# Min k connected subgraph

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## Minimum k-connected subgraph

Input: (Dir.) Graph G which is k-connected

Output: Spanning subgraph H of G
which is k-connected
has minimum number of edges

Note: Edge Connectivity

Recall:

- Graph G is k-connected if and only if for every pair of vertices $s, t \in V(G)$, there are k edge disjoint paths from s to t

- In directed graph G, we have such a collection of paths in both directions, s to t and t to s

Special Cases:

## Special Cases:

- Undir Graphs, $k = 1$
  Min Spanning Tree
- Digraphs, $k = 1$
  Min Strong Spanning Subgraph
- Undir, $k \geqslant 2$
  Min k-conn spanning subgraph

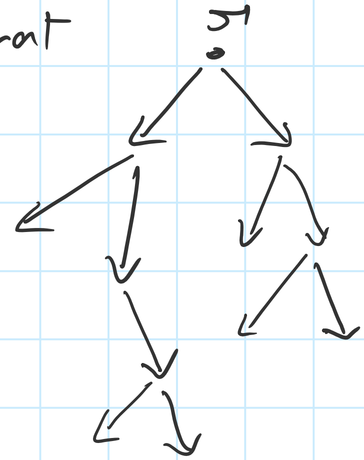
NP-hard $\longrightarrow$ Approximation Algorithms

- Given a digraph $G$, the underlying graph $U(G)$ is obtained by forgetting the directions of all edges of $G$.

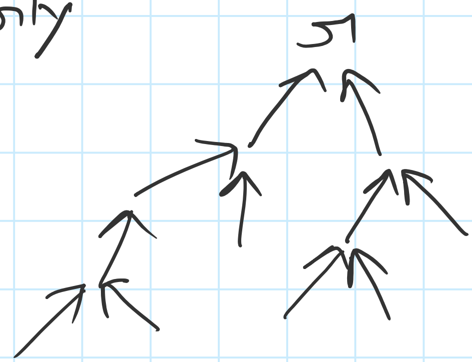
Def$^n$: Given a digraph $G$, vertex $r \in V(G)$ an out-branching rooted at $r$, denoted by $T_r$, is a spanning subgraph such that $r$

denoted by $T_r$, is a spanning subgraph such that

- the underlying graph $U(T_r)$ is a tree
- every vertex except $r$ has exactly on in-edge




An <u>in-branching</u> is similarly defined




<u>Lemma</u>: Digraph $G$ is $k$-connected if and only if, for any vertex $r \in V(G)$

- there is a collection of $k$ edge-disjoint in-branchings rooted at $r$ in $G$

$$\mathcal{T}^{in} = \{ T_1^{in}, T_2^{in}, \ldots, T_k^{in} \}$$

- and, there is a collection of $k$ edge-disjoint outbranchings rooted at $r$ in $G$

$$\mathcal{T}^{out} = \{ T_1^{out}, \ldots, T_k^{out} \}$$

Further $\mathcal{T}^{out}$, $\mathcal{T}^{in}$ can be constructed in polynomial time

- Using the above lemma, we can obtain a 2-approximation algorithm for Min k-conn Spanning Subgraph
  - In digraph G
    - Let $H^*$ be an optimum solution
      - $|E(H^*)| \geq \underbrace{k(n-1)}_{\text{number of edges in } \mathcal{T}^{out}_H}$
      - Let $H = \bigcup_{i=1}^{k} T_i^{out} \cup T_i^{in}$

        Clearly $|E(H)| \leq |\mathcal{T}^{out}| + |\mathcal{T}^{in}|$

        $\leq 2k(n-1)$
  - In undir graph G:
    - Convert to digraph G':

- convert to digraph $G'$:
  for edge $(u,v) \in E(G)$
  we have $u \to v$, $v \to u$ in $E(G')$
- apply algorithm for digraph $G'$
  and obtain sol$^n$ $H'$
- convert directed $H'$ to undir $H$
- show $H$ is $k$-conn (Exercise)
  and 2-approx sol$^n$

## Proof of above lemma:

(Reverse Dir)

Claim: Suppose we are given
$G$, $k$, $r \in V(G)$, $\mathcal{Y}_{in}$, $\mathcal{Y}_{out}$.
Then $G$ is $k$-connected

Proof: Exercise
(For every $s, t \in V(T)$ we have

(For every $s, t \in V(T)$ we have $k$ paths from $s$ to $t$, and the reverse)

(Forward Direction)

Suppose we are given $G, k, r \in V(G)$

Then we can construct $\mathcal{Y}^{out}, \mathcal{Y}^{in}$ in polynomial time.

- Let us describe the construction of $\mathcal{Y}^{out}$.
  - $\mathcal{Y}^{in}$: construct out-branchings in $G^{reverse}$

  → reverse the dir of every edge in $G$

<u>Claim:</u> $G$ has $k$ edge disjoint outbr rooted at $r \in V(G) \iff$

$$\forall X \subseteq V(G) - r, \quad d^{in}(X) \geq k$$

<u>Proof:</u> $(\Rightarrow)$ Clearly, if we have $T_1^{out}, \ldots, T_k^{out}$, for any $X \subseteq V(G) - r$, we have

at least $k$ in-coming edges to $X$, one from each $T_i^{out}$.

($\Leftarrow$) Given $G, k, r \in V(G)$, we describe the construction of an out-branching $T_k^{out}$ such that,

$\forall X \subseteq V(G) - r$,

$$d^{in}_{G - E(T_k^{out})}(X) \geq k-1$$

- Then after constructing $T_k^{out}$, we construct $T_{k-1}^{out}$ in $G - E(T_k^{out})$, $T_{k-2}^{out}$ in $G - (E(T_k^{out}) \cup E(T_{k-1}^{out}))$ ... and so on till $T_1^{out}$

Let us describe the algorithm

- Start from $T$ $\leftarrow$ only one vertex $r$
  and grow it

and grow it by adding out-edges one by one

vertex $r$ and no edges

- We will show that the following invariant is true at every step

$$\forall X \subseteq V(G) - r, \; d^{in}_{G-E(T)}(X) \geq k-1$$

- To grow $T$ by one edge we do the following:
  - Clearly $V(T) \subsetneq V(G)$, otherwise we are done.
  - Call $X \subseteq V(G)$ tight if $d^{in}_{G-E(T)}(X) = k-1$

- Claim: If $X$ and $Y$ are tight and $X \cap Y \neq \emptyset$ then $X \cap Y$ is also tight

Proof:

- Note that $X \cup Y \subseteq V(G) - r$
- Let $G' = G - E(T)$

$$d^{in}_{G'}(X \cup Y) + d^{in}_{G'}(X \cap Y) \leq d^{in}_{G'}(X) + d^{in}_{G'}(Y)$$

We use submodularity of the function $d^{in}$ in a digraph

- Claim: If $X$ is tight, then $X \cap V(T) \neq \emptyset$

Proof: Initially, $\forall X \subseteq V(G) - r$

$$d^{in}_{G}(X) \geq k$$

and $d^{in}_{G-E(T)}(X) = k-1$

$\therefore$ some in-edge of $X$ lies in $E(T)$

$\Rightarrow$ the head of this in-edge lies in $X \cap V(T)$

- Let $W \leftarrow$ Inclusion-wise minimal subset

Let $W \leftarrow$ inclusion-wise minimal subset of $V(G) - r$ such that

- $W$ is tight
- $W$ is not a subset of $V(T)$

If such a $W$ doesn't exist then let $W = V(G)$

- <u>Observe</u>: In this case, every tight set is a subset of $V(T)$

<u>Claim</u>: Given $G$, $r$, $k$, $T$, $W$ can be constructed in polynomial time

<u>Proof</u>: Let $G' = G - E(T)$.

- Find $w \in V(G) - r$ such that $|\text{mincut}(r, w)| = k-1$
  - by testing all $w \in V(G) - r$ and MaxFlow algorithm
- Compute a <u>closest-to-$w$</u> mincut $S$

$W$ = all vertices in $G' - S$ that can reach $w$

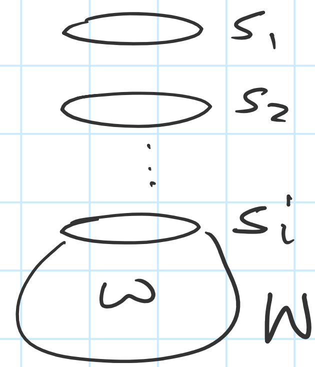
$r$

## (*) Closest-to-$w$ mincut:

- $S_1 \leftarrow (r, w)$ mincut

  $S_2 \leftarrow (S_1, w)$ mincut

  and so on, until $S_i$ where $|\text{mincut}(S_i, w)| \geq k$



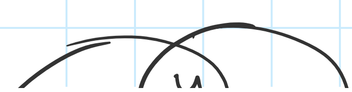

<u>Claim:</u> There is an edge $u \to v$ where $u \in W \cap V(T)$

$v \in W - V(T)$

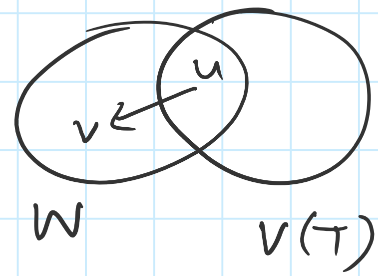
<u>Pf:</u> If $W = V(G)$ (and $V(T) \subsetneq V(G)$) then we can easily find $u \to v$

Else $W \subsetneq V(G) - r$, and suppose there is no arc from $W \cap V(T)$ to $W - V(T)$. Then,

$d^{in}_G(W - V(T))$



$$d^{in}_G(W - V(T))$$




$$= d^{in}_{G-E(T)}(W - V(T))$$

$$\leqslant d^{in}_{G-E(T)}(W)$$

$$= k-1$$

But this contradicts our premise that for

$$X = W - V(T) \subseteq V(G) - r$$

$$d^{in}_G(X) \geqslant k$$

Hence there must be an arc from $W \cap V(T)$ to $W - V(T)$

- Our goal is to show that we can add $u \to v$ to $T$

In other words, let $T' = T + (u,v)$

Then $\forall\ X \subseteq V(G) - r$, we have

$$d^{in}_{G-E(T')}(X) \geqslant k-1$$

- The above condition fails only if there is some tight set $X$ such that $v \in X$.

<u>Claim:</u> No such tight set exists

<u>Pf:</u> Suppose $X$ exists, and as $v \notin V(T)$, $X \not\subseteq V(T)$

$\therefore$ $W \neq V(G)$

as $v \in W \cap X$, $W \cap X \neq \emptyset$
and $X, W \subseteq V(G) - r$
$\Rightarrow$ $X \cup W \neq V(G)$

$\therefore$ by submodularity of $d^{in}_{G-E(T)}$ we have $X \cap W$ is tight, and it is not a subset of $V(T)$ as $v \in X \cap W$

This contradicts our choice of $W$ as a inclusion-wise minimal tight set that is not a subset of $V(T)$

minimal right set that
is not a subset of $V(T)$