1 Maxflow Mincut Theorem by Dual Analysis

Consider the following LP for flows and its cut dual.

$$
\begin{aligned}
\text{(FLOW-LP)} \quad \max \quad & \sum_{uv \in \delta^+(s)} x(uv) - \sum_{uv \in \delta^-(s)} x(uv) \\
\text{s.t.} \quad & x(uv) \leq c(uv) \text{ for all } uv \in E \\
& \sum_{uv \in E} x(uv) = \sum_{vw \in E} x(vw) \text{ for all } v \in V \setminus \{s, t\} \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{(CUT-LP)} \quad \min \quad & \sum_{uv \in E} c(uv) y(uv) \\
\text{s.t.} \quad & z(v) - z(u) \leq y(uv) \text{ for all } uv \in E \\
& z(s) = -1 \\
& z(t) = 0 \\
& y(uv) \geq 0 \text{ for all } uv \in E \\
& y \in [0,1]^{|E|}, z \in \mathbb{R}^{|V|}
\end{aligned}
$$

Let x^* be the optimal solution for the (FLOW-LP) and (y^*, z^*) be the optimal solution for the (CUT-LP). Let us recall the complimentary slackness conditions:

- If $y^*(uv) > 0$, then $x^*(uv) = c(uv)$.
- If $x^*(uv) > 0$, then $z^*(v) - z^*(u) = y^*(uv)$.

We will directly construct a cut U whose cost achieves the optimal of the primal LP, therefore establishing the max-flow/min-cut theorem, as well as the integrality of (CUT-LP).

Define the cut $U = \{v : z^*(v) < 0\}$. The cost of this cut is equal to:

$$c(U) = \sum_{uv \in \delta^+(U)} c(uv)$$

Observe that $y^*(uv) > 0$ for all edges $uv \in \delta^+(U)$: Otherwise, we must have $z^*(v) \leq z^*(u) < 1$, contradicting to the choice of $v \notin U$. This implies that $x^*(uv) = c(uv)$ for all such edges uv leaving set U. Therefore, $c(U) = \sum_{uv \in \delta^+(U)} c(uv) = \sum_{uv \in \delta^+(U)} x^*(uv)$.

Now, the second observation is that $x^*(uv) = 0$ for all $uv \in \delta^-(U)$: Otherwise, we must have $z^*(v) = z^*(u) + y^*(uv) \geq z^*(u)$, a contradiction. Combining this with the previous equation, we have:

$$c(U) = \sum_{uv \in \delta^+(U)} x^*(uv) - \sum_{uv \in \delta^-(U)} x^*(uv)$$

Using the fact that flow across every cut is the same, we have $c(U) = \sum_{uv \in \delta^+(s)} x^*(uv) - \sum_{uv \in \delta^-(s)} x^*(uv)$, as desired.