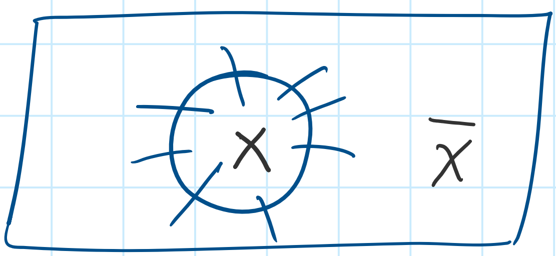
Kargars Mincut Algorithm

Sunday, June 16, 2019 4:59 PM

KARGAR'S MINCUT ALGORITHM

- All graphs are connected
- An <u>edge-cut</u> in a graph G is a partition of $V(G)$ as $(X, \bar{X})$
  - cutsize $\delta(X) = |E(X, \bar{X})|$

    edges with 1 endpoint in X and other in $\bar{X}$



G

- Equivalent characterization by subset of edges:

  $S \subseteq E(G)$ such that $G - S$ has two connected components.

- A (global) mincut in a graph G is a cut $(X, \bar{X})$ such that

A (global) mincut in a graph G
is a cut $(X, \bar{X})$ such that

$$\delta(X) \leq \delta(Y) \quad \forall\, Y \subseteq V(G)$$

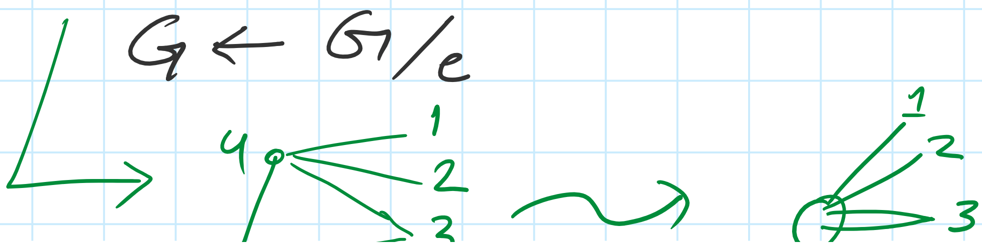
- Input: Graph G
  Output: A mincut $(X, \bar{X})$ of G

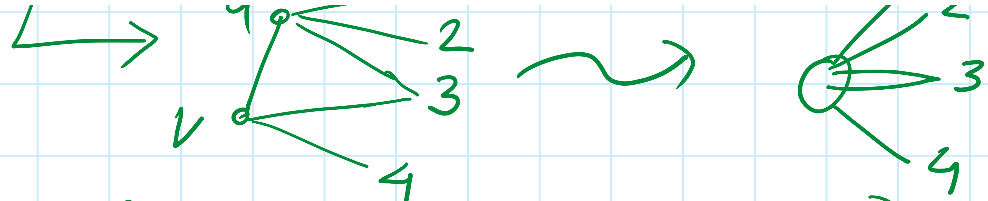
- Thm [Karger 93]: There is a $O(n^2 m \log(n))$ time randomized algorithm that outputs a mincut with prob $\geq 1/n^{100}$

<u>Algorithm:</u>

- For $i = 1$ to $t = 100\, n^2 \log n$
  - $G_i \leftarrow G$
  - while $|V(G_i)| > 2$:
    - pick an edge e at random
    - contract e in G

      $G \leftarrow G/e$





(Keep parallel edges)

- When $|V(G_i)| = 2$,
  let $S_i \leftarrow$ all edges in G_i
- Output the smallest of $S_1, S_2, \ldots S_t$ as the mincut of G

Running Time:

- 1 iteration takes $O(m)$ time
- we have $100 n^2 \log n$ iterations

Probability of Success

- Let C be a mincut in G containing k edges
- Obs: If we never pick any edge of C in our random sampling,

of C in our random sampling,
the cut C will survive,
and hence we output some mincut

- $\Pr$ we pick an edge from C

$$= \frac{k}{|E|} \le \frac{k}{\frac{nk}{2}} = \frac{2}{n}$$

- As mincut has k edges

$$\forall\, v \in V(G) \quad d(v) \geq k$$

$$\therefore \sum_{v \in V(G)} d(v) \geq nk$$

$$\Rightarrow \quad 2|E| \geq nk$$

$$\Rightarrow \quad \frac{1}{|E|} \leq \frac{nk}{2}$$

- Suppose we contract $e_1 e_2 \cdots e_\ell$ in G

Call the new graph G'

Then $\forall\, v \in V(G')$ $d_{G'}(v) \geq k$

Then $\forall\, v \in V(G')\ \ d_{G'}(v) \geq k$

Pf: Exercise

– Pr of Success in iter i

$= Pr(\text{no edge of } C \text{ was picked})$

$$= \prod_{j=1}^{n-2} Pr(j\text{th edge picked is not in } C)$$

$$\geq \underbrace{\prod_{i=1}^{n-2} \left(1 - \frac{2}{n-j+1}\right)}$$

$$\left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right) \cdots \cdots \left(1-\frac{2}{3}\right)$$

$$= \frac{n-2}{n} \ \frac{n-3}{n-1} \ \frac{n-4}{n-2} \cdots \frac{2}{4}\ \frac{1}{3}$$

$$= \frac{2}{n(n-1)} > \frac{1}{n^2}$$

- $\Pr[\text{Success in some iter}]$

$$= 1 - \Pr[\text{Fail in all iter}]$$

$$\geq 1 - \left(1 - \frac{1}{n^2}\right)^{n^2 100 \log n}$$

$$= 1 - (1/e)^{100 \log n}$$

$$= 1 - 1/n^{100}$$