Optimization	**Summer 2016**

Lecture 4: May 4

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1 Definition: Tight

Let x and p be feasible primal and dual solutions, respectively.

- i-th constraint in primal is tight if $a_j^T x = b_i$
- j-th constraing in dual is tight if $p^T A_j = c_i$

x, p optimal

- If $p_i > 0 \Rightarrow i$-th constraint in primal is tight
- If $x_j > 0 \Rightarrow j$-th constraint in dual is tight

2 Theorem: Complementary slackness

Let x and p be feasible primal and dual solutions, respectively. Then x and p are both optimal if and only if

- $p_i(a_i^T x - b_i) = 0 \; \forall i$
- $(c_j - p^T A_j)x_j = O \; \forall j$

2.1 Proof

Assume x and p are optimal.
Define: (Same as in the proof of week duality)

- $u_i = p_i(a_i^T x - b_i)$
- $v_i = (c_j - p^T A_j)x_j$

- $u_i \geq 0 \; \forall i$
- $v_j \geq 0 \; \forall j$

$\sum_i u_i + \sum_j v_j =$ (proof of weak duality theorem) $c^T x - p^T b =$ (strong duality) 0
$\Rightarrow u_i = 0 \; \forall i, v_j = 0 \; \forall i$

Assume

- $u_i = 0 \; \forall i$
- $v_j = 0 \; \forall j$

$\Rightarrow c^T x = p^T b$
$\Rightarrow x, p$ optimal by weak duality.

3 Simplex Algorithm

Key observation: Many points in polytope (every point is candidate for being optimal). But we only have to look at corners.

1. start in arbitrary vertex
2. If there is an adjacent vertex with better objective value $\Rightarrow$ more there
3. Repeat until current vertex is optimal

4 Definition: Extreme Point

Let P be a polyhedron. A point $x \in P$ is an extreme point of P if we cannof find two vectors $y, z \in P$ with $y \neq x \neq z$ and a scalar $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z$.

5 Definition: Vertex

Let $P \subseteq R^n$ be a polyhedron. A vector $x \in P$ is a vertex of P if there exists some $c \in R^n$ such that $c^T x < c^T y$ for all y satisfying $y \in P$ and $y \neq x$.

- $a_i^T x >= b_i i \in M_1$
- $a_i^T x <= b_i \in M_2$
- $a_i^T x = b_i i \in M_3$

6 Definition: Tight

If vector x satisfies $a_i^T x = b_i$ for some constraing i we say that constraint i is tight, active or binding at x. 2-dimensional polytope $\Rightarrow$ 2 tight inequalities at each vertex

7 Theorem

Let $x* \in R^n$. Let $I = \{i \mid a_i^T x* = b_i\}$ be set of indices of active constraints at $x*$. Then the following are equivalent:

1. There are n vectors in $S = \{a_i \mid i \in I\}$ that are linearly independent
2. The linear system $a_i^T x = b_i, i \in I$, has a unique solution

8 Definition: Linearly independent

Constraints are linearly independent if their corresponding vectors a_i are linearly idependent

9 Definition: Basic solution

Let P be polygon. Vector $x* \in R^n$ is basic solution if there are n linearly independent constraings that are tight at $x*$. If additionaly $x* \in P$ it is a basic feasible solution.

Receipt for finding Attempt to construct vertex of $P \subseteq R^n$

- Pick n linearly independent constraints I
- Loss for a point s.t. they are all activ $T_i x = b_i \; \forall i \in I$
- Has unique solution $x* \rightarrow$ basic solution
- If additionaly $x* \in P \rightarrow$ basic feasible solution

10 Theorem

Extreme point, Vertex and Basic feasible solution are equivalent definitions

10.1 Proof

10.1.1 (i) → (ii)

Let $x* \in P$ be a vertex $\Rightarrow \exists x x^T x* < c^T y \; \forall y \; P \; x*$ Assume by contradiction $\exists y, z \in P \lambda O < \lambda < 1. x* = \lambda y + (1-\lambda) z$
$c^T x* = c^T(\lambda y + (1-\lambda) z) > c^T \lambda x* + c^T (1-\lambda) x* = c^T x* = c^T(\lambda x* + (1-\lambda) x*)$ Contradiction!

10.1.2 (ii) → (iii)

Let $x*$ be extreme point solution Assume that x* is not basic feasible solution $I = \{i | a_i^T x* = b_i\}$ $\Rightarrow$ There are no n linearly independent vectors in $\{a_i \mid i \in I\}$
$\Rightarrow \exists d \neq 0 a_i^T d = 0 \forall i \in I$
define $y = x* + \epsilon d, z = x* - \epsilon d$
$y, z \in P$?

- if $i \in I : a_i^T y = a_i^T(x* + \epsilon d) = a_i^T x* + a_i^T \epsilon d = a_i^T x* = b_i$
  $a_i^T >= b_i$
- if $i \notin I a_i^T x* > b_i \Rightarrow$ we can choose $\epsilon > 0$ s.t. $a_i^T y > b_i$ and $a_i^T z > b_i \exists \epsilon_i if \epsilon <= \epsilon_i$
  $\Rightarrow a_i^T x* + a_i^T \epsilon d > b_i$
  choose $\epsilon = min \epsilon_i$

$\Rightarrow y, z \in P \; \frac{1}{2}(y+z) = \frac{1}{2}(x* + \epsilon d) + \frac{1}{2}(x* - \epsilon d) = x*$
$\Rightarrow x*$ is not an extreme point

10.1.3 (iii) → (i)

Let $x*$ be basic feasible solution
$I = \{i | a_i^T x* = b_i\}$
$x = \sum_{i \in I} a_i$
$c^T x* = \sum_{i \in I} a_i^T x* = \sum_{i \in I} b_i$
Let $x \in P$, for every i we have $a_i^T x >= b_i$
$c^T x = \sum_{i \in I} a_i^T x >= \sum_{i \in I} b_i => x*$ is optimal solution for $min x^T x$ s.t. $Ax >= b$
$c^T x = \sum_{i \in I} b_i \Leftrightarrow a_i^T x = b_i foreachi \in I$
$\Rightarrow if c^T x = c^T x* => x = x*$
$\Rightarrow x*$ is unique optimum for objective vector c
$\Rightarrow x*$ is a vertex