# Complexity Theory of Polynomial-Time Problems

Lecture 3: The polynomial method
Part I: Orthogonal Vectors

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# Organization of lecture

- No lecture on 26.05. (State holiday)
- 2nd exercise sheet: Next week
- Tutorials:
  - New slot: Friday, 12:15 - 14:00, U12 E1.1, biweekly
  - Fr, 13.05. (tomorrow)
  - Fr, 03.06.
  - Etc.

# The polynomial method

- Recently developed technique in Algorithm Design
- Current fastest algorithms for
  - All-Pairs Shortest Paths [Williams 14]
  - Orthogonal Vectors [Abboud/Williams/Yu 15]
  - Hamming Nearest Neighbors [Alman/Williams 15]
- Two main tools
  1. Razborov-Smolensky from Circuit Complexity
  2. Fast rectangular matrix multiplication

# Reminder: Orthogonal Vectors Problem

*Input:* Two sets $A, B \subseteq \{0,1\}^d$ of $d$-dimensional 0/1-vectors of size $n$

*Output:* Is there a pair $a \in A, b \in B$ s.t. $a$ and $b$ are orthogonal?

$$\exists a \in A, b \in B: \quad \langle a, b \rangle = 0$$

$$\sum_{k=1}^{d} a[k] \cdot b[k] = 0$$

$$\forall 1 \leq k \leq d: (a[k] = 0) \vee (b[k] = 0)$$

<u>Trivial algorithms:</u>

- $O(n^2 d)$
- $O(2^d n)$

} Interesting Regime: $d = c \log n$

# Today's result

Reminder:

**Conjecture**: There is no algorithm for the orthogonal vectors problem with running time $O(n^{2-\epsilon}\text{poly}(d))$ for any $\epsilon > 0$.

State of the art:

**Theorem**: There is an algorithm for the orthogonal vectors problem with running time $n^{2-1/O(\log(d/\log n))}$.

**In this lecture:** $n^{2-1/O(\log d)}$

Algorithm is randomized and correct with high probability, i.e., probability $\geq 1 - 1/n$

# Overview

1. Reduce problem to many subproblems of very small size
2. Precompute small **circuits** for solving subproblems
3. Evaluate circuits with **probabilistic polynomials** of **low degree**
4. Evaluate polynomials using fast **rectangular** matrix multiplication

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# Dividing into smaller subproblems

1. Divide $A$ and $B$ into $q = \lceil \frac{n}{s} \rceil$ subsets of size $\leq s$:
   $A_1, \dots, A_q$ and $B_1, \dots, B_q$
2. Construct a polynomial $P(a_1[1], \dots a_1[d], \dots, a_s[1] \dots, a_s[d],$
   $b_1[1], \dots b_1[d], \dots, b_s[1] \dots, b_s[1])$
   $P(A_i, B_j) = 1$ if and only if $A_i$, $B_j$ contains orthogonal pair
   *…only with high probability*
3. For every pair of subsets $A_i$, $B_j$: evaluate $P$ on $A_i$, $B_j$
   *…simultaneously!* $\rightarrow O(\frac{n^2}{s^2} \operatorname{polylog}(n))$
4. Return "yes" if some $A_i$, $B_j$ contains orthogonal pair, "no" otherwise

We set $s = 2^{\epsilon \log n / \log d} = n^{\epsilon / \log d}$ for some sufficiently small $\epsilon$

# Questions

1. How to construct suitable polynomial $P$?
2. How to evaluate $P$ fast on all pairs of inputs?

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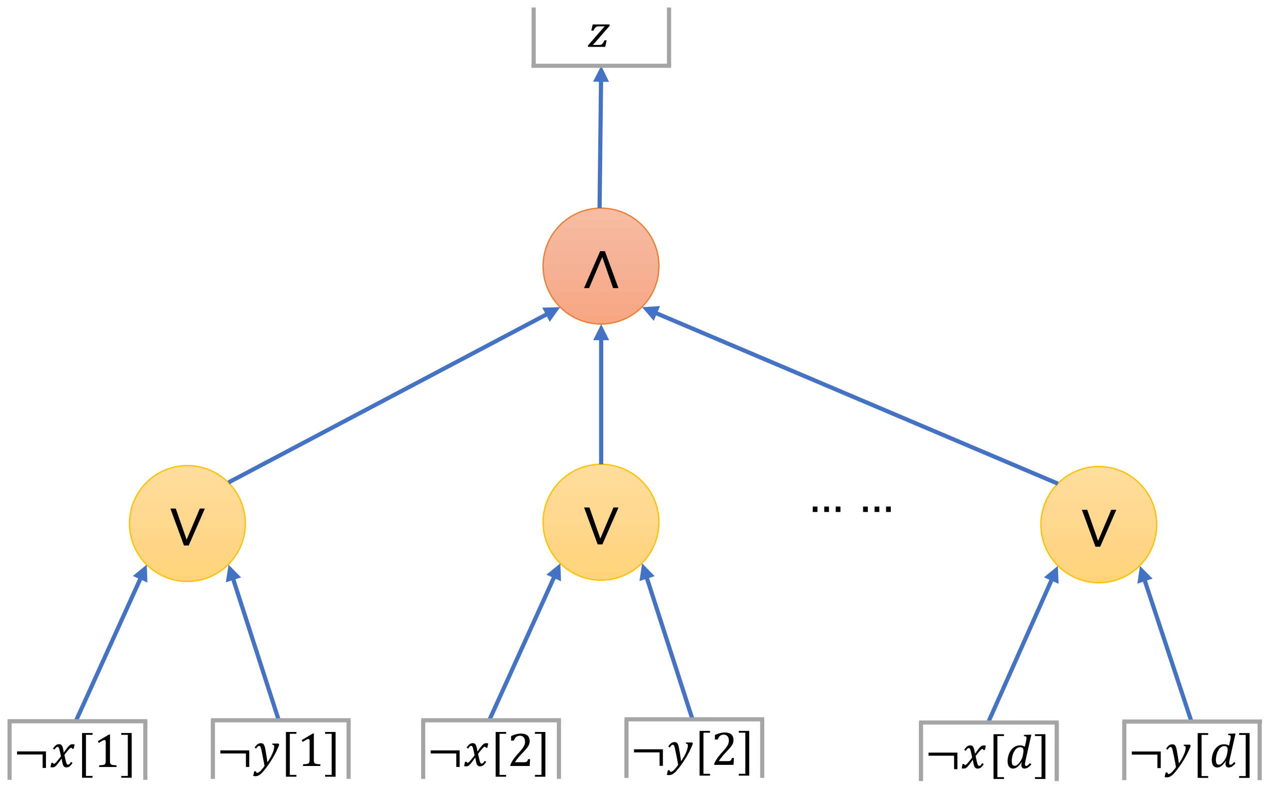
# Boolean circuits

Boolean circuit

- Directed acyclic graph
- Sources: input bits
- Sink: output bit
- Inner nodes: Boolean operations
- AND: $\wedge$
- OR: $\vee$
- Arbitrary “fan-in”

# Circuit for checking orthogonality of vectors

Are $x$ and $y$ orthogonal?



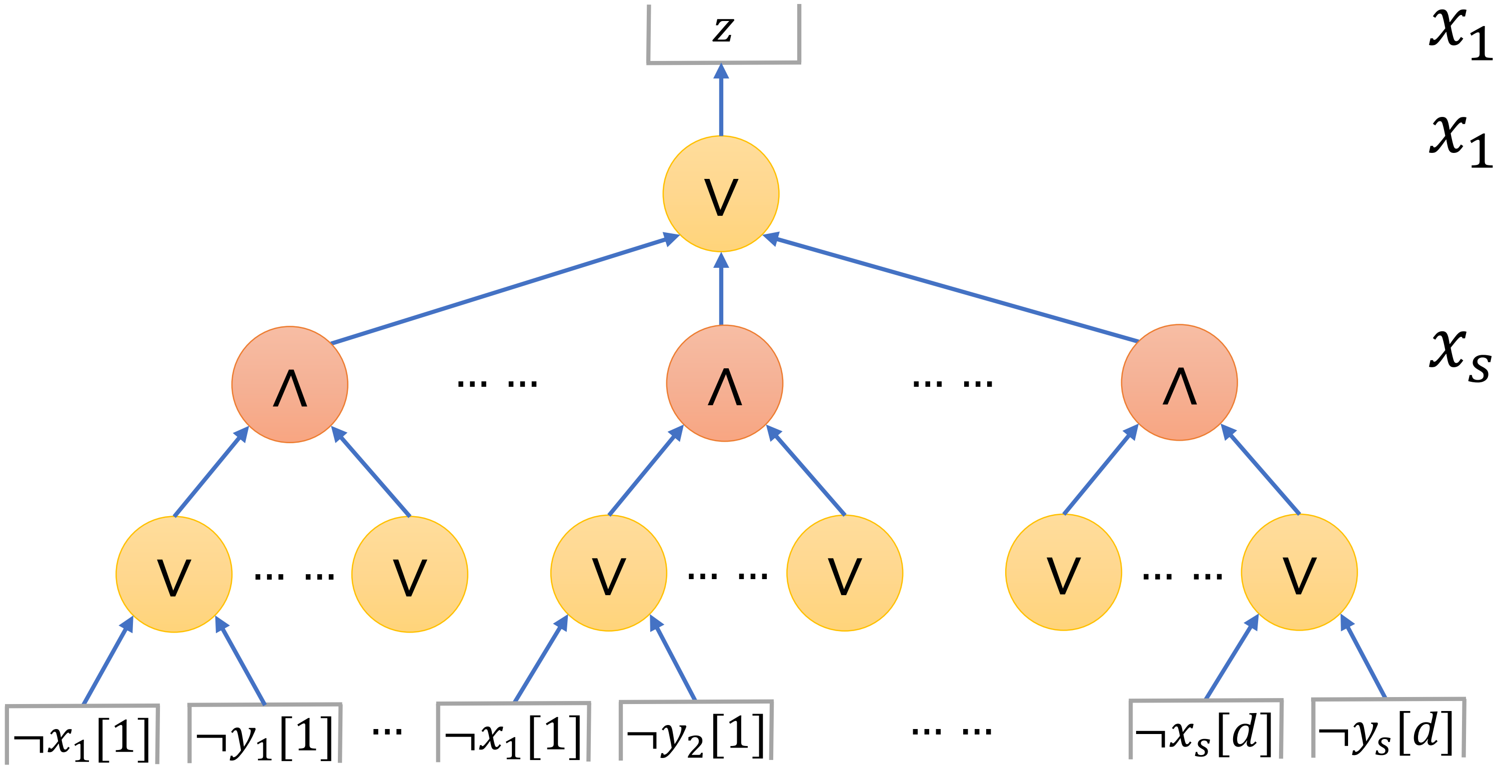

$x$ and $y$ orthogonal iff
$\neg \exists i: x[i] = 1 \wedge y[i] = 1$

Output bit $z = 1$ iff
$x$ and $y$ orthogonal

AND of ORs with
- $2d$ negated inputs
- 1 output

# Circuit for finding orthogonal pair

Is there an orthogonal pair?




Check orthogonality for **every pair**

$x_1, y_1$ orthogonal OR

$x_1, y_2$ orthogonal OR

… OR

$x_s, y_s$ orthogonal?

OR of ANDs of ORs

$2ds^2$ negated inputs

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# From circuits to polynomials

- Obtain polynomial over $F_2$ outputting 1 if and only if circuit outputs 1
- $F_2$: Field of $\{0, 1\}$ with operations $\oplus$ and $\cdot$
- $\oplus$ is XOR-operation:
  - $0 \oplus 0 = 0$ $1 \oplus 0 = 1$ $0 \oplus 1 = 1$ $1 \oplus 1 = 0$
  - XOR of multiple variables:
    $x_1 \oplus x_2 \oplus \cdots \oplus x_k = 1$ if and only if odd number of $x_i$'s is 1
- Expanded polynomials:
  - $a \cdot (b \oplus c) \cdot (a \oplus b \oplus d) = ac \oplus abc \oplus abd \oplus acd$
  - XOR of monomials
  - Goal: Few monomials allows fast evaluation later

# Representing circuit by polynomial

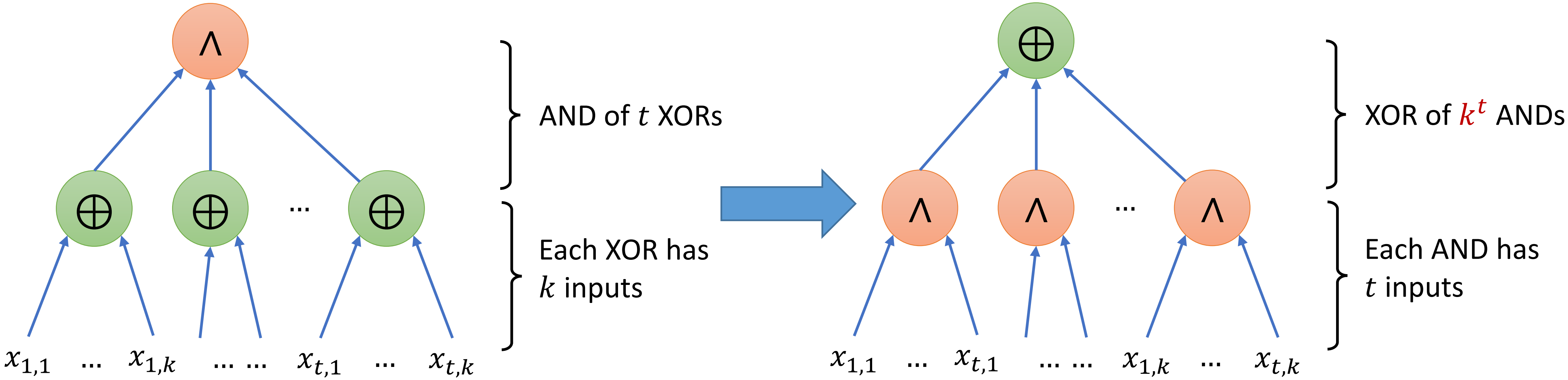
Straightforward approach:

- AND: $a \wedge b \Rightarrow a \cdot b$
- Negation: $\neg a \Rightarrow 1 \oplus a$ (addition=subtraction in $F_2$)
- OR: $a \vee b \Rightarrow \neg(\neg a \wedge \neg b)$ (DeMorgan's Law)

Example: Bottom-level of circuit

$$\neg a \vee \neg b \Rightarrow 1 \oplus a \cdot b$$

# Expanding a multiplication (distributive law)



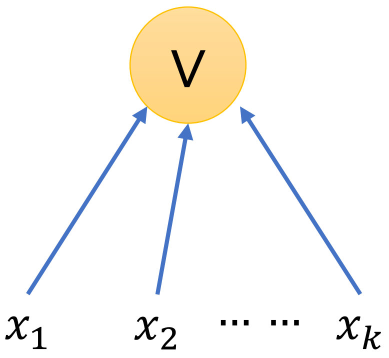

$$(x_{1,1} \oplus \cdots \oplus x_{1,k}) \cdot (x_{2,1} \oplus \cdots \oplus x_{2,k}) \cdot \ldots \cdot (x_{t,1} \oplus \cdots \oplus x_{t,k}) = x_{1,1} \cdot x_{2,1} \cdot \ldots \cdot x_{t,1} \oplus \cdots \oplus x_{1,k} \cdot x_{2,k} \cdot \ldots \cdot x_{t,k}$$

$k$ choices $k$ choices $k$ choices $k^t$ monomials

Running time: $O(k^t t \cdot \deg)$ where deg is degree after expansion (maximum size of any monomial); here $\deg \leq t$

# Razborov/Smolensky trick [Raz87] [Smo87]

**Naïve representation of OR**

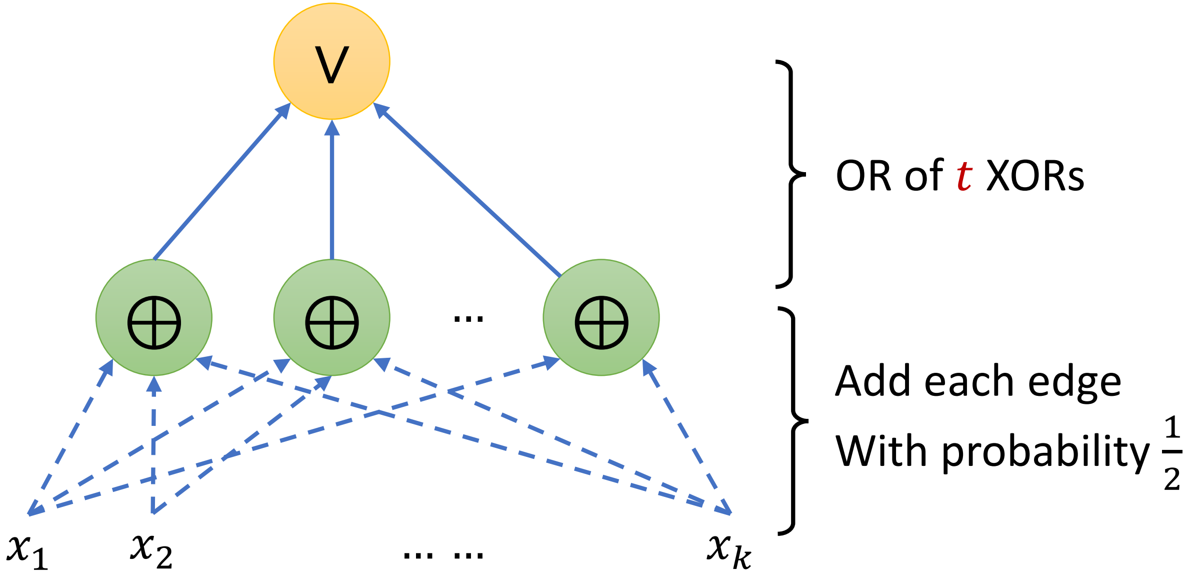



DeMorgan:

$$OR(x_1, \dots, x_k) = 1 \oplus \prod_{i=1}^{k} (1 \oplus x_i)$$

After expansion: $2^k$ monomials

**Probabilistic representation of OR**




Parameter $t$

Fewer monomials, correct whp

# Correct representation with high probability

## Case 1: $x_1 \vee \cdots \vee x_k = 0$

Easy case: each XOR outputs 0, top OR outputs 0

## Case 2: $x_1 \vee \cdots \vee x_k = 1$

Let $X$ be set of inputs with $x_i = 1$

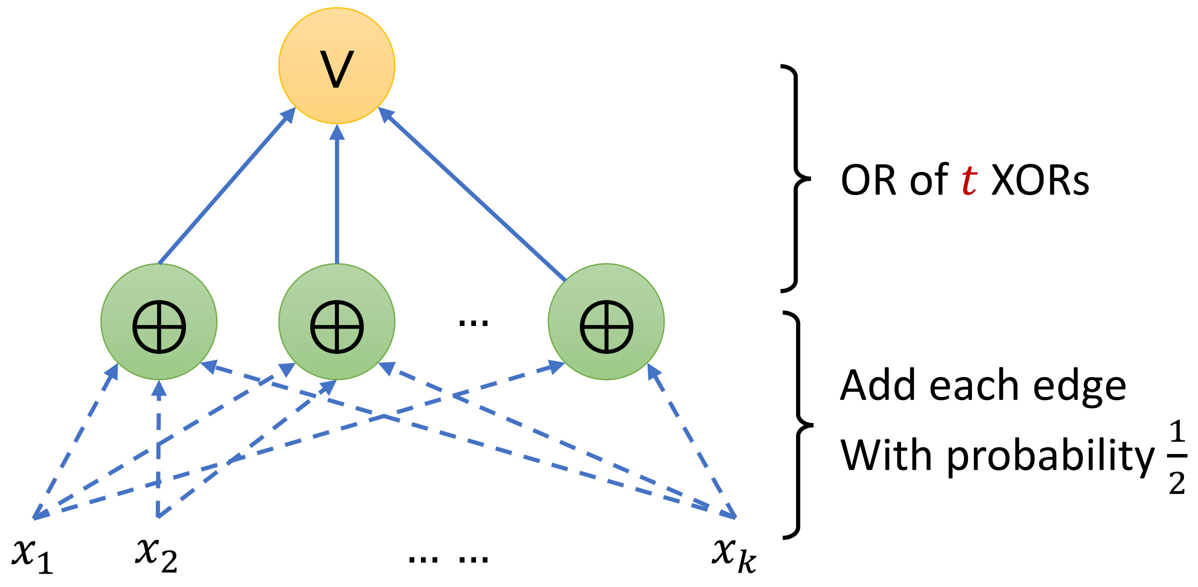
For each XOR:

- If XOR has odd number of links to $X$: XOR outputs 1 (good event: top OR outputs 1)
- If XOR has even number of links to $X$: XOR outputs 0 (bad event!)

Probability that XOR has even number of links to $X$:

$= 1/2$ because last element of X "decides" whether number of links is even or odd (each with prob. $1/2$)

**Probabilistic representation of OR**




$\Rightarrow$ Probability that all XORs output 0: $= \left(\frac{1}{2}\right)^t = \frac{1}{2^t}$

$\Rightarrow$ Probability that OR outputs 1: $= 1 - \frac{1}{2^t}$

# Bounding number of monomials

**Formal definition of polynomial**

For $i = 1 \dots t, j = 1 \dots k$:

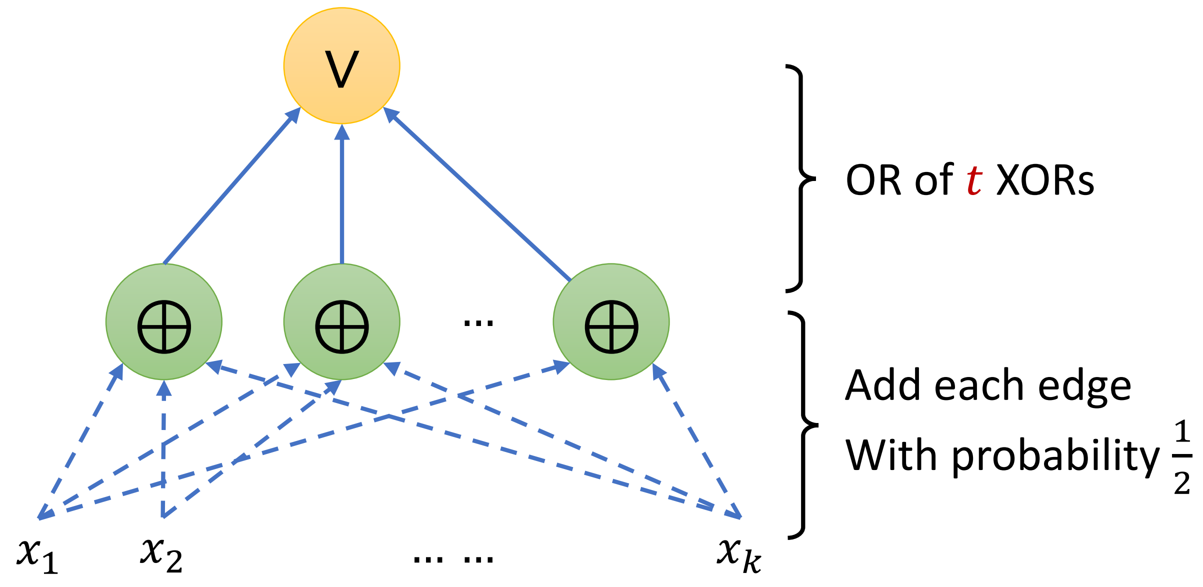
- With probability $\frac{1}{2}$: Set $r_{i,j} = 1$
- Otherwise: Set $r_{i,j} = 0$

**Polynomial:** $OR_t(x_1, \dots, x_k) =$

$$1 \oplus \prod_{i=1}^{t} \left(1 \oplus \oplus_{j=1}^{k} r_{i,j} \cdot x_i\right)$$

After expansion: $(k+1)^t$ monomials

**Probabilistic representation of OR**



# Formal definition

**Definition:** Let $C$ be a Boolean circuit with $k$ input gates and let $D$ be a finite distribution of polynomials on $k$ variables over a ring $R$ containing 0 and 1[(*)]. The distribution $D$ is a probabilistic polynomial over $R$ representing $C$ with error $\delta$ if for all $(x_1, \dots, x_k) \in \{0,1\}^k$:

$$\Pr_{p \sim D}[p(x_1, \dots, x_k) = C(x_1, \dots, x_k)] > 1 - \delta.$$

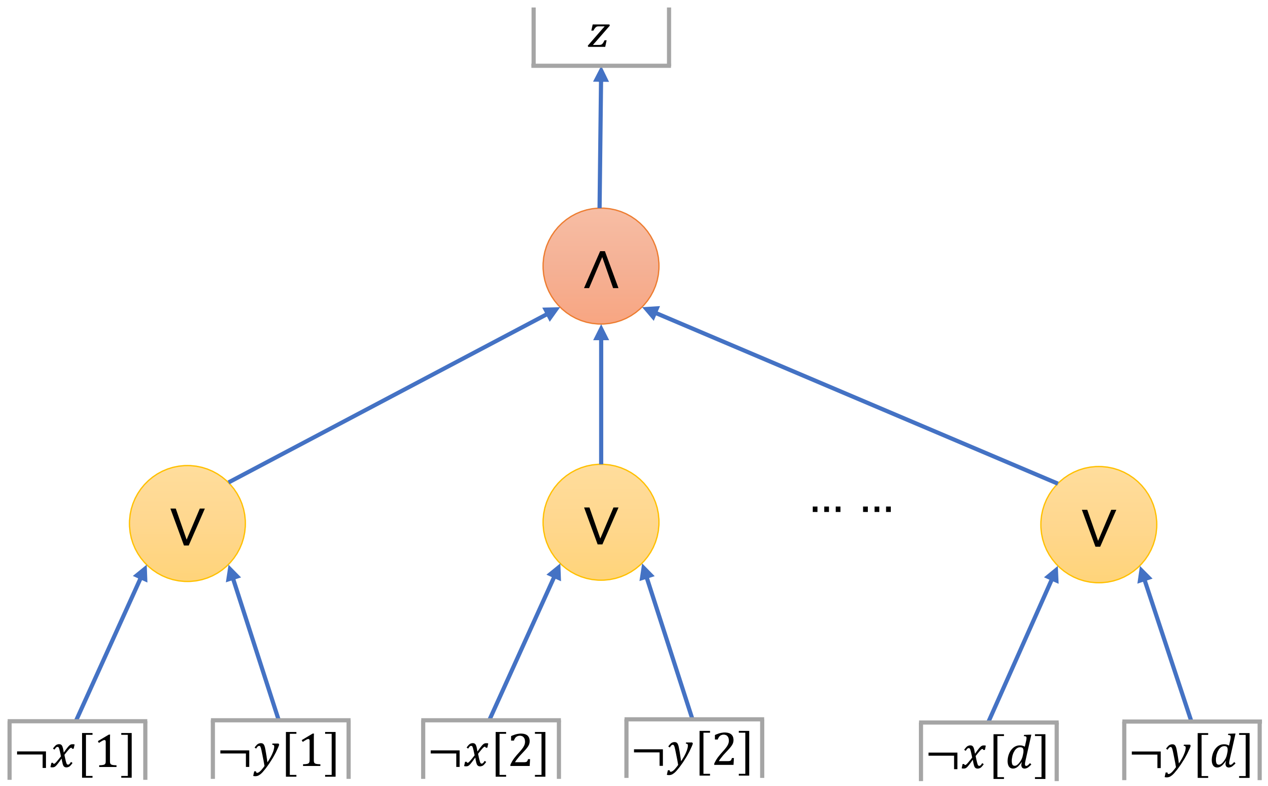
Example: OR-gate represented by

$$OR_t(x_1, \dots, x_k) = 1 \oplus \prod_{i=1}^{t} \left(1 \oplus \bigoplus_{j=1}^{k} r_{i,j} \cdot x_i\right) \text{ with error } \delta = 1 - \frac{1}{2^t}$$

(*) In our case, $R$ is the field $F_2$

# Representing OV circuit I

Are $x$ and $y$ orthogonal?




**Bottom OR:**

$\neg x[k] \vee \neg y[k]$
$\Rightarrow 1 \oplus x[k] \cdot y[k]$

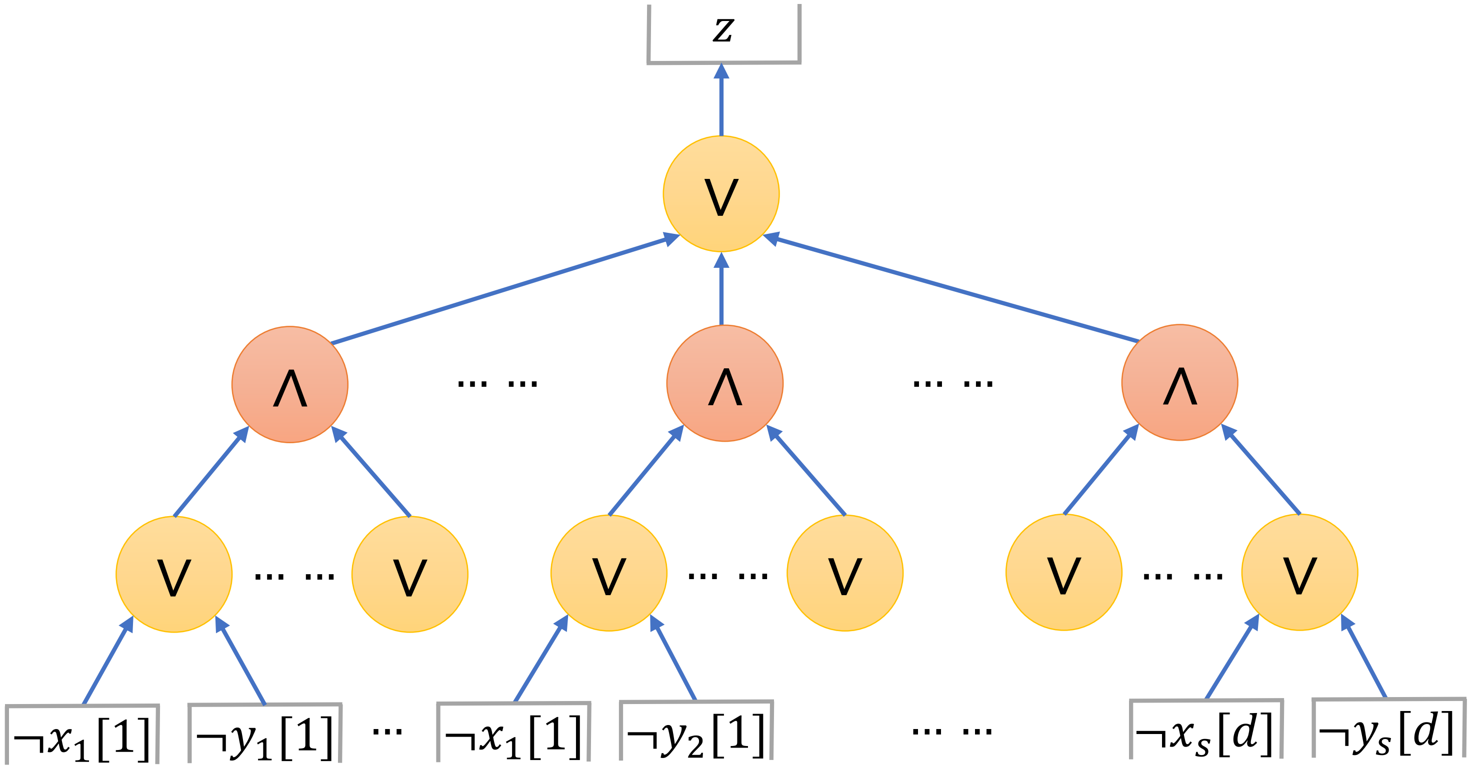
**Middle AND:**

1. DeMorgan
2. Razborov/Smolensky with $t_1 = 3 \log s$

Number of monomials:
$(d+1)^{t_1}$

# Representing OV circuit II

Is there an orthogonal pair?




**Middle ANDs:**

XOR of $s^2$ polynomials, each with $(d+1)^{t_1}$ monomials

$\Rightarrow s^2\,(d+1)^{t_1}$ monomials

**Top OR:**

Raz/Smol with $t_2 = 2$

$\Rightarrow (s^2(d+1)^{t_1})^{t_2}$ monomials

$= s^4(d+1)^{6\log s}$

# Analysis of error

We apply Razborov/Smolensky

- $s^2$ times with $t_1 = 3 \log s$
- 1 time with $t_2 = 2$

**Union Bound:** $\Pr(X \cup Y) \leq \Pr(X) + \Pr(Y)$

Probability of error: $\leq \frac{s^2}{2^{t_1}} + \frac{1}{2^{t_2}} = \frac{s^2}{s^3} + \frac{1}{4} = \frac{1}{s} + \frac{1}{4} \leq \frac{1}{3}$

($1/s$ is small enough for instances with sufficiently large $n$)

# Plugging in the right values

$$s = 2^{\epsilon \log n / \log d}$$
$$\epsilon = 1/160$$

#monomials: $m \leq s^4 (d+1)^{6 \log s} \leq s^4 (d+1)^{6\epsilon \log n / \log d}$

$$\leq 4\epsilon \log n + 12\epsilon \log n = 0.1 \log n$$

$$\Rightarrow m \leq n^{0.1}$$

# Running time for expanding polynomial

We explicitly have to expand our polynomial into XOR of monomials!
Running time dominated by applications of distributive law

1st expansion (repeated $s^2$ times):

- Degree after expansion: $O(t_1)$
- Total time: $O(s^2(d+1)^{t_1} t_1^2)$

2nd expansion :

- Degree after expansion: $O(t_1 t_2)$
- Running time: $O\left((s^2(d+1)^{t_1})^{t_2}\, t_1^2 t_2^2\right) \leq O(n^{0.1}\, t_1^2 t_2^2) \leq O(n^{0.1} \log^2 n)$

⇒ Total time: $O(n^{0.1} \log^2 n)$ (negligible)

# Summary for probabilistic polynomial

We can construct polynomial $P$ over $F_2$ with $2sd$ inputs such that, given two sets $A', B' \subseteq \{0,1\}^d$ of $d$-dimensional 0/1-vectors of size $s$, with probability $> \frac{2}{3}$: $P(A', B') = 1$ iff $A'$ and $B'$ have orthogonal pair.

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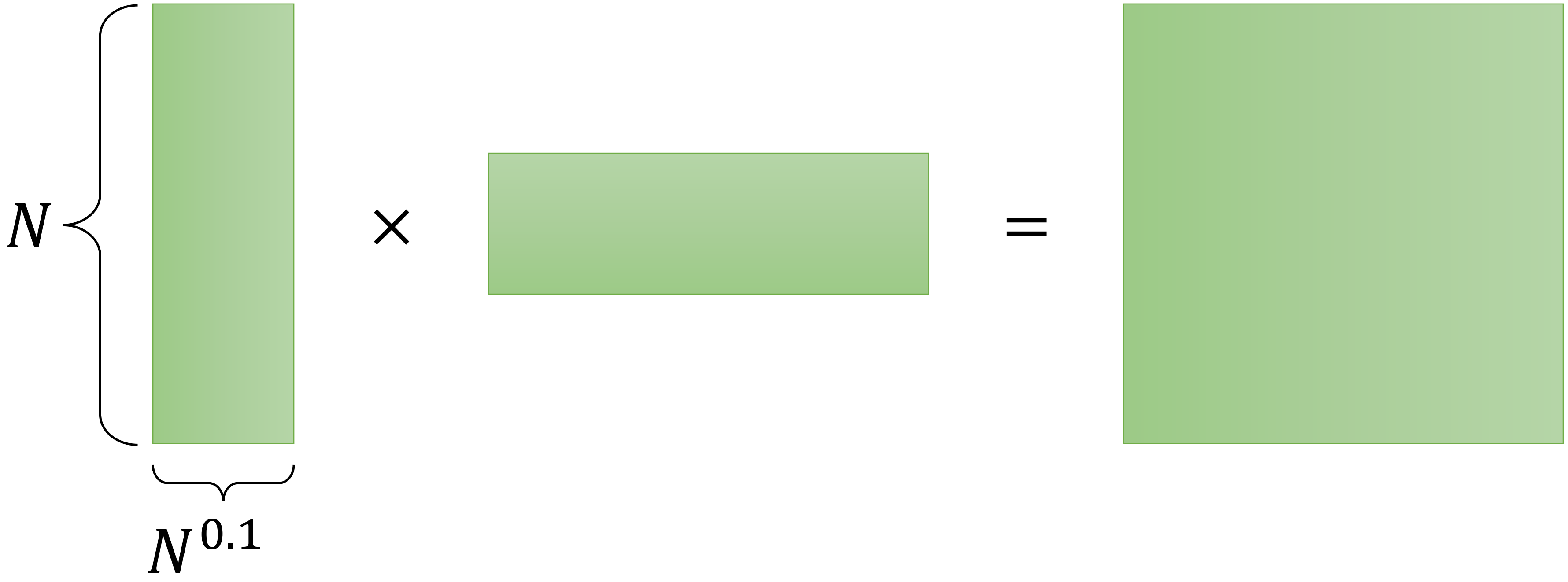
# Fast matrix multiplication

- Goal: Compute $C = A \times B$ where $A$ and $B$ are $n \times n$ matrices
- Naïve algorithm: $O(n^3)$
- Strassen's algorithm: $O(n^{2.807})$
- Current fastest: $O(n^{2.373})$
- Best we can hope for: $O(n^2)$

# Rectangular matrix multiplication




**Lemma**: There is an algorithm for multiplying an $N \times N^{0.17}$ matrix with an $N^{0.17} \times N$ matrix in time $O(N^2 \log^2 n)$.

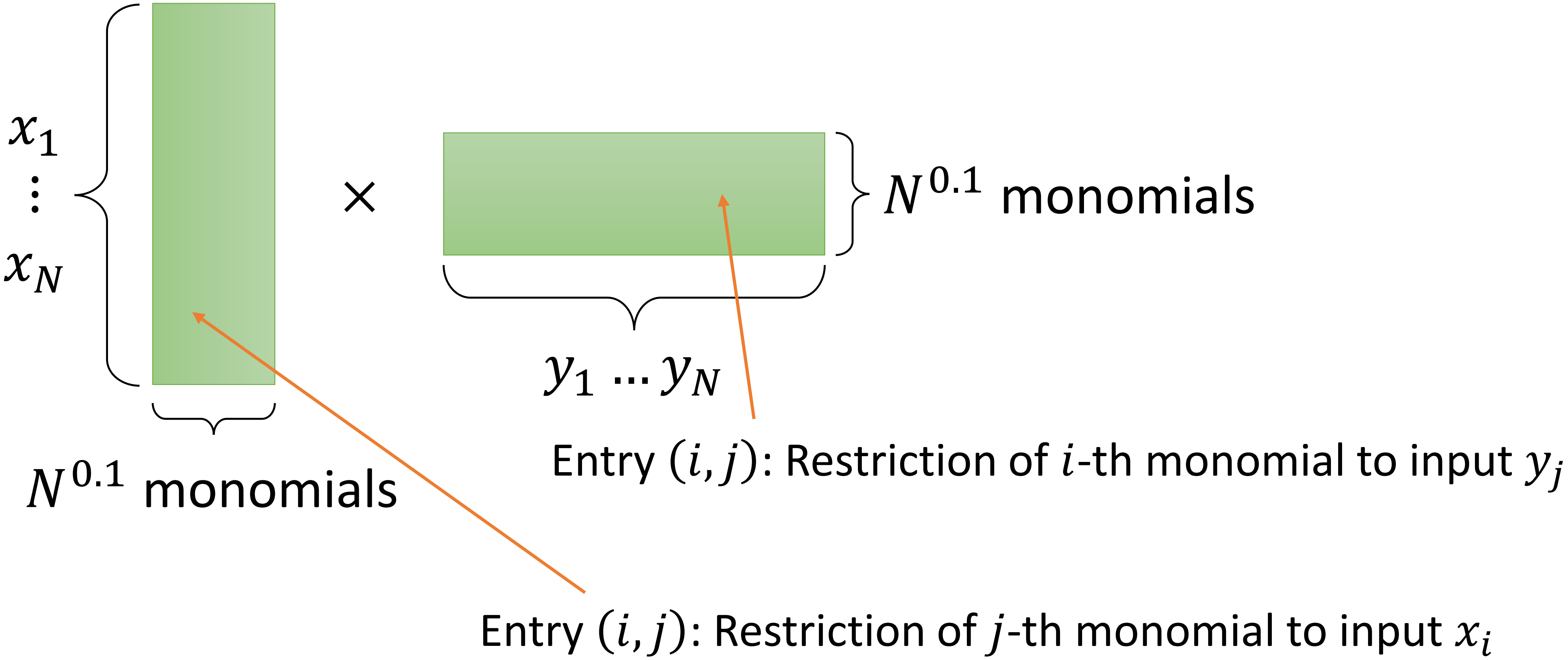
Also works for finite fields such as $F_2$!

# Fast evaluation of polynomial

- Given: Polynomial $P(x[1], \dots, x[K], y[1], \dots, y[K])$ over $F_2$
- With at most $N^{0.1}$ monomials
- Two sets of inputs:
$$X = \{x_1, \dots, x_N\} \subseteq \{0,1\}^K, Y = \{y_1, \dots, y_N\} \subseteq \{0,1\}^K$$
- Evaluate $P$ on all pairs $(x_i, y_j) \in X \times Y$ simultaneously

  in time $O(N^2 \mathrm{polylog}(n))$

# Reduction to matrix multiplication



# Evaluating OV-polynomial on all subgroups

1. Divide $A$ and $B$ into $q = \lceil \frac{n}{s} \rceil$ subsets of size $\leq s$:

   $A_1, \ldots, A_q$ and $B_1, \ldots, B_q$

2. Construct a polynomial $P(a_1[1], \ldots a_1[d], \ldots, a_q\,[1] \ldots, a_q[d],$

   $b_1[1], \ldots b_1[d], \ldots, b_q[1] \ldots, b_q[1])$

   $P(A_i, B_j) = 1$ if and only if $A_i$, $B_j$ contains orthogonal pair

3. For every pair of subsets $A_i$, $B_j$: evaluate $P$ on $A_i$, $B_j$

   $P$ has $\leq n^{0.1}$ monomials

   Simultaneous evaluation in time $O(n^2/s^2 \, \mathrm{polylog}(n)) \leq n^{2-1/O(\log d)}$

   $s = 2^{\epsilon \log n / \log d} = n^{\epsilon / \log d}$ for $\epsilon = 1/160$

# Remarks

Correctness with high probability

- Polynomial is only correct with probability $\geq 2/3$
- Amplify the success probability by repeating with $10 \log n$ independent polynomials and taking majority value
- $\Rightarrow$ Chernoff Bound

Faster algorithm:

- $n^{2-1/O(\log(d/\log n))}$
- Just needs better estimate for number of monomials and slightly different choice of $s$