Lecture 4:Testing convex position

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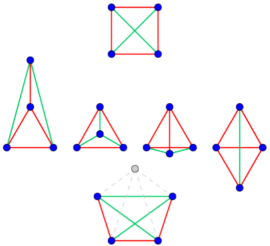
Convex position

Definition (completeness): A pointset $P \subseteq \mathbb{R}^d$ is in convex position if every point is a vertex of the convex hull (i.e extreme point)

- In addition, we assume the points are in general position
  - No $d + 1$ points on the same hyperplane.

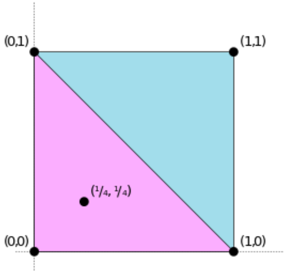


Definition (soundness): A pointset $P \subseteq \mathbb{R}^d$ is ϵ-far from convex position if $\forall Q \subseteq P, |Q| \leq \epsilon n, P \setminus Q$ is not in convex position.

Caratheodory's theorem

Theorem

Let P be a set of n points in $\mathbb{R}^d$ and $p \in CH(P)$. Then, p can be written as a convex combination of at most $d + 1$ points of P.



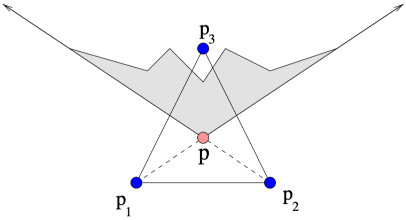
Corollary: There exist $d + 2$ points of P not in convex position.

Useful lemmas

Lemma

Let P be a set of n points in $\mathbb{R}^d$, $p \in CH_{int}(P)$ and $P \cup \{p\}$ in general position. There exist $W \subseteq P$, $U \subseteq P \setminus W$ such that $|W| = d$ and $|U| \geq \frac{n}{d+1}$ and $p \in CH_{int}(W \cup \{q\})$ for each $q \in U$.

Proof:



Useful lemmas

Lemma

Let P be a set of n points in $\mathbb{R}^d$, which are ϵ-far from being in convex position and let $k = \frac{\epsilon n}{d+1}$. There exist $W_i \subseteq P, W_i \subseteq P, i \in [k]$ such that:

1. $|W_i| = d + 1$
2. *For $i \neq j : W_i \cap W_j = \emptyset$*
3. *For $q \in U_i : W_i \cup \{q\}$ not in convex position.*
4. $|U_i| \geq (1 - \epsilon)\frac{n}{d+1}$

Proof:

Algorithm and analysis

ConvexTester (P, ϵ)

let $s = 16\left(4\sqrt[d+1]{n^d/\epsilon} + 2d + 2\right)$

Choose a set $S \subseteq P$ of size s uniformly at random

if S is in convex position **then** *accept*

else *reject*

Analysis:

Algorithm and analysis

Lemma

Let Ω be a set of n elements and $W_1, \dots W_k$ disjoint size ℓ subsets of Ω. Also let W be a set of $s \geq \frac{2n}{(2k)^{1/\ell}}$ elements chosen u.a.r from Ω. Then,

$$\Pr[\exists j \in [k] : W_j \subseteq W] \geq \frac{1}{4}$$

Claim: The query complexity of the algorithm is $O(\sqrt[d+1]{n^d/\epsilon})$.
Proof:

Lower bound

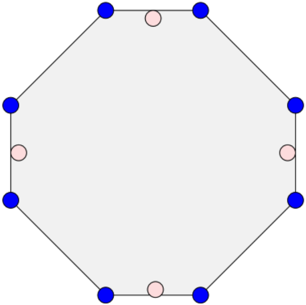
Canonical testers:

Lemma

Let A be a property tester for convex position of point sets with query complexity $q(\epsilon, n)$. Then, there is a property tester A' for convex position that samples a set S of size $q(\epsilon, n)$ u.a.r and accepts iff S is in convex position.

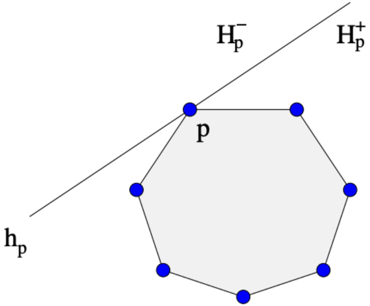
Lower bound-construction

High level:



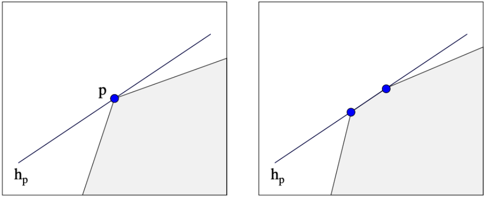
Lower bound-construction

More details:



Lower bound-construction

Constructing facets:



Lower bound-construction

Adding more points: