Optimization	**Summer 2016**

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1 Linear Programming: Algorithms for Solving

1.1 Refresh

1. objective function: $\mathbf{c}^T\mathbf{x}$ maximise/minimise
2. constraints: $\mathbf{Ax} \leq \mathbf{b}$
3. variables: $\mathbf{x} \leq, \geq 0$
4. feasible region: $\mathbf{P} := \{\mathbf{x}|\mathbf{Ax} \geq \mathbf{b}\}$
5. *Def* polytope: A polytope is an n dimensional region with flat sides. The shape of our feasible region.

1.2 Theorem

All polyhedrons are closed sets

Let $P = \{\mathbf{x}|\mathbf{Ax} \geq \mathbf{b}\}$

Let $x^{|1|}, x^{|2|}, \ldots$, be a convergent sequence in P

$\lim_{k\to\infty} x^{(k)} = x^{(*)}, \ldots , x^{(1)} \in P \ \forall\, k$

$x^{(*)} \in \{P|\mathbf{Ax}^{(*)} \leq \mathbf{b}\}$

Let i be a component.

$$\begin{aligned}(\mathbf{Ax}^{(*)})_i &= \mathbf{A}\lim_{k\to\infty} x^{(k)} \\ &= \lim_{k\to\infty} (\mathbf{Ax^{(k)}})_i\end{aligned}$$

We know for every $k\,|\mathbf{Ax^{(k)}} \in P$ (A function mapping) and every k forms a sequence. Then from the above equation, every single element will be at least, b_i. Hence, it is closed from $(b_i \to \infty)$

1.3 Projection and Fourier-Motzkin

We requier a function that maps n-points to our n-1 points. We define such function $\pi_k \mathbb{R}^n \to \mathbb{R}^k$

$\pi_k(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_k)$

for a set S

$S \subseteq \mathbb{R}^n \quad \pi_k(S) = \{\pi_k(x) \mid x \in S\}$

$\Rightarrow \{(x_1, x_2, \dots, x_k) \mid \exists\, x_k \dots x_n \mid x_k, x_n \in S\}$

Given a polyhedron $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ NOTE: if $P \neq \emptyset \Rightarrow \pi_{n-1}(P) \neq \emptyset$ If P is non-empty, then the projection of P is non-empty. The Fourier-Motzkin process goes as follows:

- Rewrite each constraint

$$\sum_{i=1}^{n} \alpha_{i,j} x_i \geq b_i$$
$$\alpha_{i,n} x_n \geq -\sum_{i=1}^{n-1} \alpha_{i,j} x_i + b_i$$

- if $\alpha_{i,n} \neq 0$ then divide by $\alpha_{i,n}$

$$x_n \geq d_i + e_i^T \mathbf{x} \quad \text{if } \alpha_{i,n} > 0 \quad, d_i = \frac{b_i}{\alpha_{i,n}} \quad, e_i = -\frac{\sum_{i=1}^{n-1} \alpha_{i,j}}{\alpha_{i,n}}$$

$$\mathbf{x} = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \quad , \quad d_i \in \mathbb{R}$$

if $\alpha_{i,n} < 0 \quad d_i + e_i^T \mathbf{x} \geq x_n$
if $\alpha_{i,n} = 0 \quad 0 \geq d_k + e_k^T \mathbf{x}$

Now, when rewriting constraints we only consider the projected points.

- Let Q be a polyhedron in $\mathbb{R}^{n-1}$ defined by. $0 \geq d_k + e_k^T \mathbf{x}$ for each $x_{i,n} = 0$

  Suppose there is a point in P. $\{\forall\, i, j \mid \alpha_{i,n} > 0\ ,\ \alpha_{j,n} < 0\}$

  $\Rightarrow d_j + e_j^T \bar{\mathbf{x}} \leq d_i + e_i^T \bar{\mathbf{x}}$

  Next: one round of Fourier - Motzkin to P ie;

  $Q = \pi_{n-1}(P)$ We will now prove that Q and $\pi_{n-1}(P)$ contain each other.

  First, we will prove the lower bound $\pi_{n-1}(P) \leq Q$.

  The inequalities denoted $iq(1), iq(2), iq(3)$ are when $\{\alpha > 0\ ,\ \alpha < 0\ ,\ \alpha = 0\}$ respectively.

$$Let\ \ \bar{\mathbf{x}} \in \pi_{n-1}(P) \Rightarrow \{\ \exists\, \mathbf{x}_n \mid (\bar{\mathbf{x}}_1, \mathbf{x}_n) \in P\}$$

In iq(3) there is no relation to $\mathbf{x}_n$ hence, satisfied.

For iq(2) and iq(1) it can be shown:

$$d_j + e_j^T \bar{\mathbf{x}} \leq d_i + e_i^T \bar{\mathbf{x}}$$
$$min_{\{j \mid \alpha_{j,n} < 0\}}\ d_j + e_j^T \bar{\mathbf{x}} \geq max_{\{i \mid \alpha_{i,n} > 0\}}\ d_i + e_i^T \bar{\mathbf{x}}$$

(Recall the variables d , e)

This is to say: $\implies [b, \alpha] \neq \emptyset$ then take an arbitrary point on this interval:

$$Let\ \ x_k \in [b, \alpha]$$
$$\implies (\bar{x}, x_n) \text{ satisfies } iq(1), iq(3)$$
$$\implies (\bar{x}, x_n) \in P$$
$$\implies \bar{x} \in \pi_{n-1}(P)$$

Initially, the polytope must be checked if it is non-empty. Apply this process:

$$\pi_{n-1}(P) \to \pi_{n-2}(P) \to ... \to \pi_1(P)$$

if $\pi_1(P) \neq \emptyset$ then, it is non-empty.

Complexity: Given M constraints it's obvious to see for first projection $O(M^2)$ 2nd, $O(M^4)$ 3rd, $O(M^6)$ ... n-th $O(M^{2n})$

The introduction of a objective function can result in the following scheme seen previously for solving a LP:

$$min \;\; c^T\mathbf{x}$$

$$\mathbf{Ax} \geq \mathbf{b}$$

introduce a dummy variable, x_0 and let the dummy variable be the initial $x_0 = c^T\mathbf{x}$

With this new polytope, first check if it is non-empty.

$P = \{(x_0, x_1, ... , x_n) \in \mathbb{R}^{n-1} \mid \mathbf{Ax} \geq \mathbf{b} \wedge x_0 = c^T\mathbf{x}\}$

Using Fourier-Metzkin elimination, we then obtain Q, where $Q = \{x_0\}$ initially, $Q = c^T\mathbf{x}$

$\implies Q = \{x_0 \mid \exists \, (x_1, x_2, ... , x_n) \mid \mathbf{Ax} \geq \mathbf{b} \wedge x_0 = c^T\mathbf{x}\}$

This polytope is useful as one dimensional and all points in Q are the values optimized.

$Q = \pi_1(P') \quad P'$ When $x_0 = c^T\mathbf{x}$ To find the result, back track from $\pi_1(P')$

The computation is not efficient but is nice in theory. Say we project from $\mathbb{R}^n \to \mathbb{R}^k$ we produce synthetic inequalities which are also polyhedrons. It is also the simplest method to prove there exists this relation.

Corollary

$$min \;\; c^T\mathbf{x}$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

Apply a transformation by introducing variables. For each x_i introduce: x_j^+, x_j^- and replace all instances: $x_i = x_j^+ - x_j^-$.
Introduce constraints: if $x_i \geq 0$: $x_j^+ \geq 0$ then $x_j^- \leq 0$. For each constraint $a_i^T\mathbf{x} \geq b_i$ introduce a new dummy variable that represents the residue from $a_i^T\mathbf{x} - b_i = s_i$ hence, we can say: $a_i^T\mathbf{x} - s_i = b_i$. We can add this residual as an extra constraint: ie; $s_i \geq 0$. This allows for a one sided, defined limit.