Lower bound-construction

High level:
Lower bound-construction

More details:
Lower bound-construction

Constructing facets:
Lower bound-construction

Adding more points: