Randomized Algorithms and Probabilistic Analysis of Algorithms
Summer 2016
Exercise Set 1

Please hand in your solutions at the beginning of the lecture on May 9.
You may work in groups of up to three students.

Exercise 1: (4 Points)
Show that in the contention-resolution algorithm with \( p = \frac{1}{n} \) it takes \( \Theta(n \log n) \) rounds in expectation until all processes have accessed the resource successfully at least once. Use linearity of expectation as we did in the coupon collector’s problem.
(Note: This includes both an upper and a lower bound.)

Exercise 2: (4 Points)
Suppose there are \( n \) totally drunken sailors returning to their ship and choosing their cabins independently, uniformly at random (i.u.r.)\(^1\). What is the expected number of sailors sleeping in their own cabin?

Exercise 3: (2+4 Points)
(a) Given an example of random variables \( X \) and \( Y \) such that \( \mathbb{E}[X \cdot Y] \gg \mathbb{E}[X] \cdot \mathbb{E}[Y] \) and an example such that \( \mathbb{E}[X \cdot Y] \ll \mathbb{E}[X] \cdot \mathbb{E}[Y] \).

(b) Show that if \((X_n)_{n \in \mathbb{N}}\) and \((Y_n)_{n \in \mathbb{N}}\) are bounded by \( O(f(n)) \) and \( O(g(n)) \) respectively with high probability, then \((X_n \cdot Y_n)_{n \in \mathbb{N}}\) is bounded by \( O(f(n) \cdot g(n)) \) with high probability. (Hint: Union Bound)

Exercise 4: (4 Points)
Consider a cut \((S, \bar{S})\) in a graph such that the number of edges crossing \((S, \bar{S})\) is at most \( \alpha \)-times the number of edges in a minimum cut. Formally: \( |E_S| \leq \alpha \cdot \min_{(S', \bar{S}')} |E_{S'}| \). Show that the probability that \((S, \bar{S})\) survives the first \( n - 2\alpha \) steps of the simple contraction algorithm with probability at least \( \frac{1}{2\alpha} \) if \( 2\alpha \in \mathbb{N} \).

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\(^1\)Even drunk, they preserve their intimacy and don’t share cabins.