Min k connected subgraph

Minimum k-connected subgraph

Input: (Diir.) Graph G which is k-connected

Output: Spanning subgraph H of G which is k-connected has minimum number of edges

Note: Edge Connectivity

Recall:
- Graph G is k-connected if and only if for every pair of vertices s, t \in V(G), there are k edge disjoint paths from s to t
- In directed graph G, we have such a collection of paths in both directions, s to t and t to s

Special Cases:
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- Undir Graphs, \( k = 1 \)
  Min Spanning Tree

- Digraphs, \( k = 1 \)
  Min Strong Spanning Subgraph

- Undir, \( k \geq 2 \)
  Min \( k \)-conn spanning subgraph

\text{NP-hard} \rightarrow \text{Approximation Algorithms}

- Given a digraph \( G_I \), the underlying graph \( U(G_I) \) is obtained by forgetting the direction of all edges of \( G_I \).

\text{Def.} \quad \text{Given a digraph } G_I, \text{ vertex } v \in V(G_I) \text{ an out-branching rooted at } v, \text{ denoted by } T_{v}, \text{ is a spanning subgraph such that } v.
denoted by \( T_n \), is a spanning subgraph such that:

- the underlying graph \( U(T_n) \) is a tree
- every vertex except \( s \) has exactly one in-edge

An in-branching is similarly defined.

**Lemma:** Digraph \( G_1 \) is \( k \)-connected if and only if, for any vertex \( s_1 \in V(G_1) \):

- there is a collection of \( k \) edge-disjoint in-branchings rooted at \( s_1 \) in \( G_1 \)
  \[ Y_{in} = \bigcup_{i=1}^{k} T_{in} \]

- and, there is a collection of \( k \) edge-disjoint out-branchings rooted at \( s_1 \) in \( G_1 \)
  \[ Y_{out} = \bigcup_{i=1}^{k} T_{out} \]
Further, $Y_{\text{out}}, Y_{\text{in}}$ can be constructed in polynomial time.

- Using the above lemma, we can obtain a 2-approximation algorithm for Min $k$-conn Spanning Subgraph.

- In digraph $G$

  - Let $H^*$ be an optimum solution

  $|E(H^*)| \geq k(n-1)$

  - Let $H = \bigcup_{i=1}^{k} T_{\text{out}}^i \cup T_{\text{in}}^i$

  Clearly $|E(H)| \leq |Y_{\text{out}}| + |Y_{\text{in}}| \leq 2k(n-1)$

- In undigraph $G'$:
  - Convert to digraph $G'$.
- Convert to digraph $G'$:
  for edge $(u,v) \in E(G)$
  we have $u \to v$, $v \to u$ in $E(G')$
- Apply algorithm for digraph $G'$
  and obtain soln $H'$
- Convert directed $H'$ to undir $H$
- Show $H$ is $k$-connected (Exercise)
  and 2-approx soln

Proof of above lemma:

(Reverse Din)

Claim: Suppose we are given $G_1$, $k$, $x \in V(G_1)$, $y \in V(G_1)$.
Then $G_1$ is $k$-connected.

Proof: Exercise

(For every $s,t \in V(T)$ we have)
(For every $s, t \in V(T)$ we have $k$ paths from $s$ to $t$, and the reverse)

(Foward Direction)

Suppose we are given $G_1, k, s \in V(G_1)$.
Then we can construct $Y_{out}, Y_{in}$ in polynomial time.

- Let us describe the construction of $Y_{out}$.
- $Y_{in}$: Construct out-brandings in $G_{\text{reverse}}$.

\[ \text{reverse the dir of every edge in } G_1 \]

Claim: $G_1$ has $k$ edge disjoint out-brandings rooted at $s \in V(G_1)$ if \[ \forall X \subseteq V(G_1) - s, \quad d_{in}(X) \geq k \]

Proof: $(\Rightarrow)$ Clearly, if we have $T_1, \ldots, T_k$ for any $X \subseteq V(G_1) - s$, we have
at least $k$ in-coming edges
to $x$, one from each $T_{i}\text{.}$

$(\Leftarrow)$ Given $G, k, \pi \in \mathcal{V}(G)$,
we describe the construction
of an out-branching $T_{k}\text{,}$
such that,

$$\forall x \in V(G) - \pi, \quad d_{G - E(T_{k})}^{-}(x) \geq k - 1$$

Then after constructing $T_{k}\text{,}$
we construct $T_{k - 1}\text{,}$ in
$G - E(T_{k})\text{,}$ $T_{out}$ in
$G - (E(T_{k}) \cup E(T_{k - 1}))$.

... and so on till $T_{1}\text{.}$

Let us describe the algorithm.

Start from $T_{\text{in}}$ and arrow it
and arrow it
and grow it by adding out-edges one by one.

We will show that the following invariant is true at every step:

\[ \forall x \in V(G) - \pi, \quad d_{G - E(T)}^\text{in}(x) \geq k - 1 \]

To grow \( T \) by one edge, we do the following:

- Clearly \( V(T) \subseteq V(G) \), otherwise we are done.
- Call \( x \in V(G) \) tight if \( d_{G - E(T)}^\text{in}(x) = k - 1 \).

Claim: If \( X \) and \( Y \) are tight and \( X \cap Y \neq \emptyset \), then \( X \cap Y \) is also tight.

Proof:...
- Note that $X \cup Y \leq V(G_i) - \pi$
- Let $G_i' = G_i - E(T)$

$$d^\text{in}_{G_i'}(X \cup Y) + d^\text{in}_{G_i'}(X \cap Y) \leq d^\text{in}_{G_i'}(X) + d^\text{in}_{G_i'}(Y)$$

We use submodularity of the function $d^\text{in}$ in a digraph

- **Claim**: If $X$ is tight, then $X \cap V(T) \neq \emptyset$

**Proof**: Initially, $\forall X \subseteq V(G_i) - \pi$

$$d^\text{in}_{G_i'}(X) \geq k$$

and $d^\text{in}_{G_i - E(T)}(X) = k - 1$

$\Rightarrow$ some in-edge of $X$ lies in $E(T)$

$\Rightarrow$ the head of this in-edge lies in $X \cap V(T)$

- Let $W \Leftarrow$ Inclusion-wise minimal subset
Let \( W \subseteq \text{inclusion-wise minimal subset of } V(G) \)-\( \mathfrak{m} \) such that
- \( W \) is tight
- \( W \) is not a subset of \( V(T) \)

If such a \( W \) doesn't exist then let \( W = V(G) \)
- **Observe**: In this case, every tight set is a subset of \( V(T) \)

**Claim**: Given \( G_1, \mathfrak{m}, k, T \), \( W \) can be constructed in polynomial time.

**Proof**: Let \( G'_1 = G_1 - E(T) \).
- Find \( w \in V(G) - \mathfrak{m} \) such that
  \[ |\text{mincut}(\mathfrak{m}, w)| = k-1 \]
  - by testing all \( w \in V(G) - \mathfrak{m} \) and MaxFlow algorithm
- Compute a closest-to-\( w \) mincut \( S \)
  \[ W = \text{all vertices in } G'_1 - S \text{ that can reach } w \]
(*) Closest-to-ω mincut:
- $S_1 \leq (G, w) \text{ mincut}$
- $S_2 \leq (S_1, w) \text{ mincut}$
and so on, until $S_i$
where $|\text{mincut}(S_i, w)| > k$

Claim: There is an edge $u \rightarrow v$
where $u \in W \cap V(T)$
$v \in W - V(T)$

Proof: If $W = V(G)$ (and $V(T) \notin V(G)$)
then we can easily find $u \rightarrow v$

Else $W \notin V(G) - \pi$, and suppose there is no arc from $W \cap V(T)$
to $W - V(T)$. Then,

$\delta_{in}(W - V(T))$
\[ d_{G_t}^{\text{in}}(W - V(T)) \]
\[ = d_{G_t - E(T)}^{\text{in}}(W - V(T)) \]
\[ \leq d_{G_t - E(T)}^{\text{in}}(W) \]
\[ = k - 1 \]

But this contradicts our premise that for
\[ X = W - V(T) \leq V(G_t) - \gamma \]
\[ d_{G_t}^{\text{in}}(X) \geq k \]

Hence, there must be an arc from \( W \cap V(T) \) to \( W - V(T) \)

Our goal is to show that we can add \( u \rightarrow v \) to \( T \)

In other words, let \( T' = T + (u,v) \)

Then \( \forall X \leq V(G_t) - \gamma \), we have
\[ d_{G_t - E(T')}^{\text{in}}(X) \geq k - 1 \]
The above condition fails only if there is some tight set \( X \) such that \( v \notin X \).

Claim: No such tight set exists

\[ \text{Proof:} \] Suppose \( X \) exists, and so
\[ v \notin V(T), \ X \notin V(T) \]
\[ \implies W \neq V(G) \]
\[ \Rightarrow v \in W \cap X, \ W \cap X \neq \emptyset \]
and \( X \cup W \subseteq V(G) - v \)
\[ \Rightarrow X \cup W \neq V(G) \]
\[ \Rightarrow \text{by submodularity of } d_{G - E(T)} \]
we have \( X \cup W \) is tight,
and it is not a subset of \( V(T) \) as \( v \notin X \cup W \).

This contradicts our choice of \( W \) as a inclusion-wise minimal tight set that is not a subset of \( V(T) \).
minimal right set that is not a subset of $V(T)$