Randomized Algorithms and Probabilistic Analysis of Algorithms
Summer 2016
Exercise Set 3

Exercise 1: (5 Points)
We now consider the more general balls-into-bins setting with \( m \) balls being thrown to \( n \) bins, \( n \neq m \). Show that the highest loaded bin contains \( O\left(\frac{mn}{n} + \log n\right) \) balls with high probability.

Exercise 2: (5 Points)
Consider a random variable \( X = X_1 + \ldots + X_n \) such that each \( X_i \) is independent and identically distributed with \( \Pr[X_i = 1] = p, \Pr[X_i = 0] = 1 - p \). To bound \( \Pr[X \geq (1 + \delta)E[X]] \), you can use Markov’s inequality, Chebyshev’s inequality, and the Chernoff bound. State the resulting bounds in terms of \( n \) and \( p \). For each of the three inequalities and each \( n \), give an example value of \( p \) and \( \delta \) such that its bound is the strongest of all three.

Exercise 3: (5 Points)
(Exercise 6.1. in Mitzenmacher/Upfal) Consider an instance of SAT with \( m \) clauses, where every clause has exactly \( k \) literals.

(1) Give a Las Vegas algorithm that finds an assignment satisfying at least \( m(1-2^{-k}) \) clauses, and analyze its expected running time.

(2) Give a derandomization of the randomized algorithm using the method of conditional expectations.

Exercise 4: (5 Points)
(Exercise 6.10 in Mitzenmacher/Upfal) A family \( \mathcal{F} \) of subsets of \( \{1, \ldots, n\} \) is an antichain if no set in \( \mathcal{F} \) is properly contained in another set of \( \mathcal{F} \).

(a) Give an example of an antichain of cardinality \( \binom{n}{\lfloor n/2 \rfloor} \).

(b) Let \( f_k \) be the number of sets in \( \mathcal{F} \) of size \( k \). Show that

\[
\sum_{0 \leq k \leq n} \frac{f_k}{\binom{n}{k}} \leq 1.
\]

(Hint: Choose a random permutation of the numbers from 1 to \( n \), and let \( X_k = 1 \) if the first \( k \) numbers in your permutation yield a set in \( \mathcal{F} \). Let \( X = \sum_{0 \leq k \leq n} X_k \). What can you say about \( X \)?)

(c) Prove that \( |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \) for every antichain \( \mathcal{F} \).
Exercise 5:  
(Exercise 6.16 in Mitzenmacher/Upfal) If
\[
4 \binom{k}{2} \binom{n}{k-2} 2^{1-\binom{k}{2}} \leq 1,
\]
then it is possible to color the edges of $K_n$ with two colors such that it has no monochromatic $K_k$ subgraph. Use the Lovasz local lemma.