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April 29, 2016
Deadline: lecture on May 9, 2016

Randomized Algorithms and Probabilistic Analysis of Algorithms

Summer 2016

Exercise Set 1

Please hand in your solutions at the beginning of the lecture on May 9.
You may work in groups of up to three students.

Exercise 1: (4 Points)
Show that in the contention-resolution algorithm with $p = \frac{1}{n}$ it takes $\Theta(n \log n)$ rounds in expectation until all processes have accessed the resource successfully at least once. Use linearity of expectation as we did in the coupon collector's problem.
(Note: This includes both an upper and a lower bound.)

Exercise 2: (4 Points)
Suppose there are n totally drunken sailors returning to their ship and choosing their cabins independently, uniformly at random (i.u.r.)[1]. What is the expected number of sailors sleeping in their own cabin?

Exercise 3: (2+4 Points)

(a) Given an example of random variables X and Y such that $\mathbf{E}[X \cdot Y] \gg \mathbf{E}[X] \cdot \mathbf{E}[Y]$ and an example such that $\mathbf{E}[X \cdot Y] \ll \mathbf{E}[X] \cdot \mathbf{E}[Y]$.

(b) Show that if $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ are bounded by $O(f(n))$ and $O(g(n))$ respectively with high probability, then $(X_n \cdot Y_n)_{n \in \mathbb{N}}$ is bounded by $O(f(n) \cdot g(n))$ with high probability. (Hint: Union Bound)

Exercise 4: (4 Points)
Consider a cut $(S, \bar{S})$ in a graph such that the number of edges crossing $(S, \bar{S})$ is at most α-times the number of edges in a minimum cut. Formally: $|E_S| \leq \alpha \cdot \min_{(S', \bar{S}') \text{ is a cut}} |E_{S'}|$. Show that the probability that $(S, \bar{S})$ survives the first $n - 2\alpha$ steps of the simple contraction algorithm with probability at least $\frac{1}{\binom{n}{2\alpha}}$ if $2\alpha \in \mathbb{N}$.

[1] Even drunk, they preserve their intimacy and don't share cabins.